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Some Chen Inequalities for Submanifolds in Trans-Sasakian Manifolds Admitting a Semi-Symmetric Non-Metric Connection

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Abstract: In the present article, we study submanifolds tangent to the Reeb vector field in trans-Sasakian manifolds. We prove Chen's first inequality and the Chen–Ricci inequality, respectively, for such submanifolds in trans-Sasakian manifolds which admit a semi-symmetric non-metric connection. Moreover, a generalized Euler inequality for special contact slant submanifolds in trans-Sasakian manifolds endowed with a semi-symmetric non-metric connection is obtained.

Keywords: Chen invariant; squared mean curvature; Ricci curvature; trans Sasakian manifold; generalized Sasakian space form; semi-symmetric connection; non-metric connection

MSC: 53C40; 53C25; 53D15

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1. Introduction

In the theory of submanifolds, one fundamental problem is to find relationships involving intrinsic invariants and extrinsic invariants of a Riemannian submanifold. B.-Y. Chen ([1,2]) introduced the Chen invariants, which are consistently important in differential geometry, a particularly intriguing research area within the study of submanifolds. He established optimal inequalities, which are known as Chen inequalities, for submanifolds of a Riemannian space form, involving basic intrinsic invariants, as the sectional curvature, scalar curvature, Ricci curvature, and the main extrinsic invariant, the mean curvature.

Subsequently, various authors have investigated Chen's theory in different ambient spaces, focusing on specific types of submanifolds. For further information, see [3–6].

The notion of semi-symmetric linear connections and metric connections on differentiable manifolds was first considered by Friedmann and Schouten [7] and H. A. Hayden [8], respectively. K. Yano further studied the properties of Riemannian manifolds admitting a semi-symmetric metric connection [9]. The concept of a semi-symmetric non-metric connection on a Riemannian manifold is due to Agashe [10]. Agashe and Chafle [11] studied submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection.

In particular, the Chen δ-invariants for submanifolds of an ambient space admitting a semi-symmetric metric connection or a semi-symmetric non-metric connection have been discussed in [12–18].

2. Preliminaries

Let $(\overline{M}, g)$ be an m-dimensional Riemannian manifold and $\overline{\nabla}$ a linear connection on $\overline{M}$. The torsion $\overline{T}$ of $\overline{\nabla}$ is defined by

$$\overline{T}(\overline{X}, \overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}], \tag{1}$$

for all vector fields $\overline{X}, \overline{Y}$ in $T\overline{M}$.

If the torsion tensor $\tilde{T}$ satisfies

$$\overline{T}(\overline{X}, \overline{Y}) = \omega(\overline{Y})\overline{X} - \omega(\overline{X})\overline{Y}, \tag{2}$$

for a 1-form ω associated with a vector field P on $\overline{M}$, i.e., $\omega(\overline{X}) = g(\overline{X}, P)$, then $\overline{\nabla}$ is called a semi-symmetric connection.

The semi-symmetric connection $\overline{\nabla}$ is said to be a semi-symmetric metric connection if the Riemannian metric g is parallel with respect to $\overline{\nabla}$, i.e., $\overline{\nabla}g = 0$. Otherwise, i.e., $\overline{\nabla}g \neq 0$, $\overline{\nabla}$ is said to be a semi-symmetric non-metric connection.

It is known (see [10]) that a semi-symmetric non-metric connection $\overline{\nabla}$ on $\overline{M}$ is related to the Levi-Civita connection $\overline{\nabla}^0$ of the Riemannian metric g by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}^0_{\overline{X}}\overline{Y} + \omega(\overline{Y})\overline{X},$$

for all vector fields $\overline{X}$, $\overline{Y}$ on $\overline{M}$.

We denote by $\overline{R}$ and $\overline{R}^0$ the curvature tensors of the Riemannian manifold $\overline{M}$ corresponding to $\overline{\nabla}$ and $\overline{\nabla}^0$, respectively. We know from [10] that $\overline{R}$ is given by

$$\overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \overline{R}^0(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) + s(\overline{X}, \overline{Z})g(\overline{Y}, \overline{W}) - s(\overline{Y}, \overline{Z})g(\overline{X}, \overline{W}), \tag{3}$$

for all vector fields $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ on $\overline{M}$, where s is the $(0,2)$-tensor given by

$$s(\overline{X}, \overline{Y}) = (\overline{\nabla}^0_X\omega)\overline{Y} - \omega(\overline{X})\omega(\overline{Y}).$$

Let M be an n-dimensional submanifold of $(\overline{M}, g)$.

The Gauss formula with respect to the semi-symmetric connection $\overline{\nabla}$ and the Gauss formula with respect to the Levi-Civita connection $\overline{\nabla}^0$, respectively, are written as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}^0_X Y = \nabla^0_X Y + h^0(X, Y),$$

for all vector fields X, Y on the submanifold M.

In the above equations, h^0 is the second fundamental form of M and h is a $(0,2)$-tensor on M. In [11], it is proven that $h^0 = h$.

An odd-dimensional Riemannian manifold $(\overline{M}, g)$ is called an almost-contact metric manifold if there exist a $(1,1)$-tensor field ϕ, a unit vector field ξ and a 1-form η on $\overline{M}$ satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on $\overline{M}$.

In addition, one has

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi).$$

An almost-contact metric manifold is called a trans-Sasakian manifold if there are two real differentiable functions α and β such that

$$(\overline{\nabla}^0_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X,Y)\xi - \eta(Y)\phi X];$$

it implies

$$\overline{\nabla}^0_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]. \tag{4}$$

A trans-Sasakian manifold becomes a Sasakian manifold when $\alpha = 1$ and $\beta = 0$, a Kenmotsu manifold when $\alpha = 0$ and $\beta = 1$, and a cosymplectic manifold if $\alpha = \beta = 0$, respectively.

See also the papers [19,20].

The notion of a generalized Sasakian space form was introduced by P. Alegre, D.E. Blair and A. Carriazo [21]. It is an almost-contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$ with the curvature tensor expressed by

$$\begin{aligned}\overline{R}^0(X,Y)Z = {} & f_1[g(Y,Z)X - g(X,Z)Y] \\ & + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi],\end{aligned} \tag{5}$$

for all vector fields X, Y, Z, with f_1, f_2, f_3 real smooth functions on $\overline{M}$. It is denoted by $\overline{M}(f_1, f_2, f_3)$. As particular cases, we mention the following:

(i) A Sasakian space form, if $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$;
(ii) A Kenmotsu space form, if $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$;
(iii) A cosymplectic space form, if $f_1 = f_2 = f_3 = \frac{c}{4}$.

Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional generalized Sasakian space form endowed with a semi-symmetric non-metric connection $\overline{\nabla}$. From (3) and (5), it follows that the curvature tensor $\overline{R}$ of the semi-symmetric non-metric connection $\overline{\nabla}$ has the expression

$$\begin{aligned}\overline{R}(X,Y)Z = {} & f_1[g(Y,Z)X - g(X,Z)Y] \\ & + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \\ & + s(X,Z)g(Y,W) - s(Y,Z)g(X,W).\end{aligned} \tag{6}$$

The vector field P on M can be written as $P = P^\top + P^\perp$, where $P^\top$ and $P^\perp$ are its tangential and normal components, respectively.

The Gauss equation for the semi-symmetric non-metric connection is (see [11])

$$\begin{aligned}\overline{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)) \\ & + g(P^\perp, h(Y,Z))g(X,W) - g(P^\perp, h(X,Z))g(Y,W),\end{aligned} \tag{7}$$

for all vector fields X, Y, Z and W on M, where $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature tensor of ∇ and $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Because the connection ∇ is not metric, $R(X,Y,Z,W) \neq R(X,Y,W,Z)$; then, we cannot define a sectional curvature on M by the standard definition. We will consider a sectional curvature for a semi-symmetric non-metric connection (for the motivation, see [22]) as follows.

If p is a point in M and $\pi \subset T_pM$ a 2-plane section at p spanned by the orthonormal vectors e_1, e_2, the sectional curvature $K(\pi)$ corresponding to the induced connection ∇ can be defined by

$$K(\pi) = \frac{1}{2}[R(e_1,e_2,e_2,e_1) - R(e_1,e_2,e_1,e_2)]. \tag{8}$$

One can see that this definition does not depend on the orthonormal basis.

The scalar curvature τ of M is defined by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K_{ij}, \tag{9}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j.

Let M be an $(n+1)$-dimensional submanifold tangent to ξ and $\{e_1, e_2, \ldots, e_n, e_{n+1} = \xi\}$, an orthonormal basis of the tangent space T_pM at $p \in M$; then, from (9), the scalar curvature τ of M at p takes the following form:

$$2\tau = \sum_{1 \leq i \neq j \leq n} K(e_i \wedge e_j) + 2\sum_{i=1}^{n} K(e_i \wedge \xi). \tag{10}$$

Denote by $(\inf K)(p) = \inf\{K(\pi); \pi \subset T_pM, \dim \pi = 2\}$.
B.-Y. Chen defined the invariant δ_M by

$$\delta_M(p) = \tau(p) - \inf K(p). \tag{11}$$

Let L be a k-plane section of T_pM and $X \in L$ a unit vector. For an orthonormal basis $\{e_1 = X, e_2, ..., e_k\}$ of L, the Ricci curvature Ric_L of L at X is defined by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \ldots + K_{1k}. \tag{12}$$

It is called the k-Ricci curvature.
Recall that the mean curvature vector $H(p)$ at $p \in M$ is defined by

$$H(p) = \frac{1}{n+1}\sum_{i=1}^{n+1} h(e_i, e_i). \tag{13}$$

Denoting by $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j = 1, \ldots, n+1, r \in \{n+2, \ldots, 2m+1\}$, the squared norm of the second fundamental form h is

$$\|h\|^2 = \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n+1} (h^r_{ij})^2.$$

Obviously, from the definition of the vector field P, one has

$$\omega(H) = \frac{1}{n+1}\sum_{i=1}^{n+1} g(P, h(e_i, e_i)) = g(P^{\perp}, H). \tag{14}$$

For any $X \in TM$, we can write $\phi X = PX + FX$, where PX and FX are the tangential and the normal parts of ϕX, respectively. Let

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

Lemma 1. *Let M be an $(n+1)$-dimensional submanifold tangent to ξ of a $(2m+1)$-dimensional trans-Sasakian manifold $\overline{M}$. Then, one has the following:*
(i) $h(\xi, \xi) = 0$;
(ii) $h(X, \xi) = -\alpha FX$, for any vector field X tangent to M orthogonal to ξ.

Proof. Let $p \in M$ and $X \in T_pM$; then, we have

$$\overline{\nabla}^0_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi).$$

By the Gauss formula, we get

$$h(X, \xi) = -\alpha F X.$$

Taking $X = \xi$, we obtain (i), and taking X orthogonal to ξ we obtain (ii). □

Lemma 2 ([12])**.** *Let $f(x_1, x_2, \ldots, x_n)$, $n \geq 3$ be a real function on $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \ldots, x_n) = (x_1 + x_2) \sum_{i=3}^{n} + \sum_{3 \leq i < j \leq n} x_i x_j.$$

If $x_1 + x_2 + \ldots + x_n = (n-1)a$, then

$$f(x_1, x_2, \ldots, x_n) \leq \frac{(n-1)(n-2)}{2} a^2.$$

The equality holds if and only if $x_1 + x_2 = x_3 = \ldots = x_n = a$.

Lemma 3 ([12])**.** *Let $f(x_1, x_2, \ldots, x_n)$, $n \geq 3$ be a real function on $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \ldots, x_n) = x_1 \sum_{i=2}^{n} x_i + \sum_{i=2}^{n} x_i.$$

If $x_1 + x_2 + \ldots + x_n = 2a$, then we have

$$f(x_1, x_2, \ldots, x_n) \leq a^2.$$

The equality holds if and only if $x_1 = x_2 + x_3 + \ldots + x_n = a$.

3. Chen First Inequality

Referring to the work of C. Özgür and A. Mihai [17], they used modifications of the Gauss equation for a semi-symmetric non-metric connection. They subsequently introduced a different concept of sectional curvature by utilizing the modified Gauss equation through the formula $\Omega(X) = s(X, X) + g(P^{\perp}, h(X, X))$. Here, we consider another sectional curvature which was defined above.

In the present section, we obtain Chen's first inequality for submanifolds of trans-Sasakian generalized Sasakian space forms admitting a semi-symmetric non-metric connection.

Theorem 1. *Let M be an $(n+1)$-dimensional $(n \geq 2)$ submanifold tangent to ξ of a (α, β) trans-Sasakian generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ admitting a semi-symmetric non-metric connection, $p \in M$ and $\pi \subset T_p M$ a 2-plane section orthogonal to ξ. Then, one has*

$$\begin{aligned} \tau(p) - K(\pi) \leq\ & \frac{(n-2)(n+1)}{2} f_1 + \frac{3}{2} f_2 \{\|P\|^2 - \psi^2(\pi)\} - n f_3 \\ & - \alpha^2 \|F\|^2 - \frac{n}{2} \operatorname{trace} s - \frac{n(n+1)}{2} \omega(H) \\ & + \frac{1}{2} \operatorname{trace}(s_{|\pi}) + \frac{1}{2} g(\operatorname{trace}(h_{|\pi}), P) \\ & + \frac{n^2(n-2)}{2(n-1)} \|H\|^2. \end{aligned} \tag{15}$$

Proof. Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form, $\overline{\nabla}$ a semi-symmetric non-metric connection on $\overline{M}(f_1, f_2, f_3)$ and M an $(n+1)$-dimensional submanifold tangent to ξ.

Let $p \in M$, $\pi \subset T_pM$ be a 2-plane section orthogonal to ξ and $\{e_1, \ldots, e_n, e_{n+1} = \xi\}$ be an orthonormal basis of the tangent space T_pM and $\{e_{n+2}, \ldots, e_{2m+1}\}$ an orthonormal basis of the normal space $T^{\perp}{}_pM$, with $Fe_j = \|Fe_j\| e_{n+j+1}$, $\forall j = 1, \ldots, n$.

We will use formula (10).

If we take $X = W = e_i$, $Y = Z = e_j$, $i, j = 1, \ldots, n$, in the Gauss equation, the scalar curvature τ is expressed by

$$\begin{aligned} 2\tau(p) = &\sum_{1 \le i \neq j \le n} \overline{R}(e_i, e_j, e_j, e_i) + 2 \sum_{r=n+2}^{2m+1} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] \\ &- \sum_{1 \le i \neq j \le n} g(P^{\perp}, h(e_j, e_j)) + 2 \sum_{j=1}^{n} K(\xi \wedge e_j). \end{aligned} \tag{16}$$

We calculate $\overline{R}(e_i, e_j, e_j, e_i)$ using formula (6) and put $X = W = e_i$, $Y = Z = e_j$, for $i, j = 1, \ldots n, i \neq j$. We have

$$\overline{R}(e_i, e_j, e_j, e_i) = f_1 + 3 f_2 g^2(\phi e_i, e_j) - s(e_j, e_j). \tag{17}$$

Introducing Equation (17) into (16), one has

$$\begin{aligned} 2\tau(p) = &n(n-1) f_1 + 3 f_2 \sum_{i \neq j} g^2(\phi e_i, e_j) - (n-1)\lambda \\ &+ 2 \sum_{r=1}^{n} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] - (n+1)(n-1)\omega(H) + 2 \sum_{j=1}^{n} K(\xi \wedge e_j), \end{aligned} \tag{18}$$

where we denoted $\lambda = \sum_{j=1}^{n} s(e_j, e_j)$.

From our definition of the sectional curvature, we obtain

$$K(\xi \wedge e_j) = \frac{1}{2} [R(\xi, e_j, e_j, \xi) - R(\xi, e_j, \xi, e_j)]. \tag{19}$$

Take $X = W = \xi$, $Y = Z = e_j$, for $j = 1, \ldots, n$, in the Gauss equation. We find

$$\begin{aligned} R(\xi, e_j, e_j, \xi) = &\overline{R}(\xi, e_j, e_j, \xi) + g(h(\xi, \xi), h(e_j, e_j)) - g(h(\xi, e_j), h(\xi, e_j)) \\ &- g(P^{\perp}, h(e_j, e_j)) g(\xi, \xi). \end{aligned} \tag{20}$$

We can rewrite the last equation as

$$R(\xi, e_j, e_j, \xi) = \overline{R}(\xi, e_j, e_j, \xi) + \sum_{r=n+2}^{2m+1} [h_{jj}^r h_{\xi\xi}^r - (h_{j\xi}^r)^2] - g(P^{\perp}, h(e_j, e_j)). \tag{21}$$

By formula (6) we have

$$\overline{R}(\xi, e_j, e_j, \xi) = f_1 - f_3 - s(e_j, e_j), \; \forall j = 1, \ldots, n. \tag{22}$$

Introducing (22) into (21), one has

$$R(\xi, e_j, e_j, \xi) = f_1 - f_3 - s(e_j, e_j) + \sum_{r=n+2}^{2m+1} [h_{jj}^r h_{\xi\xi}^r - (h_{j\xi}^r)^2] - g(P^{\perp}, h(e_j, e_j)). \tag{23}$$

By using Lemma 1, we obtain

$$\sum_{j=1}^{n} \sum_{r=n+2}^{2m+1} (h_{j\xi}^r)^2 = \sum_{j=1}^{n} \sum_{r=n+2}^{2m+1} g^2(h(e_j, \xi), e_r) = \alpha^2 \sum_{j=1}^{n} \sum_{r=n+2}^{2m+1} g^2(Fe_j, e_r)$$

$$= \alpha^2 \sum_{j=1}^{n} \|Fe_j\|^2 = \alpha^2 \|F\|^2.$$

Then, Equation (23) can be rewritten as

$$R(\xi, e_j, e_j, \xi) = f_1 - f_3 - s(e_j, e_j) - \alpha^2 \|F\|^2 - g(P^{\perp}, h(e_j, e_j)). \tag{24}$$

Similarly, from the Gauss equation, if we put $X = Z = \xi$, $Y = W = e_j$, for $j = 1, \ldots n$, we have

$$R(\xi, e_j, \xi, e_j) = -f_1 + f_3 + s(\xi, \xi) + \alpha^2 \|Fe_j\|^2. \tag{25}$$

By substituting (24) and (25) in (20), and taking summation, we find

$$\sum_{j=1} K(\xi \wedge e_j) = \frac{1}{2}[2nf_1 - 2nf_3 - 2\alpha^2 \|F\|^2 - \lambda - ns(\xi, \xi) - (n+1)\omega(H)]. \tag{26}$$

If we put (26) in (18), we obtain

$$\begin{aligned} 2\tau(p) = {} & n(n+1)f_1 + 3f_2\|P\|^2 - 2nf_3 \\ & - 2\alpha^2 \|F\|^2 - n\lambda - ns(\xi, \xi) - n(n+1)\omega(H) \\ & + 2 \sum_{r=n+2}^{2m+1} \sum_{1 \le i < j \le n} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2]. \end{aligned} \tag{27}$$

Let $\pi = \mathrm{span}\{e_1, e_2\}$. In the Gauss equation, we put $X = W = e_1$, $Y = Z = e_2$. Then,

$$\begin{aligned} R(e_1, e_2, e_2, e_1) = {} & f_1 + 3f_2 g^2(\phi e_1, e_2) - s(e_2, e_2) \\ & + \sum_{r=n+1}^{2m+1} [h^r_{11} h^r_{22} - (h^r_{12})^2] - g(P^{\perp}, h(e_2, e_2)). \end{aligned} \tag{28}$$

Similarly, if we put $X = Z = e_1$, $Y = W = e_2$, in the Gauss equation,

$$\begin{aligned} R(e_1, e_2, e_1, e_1) = {} & -f_1 - 3f_2 g^2(\phi e_1, e_2) + s(e_1, e_1) \\ & - \sum_{r=n+2}^{2m+1} [h^r_{11} h^r_{22} - (h^r_{12})^2] + g(P^{\perp}, h(e_1, e_1)). \end{aligned} \tag{29}$$

So from (8), (28) and (29), we have

$$\begin{aligned} K(\pi) = {} & f_1 + 3f_2 g^2(\phi e_1, e_2) \\ & - \frac{1}{2}\mathtt{trace}(s_{|\pi}) - \frac{1}{2} g(\mathtt{trace}(h_{|\pi}), P)] \\ & + \sum_{r=n+2}^{2m+1} [h^r_{11} h^r_{22} - (h^r_{12})^2]. \end{aligned} \tag{30}$$

We denote $\psi^2(\pi) = g^2(\phi e_1, e_2)$; then

$$
\begin{aligned}
\tau(p) - K(\pi) = {} & \frac{(n-2)(n+1)}{2} f_1 + \frac{3}{2} f_2 \{\|P\|^2 - \psi^2(\pi)\} - n f_3 \\
& - \alpha^2 \|F\|^2 - \frac{n}{2}\lambda - \frac{n}{2} s(\xi,\xi) - \frac{n(n+1)}{2}\omega(H) \\
& + \frac{1}{2}\mathtt{trace}(s_{|\pi}) + \frac{1}{2} g(\mathtt{trace}(h_{|\pi}), P)] \\
& + \sum_{r=n+2}^{2m+1} \{(h^r_{11} + h^r_{22}) \sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii} h^r_{jj} \\
& - \sum_{3\le j\le n} (h^r_{1j})^2 - \sum_{2\le i<j\le n} (h^r_{ij})^2 \},
\end{aligned} \tag{31}
$$

which implies

$$
\begin{aligned}
\tau(p) - K(\pi) \le {} & \frac{(n-2)(n+1)}{2} f_1 + \frac{3}{2} f_2 \{\|P\|^2 - \psi^2(\pi)\} - n f_3 \\
& - \alpha^2 \|F\|^2 - \frac{n}{2}\lambda - \frac{n}{2} s(\xi,\xi) - \frac{n(n+1)}{2}\omega(H) \\
& + \frac{1}{2}\mathtt{trace}(s_{|\pi}) + \frac{1}{2} g(\mathtt{trace}(h_{|\pi}), P)] \\
& + \sum_{r=n+2}^{2m+1} \{(h^r_{11} + h^r_{22}) \sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii} h^r_{jj}.
\end{aligned} \tag{32}
$$

We define the real functions $f_r : \mathbb{R}^n \to \mathbb{R}$ by

$$
f_r(h^r_{11}, h^r_{22}, \dots, h^r_{nn}) = (h^r_{11} + h^r_{22}) \sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii} h^r_{jj}
$$

We study the problem max f_r, under the condition $h^r_{11} + h^r_{22} + \ldots + h^r_{nn} = b^r$, where b^r is a real number.

Lemma 2 implies that the solution $(h^r_{11}, h^r_{22}, \dots, h^r_{nn})$ must satisfy

$$
h^r_{11} + h^r_{22} = h^r_{ii} = \frac{b^r}{(n-1)}, \quad i = 3, \ldots n,
$$

which gives

$$
f_r \le \frac{(n-2)}{2(n-1)} (b^r)^2. \tag{33}
$$

By using (32) and (33), it follows that

$$
\begin{aligned}
\tau(p) - K(\pi) \le {} & \frac{(n-2)(n+1)}{2} f_1 + \frac{3}{2} f_2 \{\|P\|^2 - \psi^2(\pi)\} - n f_3 \\
& - \alpha^2 \|F\|^2 - \frac{n}{2}\mathtt{trace}\, s - \frac{n(n+1)}{2}\omega(H) + \frac{1}{2}\mathtt{trace}(s_{|\pi}) + \frac{1}{2} g(\mathtt{trace}(h_{|\pi}), P) \\
& + \frac{n^2(n-2)}{2(n-1)} \|H\|^2.
\end{aligned} \tag{34}
$$

Then the proof is achieved. $\square$

4. Chen–Ricci Inequality

In [2], B.-Y. Chen established a sharp estimate of the mean curvature in terms of the Ricci curvature for all n-dimensional Riemannian submanifolds in a Riemannian space form $\overline{M}(c)$ of constant sectional curvature c.

$$\text{Ric}\,(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2,$$

It is known as the Chen–Ricci inequality.

One of the present authors [23] derived a Chen–Ricci inequality specifically for submanifolds in Sasakian space forms.

In this section, we obtain a Chen–Ricci inequality for submanifolds tangent to ξ in a trans-Sasakian manifold endowed with a semi-symmetric non-metric connection.

Theorem 2. *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form, $\overline{\nabla}$ a semi-symmetric non-metric connection on it and M an $(n+1)$-dimensional $(n \geq 2)$ submanifold tangent to ξ. Then, we have the following:*

(1) For any unit vector $X \in T_pM$ orthogonal to ξ,

$$\begin{aligned}\text{Ric}(X) \leq\ & \frac{n^2}{4}\|H\|^2 + nf_1 + 3f_2\|Pe_1\|^2 - f_3 - \alpha^2\|F\|^2 \\ & - \frac{1}{2}[\text{trace}\, s + (n-1)s(X,X)] \\ & - \frac{1}{2}[(n+1)\omega(H) + (n-1)g(P^{\perp}, h(X,X))]. \end{aligned} \tag{35}$$

(2) If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (35) if and only if $X \in N_p$, where $N_p = \{X \in T_pM | h(X,Y) = 0, \forall Y \in \{\xi\}^{\perp}\}$.

(3) The equality case of (35) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if either

   (i) h_p vanishes on $\{\xi\}^{\perp} \times \{\xi\}^{\perp}$ or

   (ii) $n = 2$ and $h(X,Y) = g(X,Y)H$, for any $X, Y \in T_pM$ orthogonal to ξ.

Proof.

(1) Let $p \in M$, $X \in T_pM$ a unit tangent vector orthogonal to ξ. Consider an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1} = \xi, e_{n+2}, \ldots, e_{2m+1}\}$ in $T_p\overline{M}(f_1, f_2, f_3)$, with $e_1 = X, e_2, \ldots, e_n$ tangent to M at p.

$$\mathtt{Ric}(X) = \sum_{j=2}^{n} K(e_1 \wedge e_j) + K(e_1 \wedge \xi). \tag{36}$$

If we take $X = W = e_1$ and $Y = Z = e_j$ in the Gauss equation, we have

$$\begin{aligned} R(e_1, e_j, e_j, e_1) =\ & f_1 + 3f_2 g^2(\phi e_1, e_j) - s(e_j, e_j) \\ & + \sum_{r=n+2}^{2m+1} [h^r_{11}h^r_{jj} - (h^r_{1j})^2] - g(P^{\perp}, h(e_j, e_j)), \end{aligned} \tag{37}$$

respectively. From the Gauss equation, if we put $X = Z = e_1$, $Y = W = e_j$, we have

$$\begin{aligned} R(e_1, e_j, e_1, e_j) =\ & -f_1 - 3f_2 g^2(\phi e_1, e_j) + s(e_1, e_1) \\ & - \sum_{r=1}^{n} [h^r_{11}h^r_{jj} - (h^r_{1j})^2] + g(P^{\perp}, h(e_1, e_1)). \end{aligned} \tag{38}$$

Similarly to Equation (8), we have

$$K(e_1 \wedge e_j) = \frac{1}{2}[R(e_1, e_j, e_j, e_1) - R(e_1, e_j, e_1, e_j)]. \tag{39}$$

From Equations (37)–(39), we have

$$\begin{aligned} K(e_1 \wedge e_j) = {} & f_1 + 3f_2 g^2(\phi e_1, e_j) - \frac{1}{2}[s(e_j, e_j) + s(e_1, e_1)] \\ & + \sum_{r=n+2}^{2m+1} [h^r_{11} h^r_{jj} - (h^r_{1j})^2] \\ & - \frac{1}{2}[g(P^\perp, h(e_j, e_j)) + g(P^\perp, h(e_1, e_1))]. \end{aligned} \tag{40}$$

On the other hand, one has

$$\begin{aligned} K(e_1 \wedge \xi) &= \frac{1}{2}[R(\xi, e_1, e_1, \xi) - R(\xi, e_1, \xi, e_1)] \\ &= f_1 - f_3 - \alpha^2 \|Fe_1\|^2 \\ &\quad - \frac{1}{2}[s(e_1, e_1) + g(P^\perp, h(e_1, e_1)) + s(\xi, \xi)]. \end{aligned} \tag{41}$$

By substituting Equations (40) and (41) in (36), we find

$$\begin{aligned} \mathtt{Ric}(X) = {} & n f_1 + 3f_2 \sum_{j=2}^{n} g^2(\phi e_1, e_j) - f_3 \\ & - \frac{1}{2}[\operatorname{trace} s + (n-1)s(X, X)] \\ & - \frac{1}{2}[(n+1)\omega(H) + (n-1)g(P^\perp, h(X, X))] \\ & + \sum_{j=2}^{n} \sum_{r=n+2}^{2m+1} [h^r_{11} h^r_{jj} - (h^r_{1j})^2] - \alpha^2 \|F\|^2. \end{aligned} \tag{42}$$

Obviously, one has

$$h^r_{11} \left(\sum_{i=2}^{n} h^r_{ii} \right) \leq \frac{1}{4}(h^r_{11} + h^r_{22} + \ldots + h^r_{nn})^2, \tag{43}$$

and equality holds if and only if

$$h^r_{11} = \sum_{i=2}^{n} h^r_{ii}. \tag{44}$$

From Equations (42) and (43), we have

$$\begin{aligned} \mathtt{Ric}(X) \leq {} & \frac{n^2}{4}\|H\|^2 + n f_1 + 3f_2 \|Pe_1\|^2 - f_3 - \alpha^2 \|F\|^2 \\ & - \frac{1}{2}[\operatorname{trace} s + (n-1)s(X, X)] \\ & - \frac{1}{2}[(n+1)\omega(H) + (n-1)g(P^\perp, h(X, X))]. \end{aligned} \tag{45}$$

(2) If a unit vector X at p satisfies the equality case of (35), from (42), (43) and (44), one obtains

$$\begin{cases} h^r_{1i} = 0, \ 2 \leq i \leq n, \ \forall r \in \{n+2, \ldots, 2m+1\}, \\ h^r_{11} = h^r_{22} + \ldots + h^r_{nn}, \ \forall r \in \{n+2, \ldots, 2m+1\}. \end{cases}$$

Therefore, because $H(p) = 0$, we have $h^r_{1j} = 0$ for all $j = 1, \ldots, n$, $r \in \{n+2, \ldots, 2m+1\}$; that is, $X \in N_p$.

(3) The equality case of inequality (35) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h^r_{ij} = 0, \quad 1 \leq i \neq j \leq n, \quad r \in \{n+2, \ldots, 2m+1\}, \\ h^r_{11} + \ldots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, \ldots, n\}, \quad r \in \{n+2, \ldots, 2m+1\}. \end{cases}$$

There are two cases:

(i) $n \neq 2$, $h^r_{ij} = 0$. It follows that h_p vanishes on $\{\xi\}^\perp \times \{\xi\}^\perp$.

(ii) $n = 2$; then, $h(X, Y) = h(X, Y)H$, for any $X, Y \in \{\xi\}^\perp$.

□

We recall standard definitions of certain classes of submanifolds in trans-Sasakian manifolds.

Let $\overline{M}$ be a trans-Sasakian manifold and M a submanifold of $\overline{M}$ tangent to the Reeb vector field ξ.

According to the behaviour of the tangent spaces of M under the action of ϕ, we distinguish the following classes of submanifolds.

The submanifold M of $\overline{M}$ is an invariant submanifold if all its tangent spaces are invariant by ϕ, i.e., $\phi(T_pM) \subset T_pM$, $\forall p \in M$.

The submanifold M of $\overline{M}$ is an anti-invariant submanifold if ϕ maps any tangent space into the normal space, i.e., $\phi(T_pM) \subset T_p^\perp M$, $\forall p \in M$.

The submanifold M is a slant submanifold if for any $p \in M$ and any $X \in T_pM$, linearly independent on ξ, the angle θ between ϕX and T_pM is constant. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of M in $\overline{M}$.

We state the corresponding Chen–Ricci inequalities for the above submanifolds.

Corollary 1. *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form, $\overline{\nabla}$ a semi-symmetric non-metric connection on it and M an $(n+1)$-dimensional $(n \geq 2)$ invariant submanifold.*

Then, for each unit vector $X \in T_pM$ orthogonal to ξ, we have

$$\begin{aligned} \mathrm{Ric}(X) \leq &\frac{n^2}{4}\|H\|^2 + nf_1 + 3f_2 - f_3 \\ &- \frac{1}{2}[\mathrm{trace}\, s + (n-1)s(X, X)] \\ &- \frac{1}{2}[(n+1)\omega(H) + (n-1)g(P^\perp, h(X, X))]. \end{aligned} \tag{46}$$

Corollary 2. *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form, $\overline{\nabla}$ a semi-symmetric non-metric connection on it and M an $(n+1)$-dimensional $(n \geq 2)$ anti-invariant submanifold.*

Then, for each unit vector $X \in T_pM$ orthogonal to ξ, we have

$$\begin{aligned} \mathrm{Ric}(X) \leq &\frac{n^2}{4}\|H\|^2 + nf_1 - f_3 - n\alpha^2 \\ &- \frac{1}{2}[\mathrm{trace}\, s + (n-1)s(X, X)] \\ &- \frac{1}{2}(n+1)\omega(H) + (n-1)g(P^\perp, h(X, X))]. \end{aligned} \tag{47}$$

Corollary 3. *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form, $\overline{\nabla}$ a semi-symmetric non-metric connection on it and M an $(n+1)$-dimensional $(n \geq 2)$ slant submanifold.*

Then, for each unit vector $X \in T_pM$ orthogonal to ξ, we have

$$\begin{aligned}\mathrm{Ric}(X) \leq{}& \frac{n^2}{4}\|H\|^2 + nf_1 + 3f_2\cos^2\theta - f_3 - n\alpha^2\sin^2\theta \\ &- \frac{1}{2}[\mathrm{trace}\, s + (n-1)s(X,X)] \\ &- \frac{1}{2}[(n+1)\omega(H) + (n-1)g(P^\perp, h(X,X))]. \end{aligned} \tag{48}$$

5. Generalized Euler Inequality for Special Contact Slant Submanifolds

B.Y. Chen [24] proved a generalized Euler inequality for n-dimensional submanifolds in a Riemannian space form of constant sectional curvature c:

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c,$$

with equality holding identically if and only if the submanifold is totally umbilical.

In this section, we prove a generalized Euler inequality for certain submanifolds in a trans-Sasakian manifold endowed with a semi-symmetric non-metric connection.

In [18], we extended the definition of a special slant submanifold in a Sasakian manifold to trans-Sasakian manifolds.

Let M be a proper slant submanifold ($\theta \neq 0, \frac{\pi}{2}$) of a trans-Sasakian manifold $\overline{M}$. We call M a special contact slant submanifold if

$$(\nabla^0_X P)Y = \cos^2\theta[\alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X0]. \forall X, Y \in \Gamma(TM).$$

Then, the components of the second fundamental form are symmetric, i.e.,

$$h^k_{ij} = h^i_{jk} = h^j_{ik}, \ \forall i, j, k = 1, \ldots, n.$$

For special contact slant submanifolds, we prove a generalized Euler inequality.

Theorem 3. *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form, $\overline{\nabla}$ a semi-symmetric non-metric connection on it and M an $(n+1)$-dimensional $(n \geq 2)$ special contact slant submanifold. Then,*

$$\begin{aligned}\|H\|^2 \geq{}& \frac{2(n+2)}{(n-1)(n+1)^2}\tau - \frac{n(n+2)}{n^2-1}f_1 \\ &+ 3\frac{n(n+2)}{(n-1)(n+1)^2}f_2\cos^2\theta + \frac{2n(n+2)}{(n-1)(n+1)^2}f_3 + \frac{2n(n+2)}{(n-1)(n+1)^2}\alpha^2\sin^2\theta \\ &- \frac{n(n+2)}{(n-1)(n+1)^2}[\mathrm{trace}\, s + (n+1)\omega(H)]. \end{aligned} \tag{49}$$

Proof. Consider a $(2n+1)$-dimensional (α, β) trans-Sasakian generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ with a semi-symmetric non-metric connection $\overline{\nabla}$ and M an $(n+1)$-dimensional special contact slant submanifold.

For any $p \in M$ and $\pi \subset T_pM$, a 2-plane section orthogonal to ξ, let $\{e_1, \ldots, e_n, e_{n+1} = \xi\}$ be an orthonormal basis of the tangent space T_pM and $\{e_{n+2}, \ldots, e_{2n+1}\}$ an orthonormal basis of the normal space $T^\perp{}_pM$, with $Fe_j = (\sin\theta)e_{n+j+1}, \forall j = 1, \ldots, n$.

In this case, Equation (27) becomes

$$\begin{aligned}2\tau(p) ={}& n(n+1)f_1 + 3nf_2\cos^2\theta - 2nf_3 \\ &- 2n\alpha^2\sin^2\theta - n\mathrm{trace}\, s - n(n+1)\omega(H) \\ &- \|h\|^2 + (n+1)^2\|H\|^2. \end{aligned} \tag{50}$$

On the other hand, we have

$$(n+1)^2\|H\|^2 = \sum_i g(h(e_i,e_i),h(e_i,e_i)) + \sum_{i\neq j} g(h(e_i,e_i),h(e_j,e_j))$$
$$= \sum_{i=1}^{n}[\sum_{j=1}^{n}(h^i_{jj})^2 + 2\sum_{1\leq j<k\leq n} h^i_{jj}h^i_{kk}]. \tag{51}$$

From Equations (50) and (51), we obtain

$$2\tau(p) = n(n+1)f_1 + 3nf_2\cos^2\theta - 2nf_3$$
$$-2n\alpha^2\sin^2\theta - n\mathrm{trace}\, s - n(n+1)\omega(H)$$
$$+2\sum_i\sum_{j<k} h^i_{jj}h^i_{kk} - 2\sum_{i\neq j}(h^i_{jj})^2 - 6\sum_{i<j<k}(h^k_{ij})^2, \tag{52}$$

Let us now introduce a parameter m given by $m = \frac{n+2}{n-1}$, with $n \geq 2$, for studying the inequality of $\|H\|^2$ by mimicking the technique used in ([25]). Then, we have

$$(n+1)^2\|H\|^2 - m\{2\tau - n(n+1)f_1 + 3n\cos^2\theta - 2nf_3$$
$$-2n\alpha^2\sin^2\theta - n\mathrm{trace}\, s - n(n+1)\omega(H)\}$$
$$= \sum_i (h^i_{ii})^2 + (1+2m)\sum_{i\neq j}(h^i_{jj})^2 + 6m\sum_{i<j<k}(h^k_{ij})^2$$
$$-2(m-1)\sum_i\sum_{j<k} h^i_{jj}h^i_{kk}$$
$$= \sum_i (h^i_{ii})^2 + 6m\sum_{i<j<k}(h^k_{ij})^2 + (m-1)\sum_i\sum_{j<k}(h^i_{jj}-h^i_{kk})^2$$
$$+\{1+2m-(n-2)(m-1)\}\sum_{i\neq j}(h^i_{jj})^2 - 2(m-1)\sum_{i\neq j} h^i_{ii}h^i_{jj}$$
$$= 6m\sum_{i<j<k}(h^k_{ij})^2 + (m-1)\sum_{i\neq j,k}\sum_{j<k}(h^i_{jj}-h^i_{kk})^2$$
$$+\frac{1}{n-1}\sum_{i\neq j}\Big\{h^i_{ii} - (n-1)(m-1)h^i_{jj}\Big\}^2 \geq 0. \tag{53}$$

It follows that

$$\|H\|^2 \geq \frac{2(n+2)}{(n-1)(n+1)^2}\tau - \frac{n(n+2)}{n^2-1}f_1$$
$$+3\frac{n(n+2)}{(n-1)(n+1)^2}f_2\cos^2\theta + \frac{2n(n+2)}{(n-1)(n+1)^2}f_3 + \frac{2n(n+2)}{(n-1)(n+1)^2}\alpha^2\sin^2\theta$$
$$-\frac{n+2}{(n-1)(n+1)^2}[n\,\mathrm{trace}\, s + n(n+1)\omega(H)]. \tag{54}$$

□

6. Example

We will give an example of a special contact slant submanifold in $\mathbb{R}^5$ with the standard Sasakian strucure, with a semi-symmetric non-metric connection.

Consider on $\mathbb{R}^{2m+1}$ the Sasakian structure $(\mathbb{R}^{2m+1}, \phi_0, \eta, \xi, g)$, given by

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{m} y^i dx^i\right), \quad \xi = 2\frac{\partial}{\partial z},$$
$$g = -\eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{m}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi_0\left(\sum_{i=1}^{m}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{m}\left(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{m} Y_i y^i\frac{\partial}{\partial z},$$

with $\{x^i, y^i, z\}$, $i = 1, \ldots, m$, the Cartesian coordinates on $\mathbb{R}^{2m+1}$.

A semi-symmetric non-metric connection is given by

$$\nabla_X Y = \nabla^0_X Y + \eta(Y)X.$$

In particular, one derives

$$\phi_0\left(\frac{\partial}{\partial x^i}\right) = -\frac{\partial}{\partial y^i},$$
$$\phi_0\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z},$$
$$\phi_0(\xi) = 0.$$

It is known that the ϕ_0-sectional curvature of $\mathbb{R}^{2m+1}$ is -3.

We define a three-dimensional special contact slant submanifold by the equation

$$x(u, v, t) = 2((u + v), k\cos v, v - u, k\sin v, t),$$

in $\mathbb{R}^5$ with the usual Sasakian structure, endowed with the above semi-symmetric non-metric connection.

It is special contact slant submanifold with slant angle $\theta = \cos^{-1}\sqrt{\frac{2}{2+k^2}}$.

An orthonormal frame is given by

$$e_1 = \frac{1}{\sqrt{2}}(1, 0, -1, 0, 0),$$
$$e_2 = \frac{1}{\sqrt{k^2+2}}(1, -k\sin v, 1, k\cos v, 0),$$
$$e_3 = 2(0, 0, 0, 0, 1) = \xi,$$
$$e_4 = \frac{1}{\sin\theta}Fe_1 = e_{1^*},$$
$$e_5 = \frac{1}{\sin\theta}Fe_2 = e_{2^*}.$$

We compute the slant angle and obtain

$$\cos\theta = g(\phi_0 e_2, e_1) = -g(\phi_0 e_1, e_2) = \sqrt{\frac{2}{2+k^2}}$$

Now, we compute the second fundamental form.

Obviously, $h(e_3, e_3) = 0$.

Also, we know from Lemma 1 that $h(e_i, e_3) = -\sin\theta e_{i^*}, \quad i = 1, 2$.

By standard calculations, we obtain

$$h(e_1, e_1) = h(e_1, e_2) = 0$$

and

$$h(e_2, e_2) = \frac{1}{2k^2 + 8}[2(0, -k\cos v, 0, -k\sin v, 0)].$$

Let $\pi = \text{span}\{e_1, e_2\}$. In the Gauss equation, we put $X = W = e_1$, $Y = Z = e_2$. Then,

$$R(e_1, e_2, e_2, e_1) = -3g^2(\phi e_1, e_2) - s(e_2, e_2) \\ + g(h(e_1, e_1), h(e_2, e_2)) - g(h(e_1, e_2), h(e_1, e_2)) - g(P^{\perp}, h(e_2, e_2)).$$

In our case, $s(e_2, e_2) = 0$ and $g(\xi, h(e_2, e_2)) = 0$. Then, $R(e_1, e_2, e_2, e_1) = -\frac{6}{2+k^2}$.

Similarly, $R(e_1, e_2, e_1, e_1) = \frac{6}{k^2+2}$.

Consequently, $K(\pi) = -\frac{6}{2+k^2}$ and $\tau = K(\pi) + 2$.

Also, $H = \frac{1}{3}h(e_2, e_2) \neq 0$, i.e., M is not a minimal submanifold.

7. Conclusions

In this article, we dealt with trans-Sasakian manifolds admitting a semi-symmetric non-metric connection. We considered the sectional curvature defined recently in [22].

We established Chen's first inequality, the Chen–Ricci inequality and the generalized Euler inequality for submanifolds tangent to the Reeb vector field in a trans-Sasakian manifold endowed with a semi-symmetric non-metric connection. Particular cases of such submanifolds were also discussed.

This study can be continued, for instance, to obtain other Chen inequalities or improving the present results for special classes of submanifolds in trans-Sasakian manifolds or in other ambient spaces.

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