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Article

On Proximity Spaces Constructed on Rough Sets

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Abstract: Based on equivalence relation R on X, equivalence class $[x]$ of a point and equivalence class $[A]$ of a subset represent the neighborhoods of x and A, respectively. These neighborhoods play the main role in defining separation axioms, metric spaces, proximity relations and uniformity structures on an approximation space (X, R) depending on the lower approximation and the upper approximation of rough sets. The properties and the possible implications of these definitions are studied. The generated approximation topology τ_R on X is equivalent to the generated topologies associated with metric d, proximity δ and uniformity $\mathcal{U}$ on X. Separated metric spaces, separated proximity spaces and separated uniform spaces are defined and it is proven that both are associating exactly discrete topology τ_R on X.

Keywords: approximation space; rough set; separation axioms; metric spaces; proximity relations; uniform structures

MSC: 03E02; 03E20; 54D010; 54D15; 54E35; 54E05; 54E15

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1. Introduction

Originally, Pawlak in [1] initiated the notions of lower approximation set $L(A)$ and upper approximation set $U(A)$ of subset A of universal set X depending on the equivalence classes formed by equivalence relation R on X. The pair (X, R) is then called an approximation space. From the set difference, $U(A) \setminus L(A)$, a boundary region area is formed and is called the boundary region set $B(A)$. Any subset in (X, R) is then a rough set (whenever $B(A) \neq \emptyset$) or an exact set (whenever $B(A) = \emptyset$). The importance of this boundary region set is in its role in many real applications; refs. [2,3] are samples of research work of such applications. Decision Theory and Data Mining are the most intercept branches with the concept of rough sets. Yao in [4,5] extended the research work on rough sets and explained the algebraic properties of rough sets. Some researchers paid their attention to the approximation spaces (X, R) constructed by an arbitrary (not equivalence) relation R on X. As an example, ref. [6] objected to the effects on the notion of rough sets by reflexive relations or transitive relation or both. Generating approximation topology τ_R associated with (X, R) is explained well in [7,8], whenever (X, R) is constructed by arbitrary relation R on X. Then, we obtain left approximation neighborhoods $R < x >$ and right approximation neighborhoods $< x > R$ at each point $x \in X$. That is, the notion of rough sets has a generalized form (as found in [4,9]) in which the definition of Pawlak is a special case. Kozae, in [10], introduced a generalization of rough sets using the intersection of left and right approximation neighborhoods $R < x >$ and $< x > R$, respectively, at point $x \in X$. The resulting rough sets (in [10]) have fewer boundary region sets than those defined in [1,4,9], and so it is a good generalized definition. Following that generalized definition in [10], Ibedou et al. [11,12] introduced two types of generalizations of rough sets in the fuzzy case. Also, in this paper, we follow the same strategy. For all basics in general topology, please refer to [13–15].

The aim of this paper is to construct a proximity relation and a uniformity structure on an approximation space (X, R), and also define a metric function and separation axioms based on the rough sets in (X, R). In Section 2, we present (in the sense of Pawlak) some basics of rough sets and introduce the definitions of separation axioms T_i, $i = 0, 1, 2, 3, 4$ in (X, R). In Section 3, we focus on defining metric d on approximation space (X, R) and study its usual properties. In Section 4, we define proximity relation δ on (X, R) and study its properties. In Section 5, we define a uniform structure $\mathcal{U}$, similar to that defined in [16], on (X, R). We study the relations in between notion separation axioms T_i, $i = 0, 1, 2, 3, 4$ in (X, R), metric spaces (X, d), proximity spaces (X, δ) and uniform spaces $(X, \mathcal{U})$ based on the rough sets defined by an equivalence relation R on X. Finally, in Section 6, we explain the deviations in these notions whenever R is not an equivalence relation on X.

2. Preliminaries

Throughout the paper, we let X be a universal set of objects, let $P(X)$ be the power set of X and let 2^X denote the set of all characteristic functions on X. Then, in the set theory, it is well known that there is a one-to-one correspondence between $P(X)$ and 2^X. Thus, we use subset A and characteristic function A without distinction.

Relation R on X is mapping $R : X \times X \to \{0, 1\}$ defined by the following: for any $x, y \in X$,

$$R(x, y) = 1 \text{ if } x \text{ and } y \text{ are related and } R(x, y) = 0 \text{ if } x \text{ and } y \text{ are not related.}$$

R is called an equivalence relation on X if it satisfies the following conditions:

(1) R is reflexive, that is, for all $x \in X$, we have $R(x, x) = 1$,
(2) R is symmetric, that is, $R(x, y) = R(y, x)$ for any $x, y \in X$,
(3) R is transitive, that is, $R(x, z) \leq R(x, y) \wedge R(y, z)$ for any $x, y, z \in X$,

where $R(x, y) \wedge R(y, z) = min\{R(x, y), R(y, z)\}$.

The pair (X, R) is called an approximation space (see [1]).

The equivalence relation R is partitioning X into equivalence classes $[x]$ for each $x \in X$, where an equivalence class $[x]$ is mapping $[x] : X \to \{0, 1\}$ defined, for each $y \in X$, as follows:

$$[x](y) = 1 \text{ iff } R(y, x) = 1 \text{ and } [x](y) = 0 \text{ iff } R(y, x) = 0.$$

Then, for any $x, y \in X$, we have

$$x \in [y] \text{ iff } y \in [x] \text{ iff } [x] = [y] \text{ iff } [x] \cap [y] \neq \phi,$$

and moreover, $[x]$ and $[y]$ are disjointed:

$$[x] \cap [y] = \emptyset \text{ iff } R(x, y) = 0 \text{ iff } [x](z) \neq [y](z) \text{ for all } z \in X.$$

Now, for each $A \in 2^X$, the equivalence class $[A]$ of A is defined by

$$[A] = \bigvee_{x \in A} [x].$$

Then, $[A] = \{z \in X : \text{ there exists } x \in A \text{ with } R(x, z) = 1\}$ that is,

$$[A](z) = 1 \text{ iff } R(x, z) = 1 \text{ for some } x \in A.$$

For each $x \in X$ and each $A \in 2^X$, we have $\{x\} \subseteq [x]$ and $A \subseteq [A]$, respectively, and these equivalence classes, $[x]$ and $[A]$, are called the neighborhoods of x and A, respectively.

In general, let us define an equivalence class $[B]$ as follows:

$$[B](x) = \bigvee_{y \in X} \big(B(y) \wedge R(y,x)\big) \equiv \bigvee_{y \in X} (B \cap [x])(y). \tag{1}$$

Remark 1. *For $A, B \subseteq X$ where A which is not a singleton or B is not a singleton, we have $[A] \cap [B] = \emptyset$, which implies $A \cap B = \emptyset$ but not the converse. For example, we let $X = \{a,b,c,d,e,f\}$, $R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(b,d),(d,b),(e,f),(f,e)\}$, $K = \{a,c,d\}$, $H = \{b,e\}$. Then, $[K] = \{a,b,c,d\}$, $[H] = \{b,d,e,f\}$. That is, $K \cap H = \emptyset$ while $[K] \cap [H] = \{b,d\} \neq \emptyset$. Thus, for non-singleton sets, A, B may be found $[A] \cap [B] \neq \emptyset$ but $[A], [B]$ are not identical as the case with two singletons. $A \subseteq [B]$ and $B \subseteq [A]$ implies $[A] = [B]$, and in general $[A]^c \subseteq A^c \subseteq [A^c]$, $[[A]] = [A]$. Moreover, $A \subseteq B$ implies $[A] \subseteq [B]$. We recall that*

$$\begin{array}{rcl} [A] \cap [B] = \emptyset & \textit{implies} & ([A] \cap \{x\} = \emptyset \text{ for all } x \in B) \\ & \textit{equivuivalent to} & ([B] \cap \{y\} = \emptyset \text{ for all } y \in A) \\ & \textit{implies} & ([x] \cap [y] = \emptyset \text{ for all } x \in B, y \in A). \end{array}$$

Lemma 1. *For any $A, B \in 2^X$, the following properties are fulfilled:*

(1) $[A] \subseteq B$ *implies* $A \subseteq B$,
(2) $[A \cup B] = [A] \cup [B]$,
(3) $[A] \subseteq B$ *implies* $[B^c] \subseteq A^c$, *while* $A \subseteq B$ *implies* $[B]^c \subseteq A^c$,
(4) *If* $[A] \subseteq B$, *then there is* $K \in 2^X$ *such that* $[A] \subseteq K$ *and* $[K] \subseteq B$.

Proof.

(1) This is easily proven using Remark 1.
(2) $[A] \cup [B] \subseteq [A \cup B]$ is clear. Now, we let $x \in [A \cup B]$. Then, there is $y \in A \cup B$ such that $R(x,y) = 1$; that is, there is $y \in A$ or $y \in B$ such that $R(x,y) = 1$. Thus, $x \in [A]$ or $x \in [B]$. So, $x \in [A] \cup [B]$; that is, $[A \cup B] \subseteq [A] \cup [B]$. Hence, $[A \cup B] = [A] \cup [B]$.
(3) $[A] \subseteq B$ implies $B^c \subseteq [A]^c$; that is, $[B]^c \subseteq [B^c] \subseteq [A]^c \subseteq A^c$, while $A \subseteq B$ implies that $[B]^c \subseteq [A]^c \subseteq A^c$.
(4) The proof is straightforward.

□

Based on the meaning of neighborhoods $[x], [A]$, the lower and the upper approximations of any subset of X were defined. For subset A of X, we define approximation subsets $A_*, A^* : X \to \{0,1\}$ using

$$A_* = \{x \in X : [x] \cap A^c = \emptyset\},\ A^* = \{x \in X : [x] \cap A \neq \emptyset\};\ \text{that is, for each x} \in \text{X},$$

$$A_*(x) = \begin{cases} 1 & \text{if } [x] \cap A^c = \emptyset \\ 0 & \text{if } [x] \cap A^c \neq \emptyset, \end{cases} \tag{2}$$

$$A^*(x) = \begin{cases} 0 & \text{if } [x] \cap A = \emptyset \\ 1 & \text{if } [x] \cap A \neq \emptyset. \end{cases} \tag{3}$$

Lemma 2. *If (X, R) is an approximation space with R an arbitrary relation on X, then, for any $A, B \in 2^X$,*

(1) $X_* = X$, $\emptyset^* = \emptyset$,
(2) $A \not\subseteq A_* \not\subseteq A$, $A \not\subseteq A^* \not\subseteq A$,
(3) $(A_*)_* \subseteq A_*$, $(A^*)^* \supseteq A^*$,
(4) $(A \cap B)^* \subseteq A^* \cap B^*$, $(A \cup B)_* \supseteq A_* \cup B_*$,
(5) $(A \cup B)^* \supseteq A^* \cup B^*$, $(A \cap B)_* \subseteq A_* \cap B_*$,
(6) $A \subseteq B$ implies that $A_* \subseteq B_*$, $A^* \subseteq B^*$.

Proof. The proof is direct. □

Whenever R is reflexive, for any $A, B \in 2^X$, we have $A_* \subseteq A$, $A \subseteq A^*$, $X^* = X$, $\emptyset_* = \emptyset$, $(A \cup B)^* = A^* \cup B^*$, $(A \cap B)_* = A_* \cap B_*$.

If R is also transitive, $A_{**} = A_*$, $A^{**} = A^*$. For any subset A of X, the lower approximation A_R and the upper approximation A^R are defined by

$$A_R = A \cap A_* \,, \; A^R = A \cup A^*.$$

The boundary region set A^B is defined by the set difference, $A^R \setminus A_R = A^B$, and moreover, the accuracy value $\alpha(A)$ of rough set A is given by the ratio

$$\alpha(A) = \frac{\text{number of elements of } A_R}{\text{number of elements of } A^R}.$$

Whenever $A^R \nsubseteq A_R$, A^B is not empty and set A has a roughness region. Thus, A is called a rough set. As a special case, if $A^R = X, A_R = \emptyset$. Then, $A^B = X$, and A is called a totally rough set. However, if $A^R \subseteq A_R$, then $A^B = \emptyset$, and set A is called an exact set.

From Lemma 2 and the definitions of A_R and A^R, we have the following consequences.

Lemma 3. *Let (X, R) be an approximation space with R as an arbitrary relation. Then, for any $A, B \in 2^X$, the following properties are fulfilled:*

(1) $X_R = X^R = X$, $\emptyset_R = \emptyset^R = \emptyset$,
(2) $A_R \subseteq A \subseteq A^R$,
(3) $(A_R)_R \subseteq A_R$, $(A^R)^R \supseteq A^R$,
(4) $(A \cap B)^R \subseteq A^R \cap B^R$, $(A \cup B)_R \supseteq A_R \cup B_R$,
(5) $(A \cup B)^R \supseteq A^R \cup B^R$, $(A \cap B)_R \subseteq A_R \cap B_R$,
(6) $A \subseteq B$ implies that $A_R \subseteq B_R$, $A^R \subseteq B^R$.

Proof. The proof is straightforward from Lemma 2. □

Note that if R is a reflexive relation, the equality holds in (5), Lemma 3, and moreover, if R is a transitive relation, the equality holds in (3), Lemma 3. Thus, we can deduce that approximation topology τ_R on approximation space (X, R) is associated, for each $A \subseteq X$, with the interior A° and the closure $\overline{A}$ defined by $A^\circ = A_R$ and $\overline{A} = A^R$.

Now, we recall two operators on X and both operators generate topologies on X, respectively (both are dual).

Mapping $c : 2^X \to 2^X$ is called a closure operator on X (see [14]) if it satisfies the following conditions: for any $A, B \in 2^X$,

(C.1) $c(\emptyset) = \emptyset$,
(C.2) $A \subseteq c(A)$,
(C.3) $c(c(A)) = c(A)$,
(C.4) $c(A \cup B) = c(A) \cup c(B)$.

Mapping $i : 2^X \to 2^X$ is called an interior operator on X (see [14]) if it satisfies the following conditions: for any $A, B \in 2^X$,

(I.1) $i(X) = X$,
(I.2) $i(A) \subseteq A$,
(I.3) $i(i(A)) = i(A)$,
(I.4) $i(A \cap B) = i(A) \cap i(B)$.

Lemma 4 ([14])**.** *Let c be a closure operator on X. Then, topology τ_c is generated on X such that $c(A) = \overline{A}$ for each $A \in 2^X$, where $\overline{A}$ is the closure of A with respect to topology τ_c. In fact, $\tau_c = \{F \in 2^X : c(F^c) = F^c\}$.*

Lemma 5 ([14])**.** *Let i be an interior operator on X. Then, topology τ_i is generated on X such that $i(A) = A^\circ$ for each $A \in 2^X$, where A° is the interior of A with respect to topology τ_i. In fact, $\tau_i = \{U \in 2^X : i(U) = U\}$.*

We let (X, R) be an approximation space. We define mappings $i, c : 2^X \to 2^X$, respectively, for each $A \in 2^X$, as follows:

$$i(A) = \bigcup_{[x]\cap A^c = \phi} \{x\} \equiv A_R, \tag{4}$$

$$c(A) = \bigcup_{[x]\cap A \neq \phi} \{x\} \equiv A^R. \tag{5}$$

Then, from Lemma 3, we can easily check that i is an interior operator and c is a closure operator on X. Thus, by Lemmas 5 and 4, there are topologies τ_i and τ_c on X such that $i(A) = A^\circ$ and $c(A) = \overline{A}$ for each $A \in 2^X$. Furthermore, we have $c(A^c) = i(A)^c$ and $i(A^c) = c(A)^c$. So, $\tau_i = \tau_c$, and we denote both of the topologies by $\tau_{_R}$. Hence, we consider approximation space (X, R) as the topological space equipped with the interior operator defined by (4) or the closure operator defined by (5). Moreover, the generated topology on X is given by

$$\tau_{_R} = \{A \subseteq X : \ A = A^\circ\} \equiv \{A \subseteq X : \ A^c = \overline{A^c}\}.$$

Since $A^\circ = A$ iff $[A] = A$, $[A^c] = A^c$. Also, since $\overline{A} = A$ iff $[A^c] = A^c$, $[A] = A$. In general, each $A \in 2^X$ with $[A] = A$ is an open and closed set in (X, R). That is, $A_R = A^R = A$, and then A is an exact set. That means no roughness of A.

Example 1. *Let $X = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$. Then,*

$$[a] = [b] = [\{a, b\}] = \{a, b\}, \ [c] = \{c\} \text{ and } [\{a, c\}] = [\{b, c\}] = [X] = X.$$

(1) If $A = \{a, c\}$ or $A = \{b, c\}$. Then, obtain

$$A_R = \bigcup_{[x]\cap\{b\}=\emptyset} \{x\} = \bigcup_{[x]\cap\{a\}=\emptyset} \{x\} = \{c\},$$

$$A^R = \bigcup_{[x]\cap\{a,c\}\neq\emptyset} \{x\} = \bigcup_{[x]\cap\{b,c\}\neq\emptyset} \{x\} = X.$$

(2) If $A = \{a\}$ or $A = \{b\}$. Then, obtain

$$A_R = \bigcup_{[x]\cap\{b,c\}=\emptyset} \{x\} = \bigcup_{[x]\cap\{a,c\}=\emptyset} \{x\} = \emptyset,$$

$$A^R = \bigcup_{[x]\cap\{a\}\neq\emptyset} \{x\} = \bigcup_{[x]\cap\{b\}\neq\emptyset} \{x\} = \{a, b\}.$$

(3) Since $[\{a, b\}] = \{a, b\}$, $[c] = \{c\}$ and $[X] = X$, the lower approximation and the upper approximation of any of these subsets are equal, $A_R = A = A^R$, and then only subsets $\{a, b\}, \{c\}, X$ are exact sets and the other four non-empty subsets are rough sets.

Example 2. *Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}$. Then, have*

$$\begin{aligned}
&[a] = [b] = [\{a, b\}] = \{a, b\}, [c] = [d] = [\{c, d\}] = \{c, d\},\\
&[\{a, c\}] = [\{b, c\}] = [\{a, d\}] = [\{b, d\}] = [\{a, b, c\}]\\
&\quad = [\{a, b, d\}] = [\{a, c, d\}] = [\{b, c, d\}] = [X] = X.
\end{aligned}$$

(1) If $A \in \{\{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}\}$, $A_R = \emptyset$ and $A^R = X$. Thus, these subsets are totally rough sets.

(2) If $A \in \{\{a, b, c\}, \{a, b, d\}\}$, $A_R = \{a, b\}$ and $A^R = X$.

(3) If $A \in \{\{a,c,d\},\{b,c,d\}\}$, $A_R = \{c,d\}$ and $A^R = X$.

(4) If $A \in \{\{a\},\{b\}\}$, $A_R = \emptyset$ and $A^R = \{a,b\}$.

(5) If $A \in \{\{c\},\{d\}\}$, $A_R = \emptyset$ and $A^R = \{c,d\}$. These subsets appearing in he previous items (2)–(5) are rough sets.

(6) If $A \in \{\{a,b\},\{c,d\},X\}$, determine that the lower approximation and the upper approximation of any of these subsets are equal, that is, $A_R = A = A^R$. Thus, these subsets are exact sets without roughness.

Example 3. *Let $X = \{a,b,c,d\}$ and $R = \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b),(b,d),(d,b),(c,d),(d,c)\}$. Then, have*

$$[a] = \{a\},$$
$$[b] = [c] = [d] = [\{b,c\}] = [\{b,d\}] = [\{c,d\}] = [\{b,c,d\}] = \{b,c,d\},$$
$$[\{a,b\}] = [\{a,c\}] = [\{a,d\}] = [\{a,b,c\}] = [\{a,b,d\}] = [\{a,c,d\}] = [X] = X.$$

(1) If $A \in \{\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$, $A_R = \{a\}$ and $A^R = X$. These subsets are rough sets. Moreover, the boundary set is $A^B = \{b,c,d\}$, and the accuracy is $\frac{1}{4}$.

(2) If $A \in \{\{b\},\{c\},\{d\},\{b,c\},\{b,d\},\{c,d\}\}$, $A_R = \emptyset$ and $A^R = \{b,c,d\}$. These subsets are rough sets. Moreover, the boundary set is $A^B = \{b,c,d\}$, and the accuracy is $\frac{0}{3} = 0$.

(3) If $A \in \{\{a\},\{b,c,d\},X\}$, $A_R = A = A^R$. These non-empty subsets are exact sets. Moreover, the boundary set is $A^B = \emptyset$, and the accuracy is 1.

Example 4. *Let (X,R) be a finite approximation space such that $[x] = \{x\}$ for all $x \in X$ (only equal elements are related). Then, $[A] = A$ for each $A \in 2^X$. Thus, any subset A of X is open and closed, that is, $A_R = A = A^R$ for all $A \in 2^X$, and hence the boundary set is $\emptyset$. So, each $A \in 2^X$ is an exact subset of X without roughness.*

Definition 1. *An approximation space (X,R) is said to be*

(i) a T_0-space if for all $x \neq y \in X$, then $t \notin [y]$ for all $t \in [x]$ or $s \notin [x]$ for all $s \in [y]$,

(ii) a T_1-space if for all $x \neq y \in X$, then $t \notin [y]$ for all $t \in [x]$ and $s \notin [x]$ for all $s \in [y]$, that is, $[x] \cap [y] = \emptyset$,

(iii) a T_2-space if for all $x \neq y \in X$, then $[x] \cap [y] = \emptyset$,

(iv) regular if for all $x \notin F = \overline{F}$, then $t \notin [F]$ for all $t \in [x]$ and $s \notin [x]$ for all $s \in [F]$, that is, $[x] \cap [F] = \emptyset$,

(v) a T_3 space if it is regular and T_1,

(vi) normal if for all $F = \overline{F}, G = \overline{G}$ with $F \cap G = \emptyset$, $t \notin [G]$ for all $t \in [F]$ and $s \notin [F]$ for all $s \in [G]$, that is, $[F] \cap [G] = \emptyset$,

(vii) a T_4 space if it is normal and T_1.

Remark 2.

- *(1) Suppose (X,R) is a T_0-space and let $x \neq y \in X$. Then, either $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Thus, every approximation space (X,R) cannot be a T_0-space except $[x] = \underline{\{x\}}$ for all $x \in X$.*
- *(2) (X,R) is a T_1-space if and only if $[x] = \{x\}$ for all $x \in X$ if and only if $\overline{\{x\}} = \{x\}$ for all $x \in X$ from Equation (5).*
- *(3) It is obvious that T_0, T_1 and T_2 separation axioms are equivalent definitions in an approximation space (X,R).*

Proposition 1. *From Definition 1, $T_4 \Rightarrow T_3 \Rightarrow T_2 \Leftrightarrow T_1 \Leftrightarrow T_0$.*

3. Metric Distance in Approximation Spaces

Let $d : X \times X \to \{0,1\}$ be a mapping satisfying the following conditions:

(D1) $x = y$ implies that $d(x,y) = 0$,
(D2) $d(x,y) = d(y,x)$ for all $x,y \in X$,
(D3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$,

(D4) $d(x,y) = 0$ implies that $x = y$.

d is called a metric on X if mapping d satisfies only conditions (D1)–(D3). Then, d is called a pseudo-metric on X if d satisfies only conditions (D1), (D3). Then, d is called a quasi-pseudo-metric on X, and if d satisfies only conditions (D1), (D3), (D4), d is called a quasi-metric on X.

Let (X, R) be an approximation space with an equivalence relation R on X and $d : X \times X \to \{0,1\}$ a mapping defined as a relation on X in the following way:

$$d(x,y) = \begin{cases} 1 & \text{if } [x] \cap \{y\} = \emptyset \\ 0 & \text{if } [x] \cap \{y\} \neq \emptyset. \end{cases} \tag{6}$$

From (6), it is obvious that $x = y$ implies $d(x,y) = 0$. Since $[x] \cap \{y\} = \emptyset \equiv [y] \cap \{x\} = \emptyset$, $d(x,y) = d(y,x)$. Also, it is clear that $d(x,z) \leq d(x,y) + d(y,z)$. On the other hand, if $[x] = [y] = \{x,y\}$, then, clearly, $d(x,y) = 0$ but $x \neq y$. Thus, d defines a pseudo-metric on X. In this case, the pair (X,d) is called a pseudo-metric space induced by (X,R) and we write the topology on X induced by d or associated to d as τ_d. The pair (X, τ_d) is the associated topological space.

It is clear that there is a distance between x and y in X if and only if $[x] \cap \{y\} = \emptyset$.

For each $x \in X$ and each $A \in 2^X$, the distance between x and A, denoted by $d(x, A)$, is defined as follows:

$$d(x,A) = \bigwedge_{y \in A} d(x,y)$$

which is equivalent to

$$d(x,A) = \begin{cases} 1 & \text{if } [x] \cap A = \emptyset \\ 0 & \text{if } [x] \cap A \neq \emptyset. \end{cases} \tag{7}$$

For any $A,\ B \in 2^X$, the *distance between* A *and* B, denoted by $d(A,B)$, is defined as follows:

$$d(A,B) = \bigwedge_{x \in A} \bigwedge_{y \in B} d(x,y)$$

which is equivalent to

$$d(A,B) = \begin{cases} 1 & \text{if } [A] \cap B = \emptyset \\ 0 & \text{if } [A] \cap B \neq \emptyset. \end{cases} \tag{8}$$

Then, from (7), we can rewrite Equations (2) and (3), respectively, as follows:

$$A_*(x) = \begin{cases} 1 & \text{if } d(x, A^c) = 1 \\ 0 & \text{if } d(x, A^c) = 0, \end{cases} \tag{9}$$

$$A^*(x) = \begin{cases} 0 & \text{if } d(x, A) = 1 \\ 1 & \text{if } d(x, A) = 0. \end{cases} \tag{10}$$

Thus, from Equations (4) and (5), obtain

$$\mathrm{int}_{\tau_d}(A) = A^\circ = A_R = A_* = \bigcup_{d(x,A^c)=1} \{x\}, \tag{11}$$

$$\mathrm{cl}_{\tau_d}(A) = \overline{A} = A^R = A^* = \bigcup_{d(x,A)=0} \{x\}, \tag{12}$$

where $\mathrm{int}_{\tau_d}(A)$ and $\mathrm{cl}_{\tau_d}(A)$ denote the interior and the closure of A with respect to topology τ_d, respectively. So, it can easily be seen that $\tau_d = \tau_R$.

Pseudo-metric d on the approximation space (X,R) is a metric on X, if $x \neq y \in X$ implies $d(x,y) = 1$, that is, $[x] = \{x\}$ for all $x \in X$. The associated topological space (X, τ_d) proves that it is a normal topological space. Based on the definition of a metric d, and that R is given by $R(x,x) = 1$ for all $x \in X$, otherwise $R(x,y) = 0$, (X, τ_d) is a T_1 space. Thus, (X, τ_d) is a T_4 space, which means satisfying all the T_i separation axioms; $i = 0,1,2,3$. Recall that (X, τ_d) in this case is exactly a discrete topological space, i.e., all subsets are open and closed. Moreover, Equations (7) and (8) could be rewritten as

$$d(x,A) = \begin{cases} 1 & \text{if } x \notin A \\ 0 & \text{if } x \in A, \end{cases}$$

$$d(A,B) = \begin{cases} 1 & \text{if } A \cap B = \emptyset \\ 0 & \text{if } A \cap B \neq \emptyset. \end{cases}$$

Proposition 2. *Let (X,d) be a pseudo-metric space and let τ_d be the topology associated to d. Then, (X, τ_d) is a normal space. Moreover, if d is a metric, then (X, τ_d) is a T_4 space.*

Proof. We suppose d is a metric on X. From Equation (6), we determine that $x \neq y$ if $d(x,y) = 1$ if $[x] \cap \{y\} = [y] \cap \{x\} = \emptyset$, and then $y \notin [x]$ and $x \notin [y]$. Hence, (X, τ_d) is a T_1 space.

We let $F = \mathrm{cl}_{\tau_d} F \in 2^X$, $G = \mathrm{cl}_{\tau_d} G \in 2^X$ with $F \cap G = \emptyset$. Then, we have

$$F \subseteq G^c = \mathrm{int}_{\tau_d}(G^c) \text{ and } G \subseteq F^c = \mathrm{int}_{\tau_d}(F^c).$$

Thus, $[F] \subseteq [G^c] = G^c$ and $[G] \subseteq [F^c] = F^c$. We assume that $[F] \cap [G] \neq \emptyset$, say, $t \in [F] \cap [G]$. Then, there exist $x \in F$ and $y \in G$ such that $R(x,t) = 1$ and $R(t,y) = 1$. Thus, $R(x,y) = 1$. So, $x \in [G] \subseteq F^c$ and $y \in [F] \subseteq G^c$ and both are contradictions. Hence, $[F] \cap [G] = \emptyset$. Therefore, (X, τ_d) is normal. $\square$

4. Proximity Relation in Approximation Spaces

Binary relation δ on 2^X is called a nearness relation or a proximity on X, provided that the negation of δ, denoted by $\overline{\delta}$ (called a farness relation), for any $A, B, K \in 2^X$, fulfills the following conditions (see [15]):

(P1) $A\overline{\delta}B$ implies $B\overline{\delta}A$,

(P2) $(A \cup B)\overline{\delta}K$ if and only if $A\overline{\delta}K$ and $B\overline{\delta}K$,

(P3) $A = \emptyset$ or $B = \emptyset$ implies $A\overline{\delta}B$,

(P4) $A\overline{\delta}B$ implies $A \cap B = \emptyset$,

(P5) if $A\overline{\delta}B$. Then, there is $L \in 2^X$ such that $A\overline{\delta}L$ and $L^c\,\overline{\delta}B$.

The pair (X, δ) is called a proximity space. Note that δ is the negation of $\overline{\delta}$, that is, $A\delta B \equiv A \;\not\overline{\delta} B$.

(P1) and (P2) imply the following condition:

(P2′) $K\overline{\delta}(A \cup B)$ if and only if $K\overline{\delta}A$ and $K\overline{\delta}B$.

In the following proposition, we show that there is a proximity on an approximation space (X, R).

Proposition 3. *Let (X,R) be an approximation space and let δ be a binary relation on 2^X defined, for any $A, B \in 2^X$, as follows:*

$$A\overline{\delta}B \quad \text{if and only if} \quad [A] \cap B = \emptyset.$$

Then, δ is a proximity on X. In this case, δ is called a proximity on X induced by R and the pair (X, δ) is called a proximity space of (X, R).

Proof. (P1) Suppose $A\overline{\delta}B$ for any $A,\ B \in 2^X$. Then, by the definition of δ, $[A] \cap B = \emptyset$. Thus, $[A] \subseteq B^c$. So, by Lemma 1 (3), $[B] \subseteq A^c$. Hence, $B\overline{\delta}A$.

(P2) Suppose $(A \cup B)\overline{\delta}K$ for any $A, B, K \in 2^X$. Then, clearly, $[A \cup B] \cap K = \emptyset$. Thus, by Lemma 1 (2), $([A] \cup [B]) \cap K = \emptyset$, that is, $([A] \cap K) \cup ([B] \cap K) = \emptyset$. So, $[A] \cap K = \emptyset$ and $[B] \cap K = \emptyset$. Hence, $A\overline{\delta}K$ and $B\overline{\delta}K$.

Conversely, suppose $A\overline{\delta}K$ and $B\overline{\delta}K$. Assume that $(A \cup B)\delta K$, that is, $[A \cup B] \cap K \neq \emptyset$. Then, there is $x \in K$ and $R(x, y) = 1$ for some $y \in A \cup B$. Thus, $R(x, y) = 1$ for some $y \in A$ or $y \in B$. So, $x \in [A]$ or $x \in [B]$, that is, $[A] \cap K \neq \emptyset$ or $[B] \cap K \neq \emptyset$. Both are contradicting $A\overline{\delta}K$ and $B\overline{\delta}K$. Hence, $(A \cup B)\overline{\delta}K$.

(P3), (P4) The proofs are straightforward.

(P5) Suppose $A\overline{\delta}B$ for any $A,\ B \in 2^X$. Then, clearly, $[A] \cap B = \emptyset$, that is, $[A] \subseteq B^c$. Thus, there is $H \subseteq B^c$ such that $[A] \subseteq H \subseteq [H] \subseteq B^c$. Thus, $A\overline{\delta}H^c$ and $H\overline{\delta}B$, which is equivalent to there is $L \in 2^X$ such that $A\overline{\delta}L$ and $L^c\,\overline{\delta}B$. □

Let δ be a proximity on an approximation space (X, R). Consider two mappings, $int_\delta,\ cl_\delta : 2^X \to 2^X$ defined, for each $A \in 2^X$, respectively, as follows:

$$\mathrm{int}_\delta A = \bigcup_{\{x\}\overline{\delta}A^c} \{x\} \equiv \bigcup_{[x]\cap A^c=\emptyset} \{x\} \equiv \bigcup_{d(x,A^c)=1} \{x\} \equiv A_R \equiv A^\circ \tag{13}$$

and

$$\mathrm{cl}_\delta A = \bigcup_{\{x\}\not{\overline{\delta}}A} \{x\} \equiv \bigcup_{[x]\cap A\neq\emptyset} \{x\} \equiv \bigcup_{d(x,A)=0} \{x\} \equiv A^R \equiv \overline{A}. \tag{14}$$

Then, it can easily be checked that int_δ is an interior operator and cl_δ a closure operator on X. Thus, by Lemmas 4 and 5, there is topology τ_δ (called the topology associated to) on X. In fact,

$$\tau_\delta = \{K \subseteq X : K = \mathrm{int}_\delta K\} \equiv \{K \subseteq X : K^c = \mathrm{cl}_\delta(K^c)\}.$$

The pair (X, τ_δ) is the associated topological space to (X, δ). It is obvious that $\tau_\delta = \tau_R$.

Proximity δ on approximation space (X, R) is said to be *separated* if $x \neq y \in X$ implies $\{x\}\overline{\delta}\{y\}$. It is obvious that δ is a separated proximity if and only if $[x] = \{x\}$ for all $x \in X$, that is, (X, τ_δ) is a T_1-space if and only if the pseudo-metric d is a metric.

In the following Proposition, it is proven that topological space (X, τ_δ) associated to proximity space (X, δ) is a T_4 space.

Proposition 4. *Let (X, δ) be the proximity space for an approximation space (X, R) and let τ_δ be the topology associated to δ. Then, (X, τ_δ) is a normal space. Moreover, if δ is separated, (X, τ_δ) is a T_4 space.*

Proof. Clear as given in Proposition 2 and from Equations (13) and (14). □

Proposition 5. *Let (X, τ_R) be a topological approximation space. Then, the constructed proximity δ on X fulfills, for any $A, B \in 2^X$, the following property:*

$$A\overline{\delta}B \text{ if and only if } \overline{A}\,\overline{\delta}\,\overline{B}.$$

Proof. From conditions (P1), (P2), $\overline{A}\,\overline{\delta}\,\overline{B}$ if $A\,\overline{\delta}\,\overline{B}$ if $A\overline{\delta}B$. Also, $\overline{A}\,\overline{\delta}\,\overline{B}$ if $\overline{A}\,\overline{\delta}\,B$ if $A\overline{\delta}B$. □

Let (X, d) be the pseudo-metric space induced by an approximation space (X, R). Then, we can define proximity δ on X in the following way: for any $A,\ B \in 2^X$,

$$A\overline{\delta}B \text{ iff } d(A, B) = 1 \quad \text{or} \quad A\delta B \text{ iff } d(A, B) = 0. \tag{15}$$

It is easy to see that δ satisfies Conditions (P1)–(P5) depending on the properties of the pseudo-metric d. Moreover, if d is a metric on X, δ is a separated proximity on X. Thus, the resulting interior operators and closure operators in both of (X, d) and (X, δ) (as shown in Equations (11)–(14)) generate equivalent topologies τ_d and τ_δ. So, both of them are equivalent to discrete topology τ_R generated on X. Hence, all subsets of X have identical lower approximations and upper approximations.

In this case,

$$d(A,B) = 1 \text{ iff } A\bar{\delta}B \text{ iff } A \cap B = \emptyset, \quad d(A,B) = 0 \text{ iff } A\delta B \text{ iff } A \cap B \neq \emptyset.$$

5. Uniform Structure in Approximation Spaces

In this section, we study the relation between the uniform spaces and the T_i separation axioms given in Section 2, the defined pseudo-metric in Section 3 and the defined proximity in Section 4.

For a non-empty set X, the top relation and the bottom relation on X, denoted by $\mathbf{T}$ and $\mathbf{B}$, are relations on X, respectively, defined, for any $x, y \in X$, as follows:

$$\mathbf{T}(x,y) = 1 \text{ and } \mathbf{B}(x,y) = 0.$$

$2^{X\times X}$ denotes the bounded set of all relations on X.

For each $R \in 2^{X\times X}$, the inverse relation of R, denoted by R^{-1}, is a relation on X defined, for any $x, y \in X$, as follows:

$$R^{-1}(x,y) = R(y,x).$$

Binary operations $\wedge$ and $\vee$ on $2^{X\times X}$ between arbitrary relations are defined, for any $R_1, R_2 \in 2^{X\times X}$ and any $x, y \in X$, by

$$(R_1 \wedge R_2)(x,y) = R_1(x,y) \wedge R_2(x,y) \text{ and } (R_1 \vee R_2)(x,y) = R_1(x,y) \vee R_2(x,y).$$

For any $R_1, R_2 \in 2^{X\times X}$, the composition of R_1 and R_2, denoted by $R_1 \circ R_2$, is a relation on X defined as follows: for any $x, z \in X$,

$$(R_1 \circ R_2)(x,z) \;=\; \bigvee_{y\in X} R_1(x,y) \wedge R_2(y,z). \tag{16}$$

The order relation $\leq$ on $2^{X\times X}$ is defined, for any $R_1, R_2 \in 2^{X\times X}$ and $x, y \in X$, by

$$R_1 \leq R_2 \text{ iff } R_1(x,y) \leq R_2(x,y).$$

Definition 2. *Filter $\mathcal{M}$ on $X \times X$ is mapping $\mathcal{M} : 2^{X\times X} \to \{0,1\}$ satisfying the following conditions:*

(i) $\mathcal{M}(\mathbf{T}) = 1$, ($\mathcal{M}(\mathbf{B}) = 0$ to be a proper filter),
(ii) $R_1 \leq R_2$ implies $\mathcal{M}(R_1) \leq \mathcal{M}(R_2)$ for all $R_1, R_2 \in 2^{X\times X}$,
(iii) $\mathcal{M}(R_1 \wedge R_2) \geq \mathcal{M}(R_1) \wedge \mathcal{M}(R_2)$ for all $R_1, R_2 \in 2^{X\times X}$.

The inverse $\mathcal{M}^{-1}$ of $\mathcal{M}$ is defined by $\mathcal{M}^{-1}(R) = \mathcal{M}(R^{-1})$ for all $R \in 2^{X\times X}$.

The principal filter $[x,y]$ on $X \times X$ of a pair (x,y) in $X \times X$ is defined, for each $R \in 2^{X\times X}$, by

$$[x,y](R) = R(x,y).$$

It is clear that $[x,x](R) = R(x,x)$ for all $R \in 2^{X\times X}$. Then, $R_{\mathrm{ref}}(X) \subseteq [x,x]$, where $R_{\mathrm{ref}}(X)$ denotes the set of all reflexive relations on X.

For any two filters $\mathcal{M}$ and $\mathcal{K}$, we say that $\mathcal{M}$ *is finer than* $\mathcal{K}$, denoted by $\mathcal{M} \prec \mathcal{K}$, if for each $R \in 2^{X\times X}$,

$$\mathcal{M} \prec \mathcal{K} \text{ iff } \mathcal{M}(R) \leq \mathcal{K}(R).$$

Definition 3. *Let $\mathcal{M}$ and $\mathcal{K}$ be two filters on $X \times X$ such that $[x, y] \prec \mathcal{M}$ and $[y, z] \prec \mathcal{K}$ for any $x, y, z \in X$. Then, the composition of $\mathcal{M}$ and $\mathcal{K}$, denoted by $\mathcal{M} \circ \mathcal{K}$, is a filter on $X \times X$ defined, for each $R \in 2^{X \times X}$, by*

$$(\mathcal{M} \circ \mathcal{K})(R) = \bigvee_{(R_1 \circ R_2) \leq R} \mathcal{M}(R_1) \wedge \mathcal{K}(R_2). \tag{17}$$

The notion of uniformity was introduced by Weil in [15]. Here, we construct a uniform structure in an approximation space (X, R).

Definition 4. *Uniformity $\mathcal{U}$ on X is a filter on $X \times X$ satisfying the following conditions:*
(U1) $[x, x] \prec \mathcal{U}$ for all $x \in X$,
(U2) $\mathcal{U} = \mathcal{U}^{-1}$,
(U3) $(\mathcal{U} \circ \mathcal{U}) \prec \mathcal{U}$.
The pair $(X, \mathcal{U})$ is called a uniform space.

From the above definition, we can easily see that $R_{\text{eq}}(X) \subseteq \mathcal{U}$, where $R_{\text{eq}}(X)$ denotes the set of all equivalence relations on X.

Definition 5. *Let $\mathcal{U}$ be a filter on $X \times X$ such that $[x, x] \prec \mathcal{U}$ for all $x \in X$ and let $\mathcal{M} : 2^X \to 2$ be a filter on X. Then, the image of $\mathcal{M}$ with respect to $\mathcal{U}$, denoted by $\mathcal{U}[\mathcal{M}]$, is the mapping $\mathcal{U}[\mathcal{M}] : 2^X \to 2$ defined in [16], for each $R \in 2^{X \times X}$ and each $B \in 2^X$, by*

$$(\mathcal{U}[\mathcal{M}])(A) = \bigvee_{R[B] \cap A^c = \emptyset} (\mathcal{U}(R) \wedge \mathcal{M}(B)), \tag{18}$$

where $R \in 2^{X \times X}$, $B \in 2^X$ and set $R[B] \in 2^X$ is defined so that

$$(R[B])(x) = \bigvee_{y \in X} (B(y) \wedge R(y, x)) \equiv [B](x). \tag{19}$$

From Equation (1), determine that $R[B] \equiv [B]$ for all $B \in 2^X$.

It is obvious that $\mathcal{U}[\mathcal{M}]$ is a filter on X.

The principal filter $[\dot{x}]$ on X at a point $x \in X$ is defined by $[\dot{x}](A) = A(x)$ for all $A \in 2^X$. It is clear that $[\dot{x}](\{x\}) = 1$ for all $x \in X$.

Let $\mathcal{U}$ be a uniformity on a set X and let $\text{int}_{\mathcal{U}}, \text{cl}_{\mathcal{U}} : 2^X \to 2^X$ be the mappings defined, respectively, as follows: for each $R \in 2^{X \times X}$, any $A, B \in 2^X$ and each $x \in X$:

$$(\text{int}_{\mathcal{U}} A)(x) = (\mathcal{U}[\dot{x}])(A) \equiv \bigvee_{[B] \cap A^c = \emptyset} (\mathcal{U}(R) \wedge B(x)), \tag{20}$$

$$(\text{cl}_{\mathcal{U}} A)(x) = \bigvee_{[B] \cap A \neq \emptyset} (\mathcal{U}(R) \wedge B(x)). \tag{21}$$

Then, it can easily be proven that $\text{int}_{\mathcal{U}}$ and $\text{cl}_{\mathcal{U}}$ are the interior and the closure operators on X, respectively. Thus, there is topology $\tau_{\mathcal{U}}$ on X induced by $\text{int}_{\mathcal{U}}$ or $\text{cl}_{\mathcal{U}}$.

Since any equivalence relation R on X is an element of a uniformity $\mathcal{U}$ on X, in an approximation space (X, R), from Equations (4) and (5), obtain

$$\text{int}_{\mathcal{U}} A \equiv \text{int}_{\delta} A \equiv \text{int}_{\tau_d} A \equiv A^{\circ} \equiv A_R \tag{22}$$

and

$$\text{cl}_{\mathcal{U}} A \equiv \text{cl}_{\delta} A \equiv \text{cl}_{\tau_d} A \equiv \overline{A} \equiv A^R. \tag{23}$$

Uniformity $\mathcal{U}$ on X is said to be *separated*, if for all $x \neq y \in X$ there is $R \in R_E(X)$ such that $\mathcal{U}(R) = 1$ and $R(x, y) = 0$, that is, $[x] \cap [y] = \emptyset$. In this case, pair $(X, \mathcal{U})$ is called a separated uniform space.

As in Section 2, $T_2 \equiv T_1 \equiv T_0$ as separation axioms. So, separated uniform spaces satisfy all these axioms.

Generated topology τ_R on approximation space (X, R) is explained during the lower and the upper sets of a rough set. It is equivalent to induced topology τ_δ generated by constructed proximity δ on X, and also is equivalent to the generated topology τ_d by pseudo-metric d constructed on X. Moreover, all these topologies are equivalent to generated topology $\tau_\mathcal{U}$ the constructed uniformity $\mathcal{U}$ on X. According to the definitions of a metric, a separated proximity and a separated uniformity, obtain a similar result to Proposition 2 and Proposition 4 related to the defined separation axioms in Section 2.

Proposition 6. *Let X be a set, $\mathcal{U}$ a uniform structure on X and $\tau_\mathcal{U}$ the topology induced by $\mathcal{U}$. Then, $(X, \tau_\mathcal{U})$ is a normal space, and moreover*

$$(X, \mathcal{U}) \text{ separated if and only if } (X, \tau_\mathcal{U}) \text{ is a } T_4\text{-space.}$$

Proof. The proof is coming from Equations (22) and (23) and from the proofs of Proposition 2 and Proposition 4. □

6. Arbitrary Relation in Approximation Spaces

In this section, we recall the strategy of Kozae in [10]. We let R be an arbitrary relation on X. Then, the right and left neighborhoods (the after and fore sets) of element $x \in X$ are sets in 2^X given, respectively, by

$$xR = \{y \in X : R(x, y) = 1\}, \quad Rx = \{y \in X : R(y, x) = 1\}.$$

We let $< x > R \in 2^X$ be defined as

$$< x > R \; = \begin{cases} \bigcap\limits_{x \in pR} pR & \text{if there exists } p : x \in pR, \\ \emptyset & \text{otherwise} \end{cases} \tag{24}$$

and $R < x > \in 2^X$ be defined as

$$R < x > = \begin{cases} \bigcap\limits_{x \in Rp} Rp & \text{if there exists } p : x \in Rp, \\ \emptyset & \text{otherwise.} \end{cases} \tag{25}$$

$< x > R$, $R < x >$ are called minimal right neighborhoods and minimal left neighborhoods of $x \in X$;

$$R < x > R \; = < x > R \; \cap \; R < x > \tag{26}$$

is called the minimal neighborhood of $x \in X$.

For any subset A of X, the lower approximation A_R and the upper approximation A^R are defined by $A_R = A \cap A_*$, $A^R = A \cup A^*$, where

$$A_* = \{x \in X : \; R < x > R \, \cap A^c = \emptyset\}, \;\; A^* = \{x \in X : \; R < x > R \, \cap A \neq \emptyset\} \tag{27}$$

The resulting lower and upper approximation sets A_R, A^R of set A are typically those defined by Kozae in [10]. The interior operator and the closure operator defined, respectively, in Equations (4) and (5) did not satisfy the common properties of interior and closure operators to generate a topology on (X, R). In the case R is a reflexive relation, $A^\circ = A_R = A_*$, $\overline{A} = A^R = A^*$, but this is still not sufficient to generate a topology on (X, R). At least, in Equations (4) and (5), R needs to be reflexive and transitive to produce topology τ_R on (X, R). In the case R is an equivalence relation, the well-known definition of Pawlak [1] is obtained, and Equations (4) and (5) define topology τ_R on X.

In the case R is an arbitrary relation on (X, R), the separation axiom T_0 could be satisfied and the separation axiom T_1 is not satisfied. That is, the given equivalence T_0 iff T_1 iff T_2 in Section 2 is not correct.

Remark 3. *Whenever R is arbitrary relation on X, we have to replace $[x]$ with $R < x > R$ in all the notations introduced in Sections 2–5. If R is not reflexive, it may be $R(x, x) = 0$, that is, $R < x > R \cap \{x\} = \emptyset$. Hence, condition (D1) is not satisfied and we can not build pseudo-metric d on (X, R) according to Equation (6). According to Equation (13), we may have $\{x\}\overline{\delta}\{x\}$ which is a contradiction to condition (P4), and then we cannot build proximity δ on (X, R). Also, condition (U1) is not satisfied, and so construction of uniformity $\mathcal{U}$ on (X, R) is not possible. If R is not symmetric, Conditions (D2), (P1) and (U2) are not satisfied, and thus it fails to build a metric (pseudo-metric), a proximity or a uniformity in (X, R), but it could be a quasi-metric (quasi-pseudo-metric), a quasi-proximity or a quasi-uniformity in (X, R). Also, if R is not transitive, Conditions (D3), (P5) and (U3) are not satisfied, and thus it fails to build any of metric (pseudo-metric), proximity or uniformity in (X, R).*

Examples 1–4 are given for equivalence relations. Now, we offer an example of arbitrary relation R on X.

Example 5. *Let R be a relation on set $X = \{a, b, c, d\}$ as shown below.*

R	a	b	c	d
a	1	1	0	0
b	1	0	1	1
c	0	1	0	0
d	0	1	0	1

$aR = \{1,1,0,0\}$, $bR = \{1,0,1,1\}$, $cR = \{0,1,0,0\}$, $dR = \{0,1,0,1\}$ and $Ra = \{1,1,0,0\}$, $Rb = \{1,0,1,1\}$, $Rc = \{0,1,0,0\}$, $Rd = \{0,1,0,1\}$. Then, $< a > R = \{1,0,0,0\}$, $< b > R = \{0,1,0,0\}$, $< c > R = \{1,0,1,1\}$, $< d > R = \{0,0,0,1\}$ and $R < a >= \{1,0,0,0\}$, $R < b >= \{0,1,0,0\}$, $R < c >= \{1,0,1,1\}$, $R < d >= \{0,0,0,1\}$ and then, $R < a > R = \{1,0,0,0\}$, $R < b > R = \{0,1,0,0\}$, $R < c > R = \{1,0,1,1\}$, $R < d > R = \{0,0,0,1\}$.

(1) For subset $A = \{1,1,0,0\}$, we compute A_*, A^* as follows: $A_* = A_R = \{1,1,0,0\} = A$, $A^* = A^R = \{1,1,1,0\}$, and thus $A^B = \{0,0,1,0\}$, and the accuracy value is $\frac{2}{3}$.

(2) For subset $K = \{0,0,1,0\}$, we compute K_*, K^* as follows: $K_* = \{0,0,0,0\} \equiv \emptyset$, $K^* = K = \{0,0,1,0\}$, and then $K_R = \{0,0,0,0\}, K^R = K = \{0,0,1,0\}$, and thus $K^B = \{0,0,1,0\}$, and the accuracy value is $\frac{0}{1} = 0$.

(3) For subset $H = \{1,1,0,1\}$ we have $H_* = \{1,1,0,1\} = H_R = H$, $H^* = H^R = \{1,1,1,1\} \equiv X$, and thus $H^B = \{0,0,1,0\}$, and the accuracy value is $\frac{3}{4}$.

From Remark 3, we determine that $R < c > R = \{1,0,1,1\} \neq \{0,0,1,0\}$, and thus this example cannot satisfy any axiom of the separation axioms as given in Definition 1.

Also, from $R < a > R, R < b > R, R < c > R, R < d > R$ computed in this example, we can deduce function ρ (neither a metric nor a pseudo-metric) as follows:

ρ	a	b	c	d
a	0	1	1	1
b	1	0	1	1
c	0	1	0	0
d	1	1	0	0

7. Conclusions

This aim of paper was to construct a proximity relation and a uniformity structure on approximation space (X, R) and also define metric function and separation axioms based on the rough sets in (X, R). We presented some basics of rough sets and introduced the definitions of separation axioms T_i, $i = 0, 1, 2, 3, 4$ in (X, R). We focused on defining metric d on approximation space (X, R) and studied its usual properties. We defined proximity relation δ on (X, R) and studied its properties. Following the definition of uniformity structure $\mathcal{U}$ introduced by Gahler on (X, R), we studied the relations in between notion separation axioms T_i, $i = 0, 1, 2, 3, 4$ in (X, R), metric spaces (X, d), proximity spaces (X, δ) and uniform spaces $(X, \mathcal{U})$ based on the rough sets defined by an equivalence relation R on X. At last, we explained the deviations in these notions whenever R is not an equivalence relation on X. In a future work, we will discuss these results and their applications in the fuzzy approximation spaces, the soft approximation spaces and the soft fuzzy approximation spaces.

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