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On Blow-Up Solutions for the Fourth-Order Nonlinear Schrödinger Equation with Mixed Dispersions

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Abstract: In this paper, we consider blow-up solutions for the fourth-order nonlinear Schrödinger equation with mixed dispersions. We study the dynamical properties of blow-up solutions for this equation, including the $\dot{H}^{\gamma_c}$-concentration and limiting profiles, which extend and improve the existing results in the literature.

Keywords: fourth-order nonlinear Schrödinger equation; blow-up behavior; minimizers; $\dot{H}^{\gamma_c}$-critical

MSC: 35Q55; 35A15; 35B44

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1. Introduction

In this paper, we study the nonlinear fourth-order Schrödinger equation with mixed dispersions

$$\begin{cases} i\psi_t - \Delta^2\psi + \mu\Delta\psi + |\psi|^p\psi = 0, \ (t,x) \in [0,T^*) \times \mathbb{R}^N, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$, $\psi : [0,T^*) \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $0 < T^* \leq \infty$, $0 < p < 4^*$ (where $4^* = +\infty$ if $N = 1,2,3,4$ and $4^* = \frac{8}{N-4}$ if $N \geq 5$). Karpman in [1] first introduced the fourth-order Schrödinger Equation (1) to stabilize soliton instabilities. Karpman and Shagalov in [2] also proposed a small fourth-order dispersion term to describe the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. In recent years, there has been a great deal of interest in using higher-order operators to model physical phenomena (see [3–8]).

When $\mu = 0$, Equation (1) entails the scaling invariance

$$\psi_\lambda(t,x) = \lambda^{\frac{4}{p}}\psi(\lambda^4 t, \lambda x), \ \ \lambda > 0.$$

This implies that if ψ solves (1) with $\mu = 0$, then ψ_λ solves the same equation with the initial data $\psi_\lambda(0,x) = \lambda^{\frac{4}{p}}\psi_0(\lambda x)$. A direct computation shows

$$\|\psi_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\gamma+\frac{4}{p}-\frac{N}{2}}\|\psi_0\|_{\dot{H}^\gamma}.$$

This implies that the Sobolev $\dot{H}^{\gamma_c}$-norm and Lebesgue L^{p_c}-norm are invariant under the scaling $\psi \mapsto \psi_\lambda$, where

$$\gamma_c := \frac{N}{2} - \frac{4}{p} \ \text{and} \ p_c := \frac{2N}{N-2\gamma_c} = \frac{Np}{4}.$$

Although there is not any scaling invariance for Equation (1) with $\mu \neq 0$, γ_c and p_c are referred to as the critical Sobolev and Lebesgue exponents of (1), respectively. When $0 \leq \gamma_c \leq 2$, i.e., $\frac{8}{N} \leq p \leq 4^*$, Equation (1) is referred to as $\dot{H}^{\gamma_c}$-critical. In particular, when $\gamma_c = 0$ and $\gamma_c = 2$, Equation (1) is referred to as L^2-critical (or mass-critical) and $\dot{H}^2$-critical (or energy-critical), respectively.

If the initial data $\psi_0 \in H^2$, then Equation (1) reflects the mass and energy conservation laws:

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \;\; E(\psi(t)) = E(\psi_0),$$

where the energy E is defined by

$$E(\psi(t)) = \frac{1}{2}\|\Delta\psi(t)\|_{L^2}^2 + \frac{\mu}{2}\|\nabla\psi(t)\|_{L^2}^2 - \frac{1}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2}. \tag{2}$$

If the initial data $\psi_0 \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ with $\gamma_c \leq 1$, then the equation only assumes energy conservation. The conservation of mass is no longer available in this setting.

Recently, Equation (1) was investigated extensively in [9–18]. The local well-posedness in H^2 was studied in [9,13,15]. The global well-posedness for (1) in H^2 was studied by Fibich, Ilan, and Papanicolaou in [19]. The global properties, including the sharp threshold of scattering and blow-up, asymptotical behavior, and scattering were investigated in [12,15–18,20]. When $0 < p < \frac{8}{N}$, it follows that all the solutions of (1) exist globally using the mass conservation. Boulenger and Lenzmann in [21] proved the existence of radial blow-up solutions for (1) with $\frac{8}{N} \leq p \leq 4^*$. When $\mu = 0$, the dynamical properties of the blow-up solutions of (1) were investigated in [22–28]. However, when $\mu \neq 0$, the dynamical properties of the blow-up solutions of (1) have not yet been discussed.

The aim of this paper is to consider the dynamical properties of the blow-up solutions of (1) with $\mu \neq 0$. However, compared with the case $\mu = 0$ considered in [25,26,28], there are two major difficulties in the analysis of the blow-up solutions of (1). One is the loss of mass conservation due to the initial data $\psi_0 \in \dot{H}^{\gamma_c} \cap \dot{H}^2$; the other is the loss of scaling invariance to (1) with $\mu \neq 0$. Since there is no scaling invariance for $\mu \neq 0$, we choose the ground states of the equations

$$\Delta^2 Q + (-\Delta)^{\gamma_c} Q - |Q|^p Q = 0, \tag{3}$$

and

$$\Delta^2 R + |R|^{p_c-2} R - |R|^p R = 0, \tag{4}$$

to describe some of the concentration properties and limiting profiles of the blow-up solutions to (1), respectively, where (3) and (4) arise in the study of the optimal constants of inequalities (12) and (14) (see [25]).

The structure of this paper is as follows: In Section 2, we provide some preliminary information, including the local well-posedness of (1), the profile decomposition of the bounded sequences in $\dot{H}^{\gamma_c} \cap \dot{H}^2$, and the localized virial to (1). In Section 3, we investigate the dynamical properties of the blow-up solutions of (1) with $\mu \neq 0$ in the L^2-critical and L^2-supercritical cases, including the concentration properties and limiting profiles.

2. Preliminaries

First, we recall the local well-posedness for the Cauchy problem (1).

Lemma 1 ([14]). *Let $0 < p < 4^*$ and $\psi_0 \in H^2$. Then, there exists $T = T(\|\psi_0\|_{H^2})$, such that (1) admits a unique solution $\psi \in C([0,T), H^2)$. If $T^* < \infty$, then $\|\Delta\psi(t)\|_{L^2} \to \infty$ as $t \uparrow T^*$, where*

T^* is the maximal existence time of solution $\psi(t)$. Moreover, for all $[0, T^*)$, the following mass and energy conservation laws follow:

$$M(\psi(t)) = \int_{\mathbb{R}^N} |\psi(t,x)|^2 dx = M(\psi_0), \tag{5}$$

$$E(\psi(t)) = E(\psi_0), \tag{6}$$

where $E(\psi(t))$ is defined by (2).

Next, in order to study the existence of the blow-up solutions, we recall the localized virial to (1) established in [21]. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a radial function which satisfies $\nabla^j \varphi \in L^\infty$, for $1 \le j \le 6$,

$$\varphi(r) := \begin{cases} \frac{r^2}{2} & for\ r \le 1 \\ const. & for\ r \ge 10, \end{cases} \quad and\ \varphi''(r) \le 1,\ for\ r \ge 0.$$

For $R > 0$, we define $\varphi_R(r) := R^2 \varphi(\frac{r}{R})$. When $\psi \in C([0, T^*); H^2)$, we define the localized virial of $\psi(t)$ by

$$M_{\varphi_R}(\psi(t)) := 2Im \int_{\mathbb{R}^N} \overline{\psi}(t,x) \nabla \varphi_R(x) \nabla \psi(t,x) dx. \tag{7}$$

Boulenger and Lenzmann in [21] obtained the following time evolution of $M_{\varphi_R}(\psi(t))$.

Lemma 2 ([21], Lemma 3.1)**.** *Let $0 < p < 4^*$ and $R > 0$. Let $\psi \in C([0, T^*); H^2)$ be a radial solution to (1), then,*

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(\psi(t)) \le & 2NpE(\psi(t)) - (Np-8)\|\Delta\psi(t)\|_{L^2}^2 - (Np-4)\mu\|\nabla\psi(t)\|_{L^2}^2 + X_\mu[\psi(t)] \\ & + \mathcal{O}\left(\frac{1}{R^4} + \frac{\|\nabla\psi(t)\|_{L^2}^2}{R^2} + \frac{\|\nabla\psi(t)\|_{L^2}^{p/2}}{R^{\frac{(N-1)p}{2}}} + \frac{|\mu|}{R^2} \right) \\ = & 4Q(\psi(t)) + X_\mu[\psi(t)] + \mathcal{O}\left(\frac{1}{R^4} + \frac{\|\nabla\psi(t)\|_{L^2}^2}{R^2} + \frac{\|\nabla\psi(t)\|_{L^2}^{p/2}}{R^{\frac{(N-1)p}{2}}} + \frac{|\mu|}{R^2} \right), \end{aligned}$$

for any $t \in [0, T^*)$, where

$$X_\mu[\psi(t)] \le \begin{cases} 0 & for\ \mu \le 0, \\ A_0|\mu|\|\nabla\psi(t)\|_{L^2}^2 & for\ \mu < 0, \end{cases}$$

with some constant $A_0 > 0$.

Lemma 3 ([29], Proposition 1.32)**.** *Let $s_0 \le s \le s_1$. Then, $\dot{H}^{s_0} \cap \dot{H}^{s_1}$ is included in $\dot{H}^s$, and*

$$\|v\|_{\dot{H}^s} \le \|v\|_{\dot{H}^{s_0}}^{1-\theta} \|v\|_{\dot{H}^{s_1}}^{\theta}, \quad for\ all\ v \in \dot{H}^{s_0} \cap \dot{H}^{s_1}, \tag{8}$$

where $s = (1-\theta)s_0 + \theta s_1$.

Lemma 4 ([26], Theorem 1.1)**.** *If $0 < p < 4^*$, then*

$$\|v\|_{L^{p+2}}^{p+2} \le \frac{4(p+2)}{4(p+2)-Np} \left(\frac{4(p+2)-Np}{Np} \right)^{\frac{Np}{8}} \frac{1}{\|R\|_{L^2}^{\frac{p}{2}}} \|v\|_{L^2}^{\frac{4(p+2)-Np}{4}} \|\Delta v\|_{L^2}^{\frac{Np}{4}}, \tag{9}$$

for all $v \in H^2$, where $R \in H^2$ is a ground state of the equation

$$\Delta^2 R + R - |R|^p R = 0. \tag{10}$$

Moreover, the following Pohozaev's identities follow.

$$\|\Delta R\|_{L^2}^2 = \frac{Np}{4(p+2)}\|R\|_{L^{p+2}}^{p+2} = \frac{Np}{8-p(N-4)}\|R\|_{L^2}^2. \tag{11}$$

Lemma 5 ([25], Proposition 3.2)**.** *Let $\frac{8}{N} < p < 4^*$. Then, for all $v \in \dot{H}^{\gamma_c} \cap \dot{H}^2$*

$$\|v\|_{L^{p+2}}^{p+2} \le \frac{p+2}{2}\frac{1}{\|Q\|_{\dot{H}^{\gamma_c}}^p}\|v\|_{\dot{H}^{\gamma_c}}^p\|\Delta v\|_{L^2}^2, \tag{12}$$

where $Q \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ is a ground state of (3)*. Moreover, the following Pohozaev's identities follow.*

$$\|Q\|_{\dot{H}^2}^2 = \frac{2}{p+2}\|Q\|_{L^{p+2}}^{p+2} = \frac{2}{p}\|Q\|_{\dot{H}^{\gamma_c}}^2. \tag{13}$$

Lemma 6 ([25], Proposition 3.2)**.** *Let $\frac{8}{N} < p < 4^*$. Then, for all $v \in L^{p_c} \cap \dot{H}^2$*

$$\|v\|_{L^{p+2}}^{p+2} \le \frac{p+2}{2}\frac{1}{\|R\|_{L^{p_c}}^p}\|v\|_{L^{p_c}}^p\|\Delta v\|_{L^2}^2, \tag{14}$$

where $R \in L^{p_c} \cap \dot{H}^2$ is a ground state solution of the elliptic Equation (4)*. Moreover, the following Pohozaev's identities hold true:*

$$\|R\|_{\dot{H}^2}^2 = \frac{2}{p+2}\|R\|_{L^{p+2}}^{p+2} = \frac{2}{p}\|R\|_{L^{p_c}}^2. \tag{15}$$

Since the uniqueness of the ground state solutions to (10), (3) and (4) is still unknown, to study the dynamical properties of blow-up solutions, we introduce the notions of Sobolev and Lebesgue ground states. Denote

$$\begin{aligned} G_0(u) &:= \|u\|_{L^{\frac{8}{N}+2}}^{\frac{8}{N}+2} \div \left[\|u\|_{L^2}^{\frac{8}{N}}\|\Delta u\|_{L^2}^2\right], \quad u \in H^2, \\ G(u) &:= \|u\|_{L^{p+2}}^{p+2} \div \left[\|u\|_{\dot{H}^{\gamma_c}}^p\|\Delta u\|_{L^2}^2\right], \quad u \in \dot{H}^{\gamma_c} \cap \dot{H}^2, \\ K(u) &:= \|u\|_{L^{p+2}}^{p+2} \div \left[\|u\|_{L^{p_c}}^p\|\Delta u\|_{L^2}^2\right], \quad u \in L^{p_c} \cap \dot{H}^2. \end{aligned}$$

Definition 1 (Ground states)**.**

1 *We call the* **Sobolev ground states** *the maximizers of G_0 and G, which are solutions to* (10) *and* (3)*, respectively. We denote the set of Sobolev ground states of G_0 and G by $\mathcal{G}_0$ and $\mathcal{G}$, respectively.*
2 *We call the* **Lebesgue ground states** *the maximizers of K, which are solutions to* (4)*. We denote the set of Lebesgue ground states by $\mathcal{K}$.*

It follows from the optimal constants in (9), (12), and (14) that all the Sobolev ground states have the same $\dot{H}^{\gamma_c}$-norm and all the Lebesgue ground states have the same L^{p_c}-norm. We thus denote

$$G_0 := \|Q\|_{L^2}, \ \forall Q \in \mathcal{G}_0, \quad G_1 := \|Q\|_{\dot{H}^{\gamma_c}}, \ \forall Q \in \mathcal{G}, \quad G_2 := \|R\|_{L^{p_c}}, \ \forall R \in \mathcal{K}. \tag{16}$$

Finally, we recall the following two compactness lemmas:

Lemma 7 ([28], Compactness lemma I)**.** *Suppose that $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in H^2 and satisfies*

$$\limsup_{n\to\infty}\|\Delta u_n\|_{L^2} \le M, \quad \limsup_{n\to\infty}\|u_n\|_{L^{8/N+2}} \ge m > 0.$$

Then, there exist $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ and $V \in H^2$, such that, up to a subsequence,

$$u_n(\cdot + x_n) \rightharpoonup V \ \ weakly\ in\ H^2$$

with

$$\|V\|_{L^2}^p \geq \frac{2}{p+2}\frac{m^{p+2}}{M^2}G_0^p.$$

Lemma 8 ([25], Compactness lemma II). *Let $\frac{8}{N} < p < 4^*$. Let $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence in $\dot{H}^{\gamma_c} \cap \dot{H}^2$, such that*

$$\limsup_{n\to\infty} \|v_n\|_{\dot{H}^2} \leq M, \quad \limsup_{n\to\infty} \|v_n\|_{L^{p+2}} \geq m.$$

- *Then, there exist $V \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ and a sequence $\{y_n\}_{n=1}^{\infty}$ in $\mathbb{R}^N$, such that up to a subsequence,*

$$v_n(\cdot + y_n) \rightharpoonup V \text{ weakly in } \dot{H}^{\gamma_c} \cap \dot{H}^2,$$

with

$$\|V\|_{\dot{H}^{\gamma_c}}^p \geq \frac{2}{p+2}\frac{m^{p+2}}{M^2}G_1^p. \tag{17}$$

- *Then, there exist $W \in L^{p_c} \cap \dot{H}^2$ and a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{R}^N$, such that up to a subsequence,*

$$v_n(\cdot + z_n) \rightharpoonup W \text{ weakly in } L^{p_c} \cap \dot{H}^2,$$

with

$$\|W\|_{L^{p_c}}^p \geq \frac{2}{p+2}\frac{m^{p+2}}{M^2}G_2^p. \tag{18}$$

Remark 1. *The lower bounds (17) and (18) are optimal. Indeed, taking $v_n = Q$ in the first case and $v_n = R$ in the second case where $Q \in \mathcal{G}$ and $R \in \mathcal{K}$, we obtain the equalities.*

3. Dynamic of Blow-Up Solutions in the L^2-Critical and L^2-Supercritical Cases

In this section, we study the dynamical properties of the blow-up solutions for (1) in the L^2-critical and L^2-supercritical cases.

3.1. The Sharp Threshold Mass of Blow-Up and Global Existence

It easily follows from the local well-posedness that the solution of (1) with small initial data exists globally, and the solution may blow up in finite time for some large initial data. Therefore, whether there is a sharp threshold of global existence and blow-up for (1) is of particular interest. Next, we obtain the sharp threshold mass of global existence and blow-up for (1) by using the scaling argument and the inequality (9).

Theorem 1. *Let $\psi_0 \in H^2$, $\mu > 0$, $p = \frac{8}{N}$. Then, we obtain the following sharp threshold mass of the global existence and blow-up :*

(i) If $\|\psi_0\|_{L^2} \leq G_0$, then all solutions of (1) exist globally.

(ii) For any $\rho > G_0$, there exist initial data ψ_0, such that $\|\psi_0\|_{L^2} = \rho$ and the corresponding solution $\psi(t)$ of (1) blows up in finite time.

Remark 2. *When $\mu = 0$, Fibich, Ilan, and Papanicolaou in [19] proved that all solutions of (1) with initial data $\|\psi_0\|_{L^2} < G_0$ exist globally. When $\mu > 0$, we prove that all solutions of (1) with initial data $\|\psi_0\|_{L^2} \leq G_0$ exist globally. This suggest that the defocusing second-order dispersion term may prevent the occurrence blow-up.*

Proof. (i) When $\|\psi_0\|_{L^2} < G_0$, we deduce from (2) and (9) that

$$\begin{aligned} E(\psi_0) = E(\psi(t)) &= \frac{1}{2}\|\Delta\psi(t)\|_{L^2}^2 + \frac{\mu}{2}\|\nabla\psi(t)\|_{L^2}^2 - \frac{1}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2} \\ &\geq \left(\frac{1}{2} - \frac{\|\psi_0\|_{L^2}^p}{2G_0^p}\right)\|\Delta\psi(t)\|_{L^2}^2. \end{aligned}$$

Due to $\|\psi_0\|_{L^2} < G_0$, we have that $\|\Delta\psi(t)\|_{L^2}$ is uniformly bounded for all times t. Therefore, (i) follows from the conservation of mass and Lemma 1.

When $\|\psi_0\|_{L^2} = G_0$, we prove this result by contradiction. If the solution $\psi(t)$ of (1) blows up in finite time, then there exists $T^* > 0$, such that $\lim_{t\to T^*}\|\Delta\psi(t)\|_{L^2} = \infty$. Set

$$\rho^2(t) = \|\Delta R\|_{L^2}/\|\Delta\psi(t)\|_{L^2} \ \ and \ \ v(t,x) = \rho(t)^{N/2}\psi(t,\rho(t)x).$$

Let $\{t_n\}_{n=1}^{\infty}$ be any time sequence, such that $t_n \to T^*$, $\rho_n := \rho(t_n)$ and $v_n(x) := v(t_n, x)$. Then, the sequence $\{v_n\}$ satisfies

$$\|v_n\|_{L^2} = \|\psi(t_n)\|_{L^2} = \|\psi_0\|_{L^2} = G_0, \ \ \|\Delta v_n\|_{L^2} = \rho_n^2\|\Delta\psi(t_n)\|_{L^2} = \|\Delta R\|_{L^2}. \tag{19}$$

Observe that

$$\begin{aligned} 0 \leq \frac{1}{2}\|\Delta v_n\|_{L^2}^2 - \frac{1}{p+2}\|v_n\|_{L^{p+2}}^{p+2} =& \rho_n^4\left(\frac{1}{2}\|\Delta\psi(t_n)\|_{L^2}^2 - \frac{1}{p+2}\|\psi(t_n)\|_{L^{p+2}}^{p+2}\right) \\ =& \rho_n^4\left(E(\psi_0) - \frac{\mu}{2}\|\nabla\psi(t_n)\|_{L^2}^2\right) \\ \leq& \rho_n^4 E(\psi_0) \to 0, \ \ as\ n \to \infty. \end{aligned} \tag{20}$$

This implies that

$$\lim_{n\to\infty}\|v_n\|_{L^{p+2}}^{p+2} = \frac{p+2}{2}\|\Delta R\|_{L^2}^2.$$

Thus, we deduce from (19) that there exist subsequences, still denoted by $\{v_n\}$ and $u \in H^2\backslash\{0\}$, such that

$$u_n := \tau_{x_n}v_n \rightharpoonup u \neq 0 \ \ weakly\ in\ H^2,$$

for some $\{x_n\} \subseteq \mathbb{R}^N$. This implies that there exists $C_0 > 0$, such that

$$\lim_{n\to\infty}\|\nabla v_n\|_{L^2}^2 = \lim_{n\to\infty}\|\nabla u_n\|_{L^2}^2 \geq C_0 > 0. \tag{21}$$

On the other hand, we deduce from (9) and $\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2} = \|R\|_{L^2}$ that

$$\frac{1}{2}\|\Delta\psi(t)\|_{L^2}^2 - \frac{1}{p+2}\|\psi(t)\|_{L^{p+2}}^{p+2} \geq 0,$$

for all $t \in [0, T^*)$. This implies that

$$\frac{\mu}{2}\|\nabla\psi(t)\|_{L^2}^2 \leq E(\psi_0),$$

for all $t \in [0, T^*)$. We consequently obtain that

$$\|\nabla v_n\|_{L^2}^2 = \rho_n^2\|\nabla\psi(t_n)\|_{L^2}^2 \leq \frac{2\rho_n^2}{\mu}E(\psi_0) \to 0, \ \ as\ n \to \infty,$$

which is a contradiction with (21). Thus, the solution $\psi(t)$ of (1) exists globally.

(ii) Let $R \in \mathcal{G}_0$ be radial. We define the initial data $\psi_0(x) = c\lambda^{\frac{N}{2}} R(\lambda x)$ with $c = \frac{\rho}{G_0}$ and some $\lambda > 1$. Then, $\|\psi_0\|_{L^2} = \rho$. Applying the Pohožaev identity for the following equation:

$$\Delta^2 R + R - |R|^p R = 0, \tag{22}$$

i.e., $\frac{1}{2}\|\Delta R\|_{L^2}^2 = \frac{1}{p+2}\|R\|_{L^{p+2}}^{p+2}$, we deduce that

$$\begin{aligned} E(\psi_0) &= \frac{|c|^2\lambda^4}{2}\|\Delta R\|_{L^2}^2 + \frac{\mu|c|^2\lambda^2}{2}\|\nabla R\|_{L^2}^2 - \frac{|c|^{p+2}\lambda^{\frac{Np}{2}}}{p+2}\|R\|_{L^{p+2}}^{p+2} \\ &= -\frac{|c|^2\lambda^4}{2}(|c|^p - 1)\|\Delta R\|_{L^2}^2 + \frac{\mu|c|^2\lambda^2}{2}\|\nabla R\|_{L^2}^2. \end{aligned} \tag{23}$$

Now, taking λ, such that

$$\frac{\mu\|\nabla R\|_{L^2}^2}{(|c|^p - 1)\|\Delta R\|_{L^2}^2} < \lambda^2.$$

This implies $E(\psi_0) < 0$. Thus, the solution ψ of (1) with initial data ψ_0 blows up by applying the same method as that of Theorem 3 in [21]. □

3.2. The L^2-Critical Case

In this subsection, we investigate some dynamical properties of the blow-up solutions for (1) with $\mu \neq 0$ in the L^2-critical case.

Theorem 2. *(L^2-concentration) Let $\psi_0 \in H^2$, $\mu \neq 0$, $p = \frac{8}{N}$. If the solution $\psi(t)$ of (1) blows up in finite time $T^* > 0$. Let $a(t)$ be a real-valued non-negative function defined on $[0, T^*)$ satisfying $a(t)\|\Delta\psi(t)\|_{L^2}^{\frac{1}{2}} \to \infty$ as $t \to T^*$. Then, there exists $x(t) \in \mathbb{R}^N$, such that*

$$\liminf_{t \to T^*} \int_{|x - x(t)| \le a(t)} |\psi(t,x)|^2 dx \ge G_0^2, \tag{24}$$

where G_0 is defined by (16).

Remark 3. *By a similar analysis as that in Remark 2, this theorem gives the L^2-concentration and rate of L^2-concentration of the blow-up solutions of (1).*

Proof. Let $R \in \mathcal{G}_0$; we set

$$\rho^2(t) = \|\Delta R\|_{L^2}/\|\Delta\psi(t)\|_{L^2} \ \ and \ \ v(t,x) = \rho^{\frac{N}{2}}(t)\psi(t, \rho(t)x).$$

Let $\{t_n\}_{n=1}^{\infty}$ be any time sequence, such that $t_n \to T^*$, $\rho_n := \rho(t_n)$ and $v_n(x) := v(t_n, x)$. Then, the sequence $\{v_n\}$ satisfies

$$\|v_n\|_{L^2} = \|\psi(t_n)\|_{L^2} = \|\psi_0\|_{L^2}, \ \ \|\Delta v_n\|_{L^2} = \rho_n^2\|\Delta\psi(t_n)\|_{L^2} = \|\Delta R\|_{L^2}. \tag{25}$$

Observe that

$$\begin{aligned} |E_0(v_n)| =& \left|\frac{1}{2}\int_{\mathbb{R}^N} |\Delta v_n(x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^N} |v_n(x)|^{p+2} dx\right| \\ =& \rho_n^4\left|\frac{1}{2}\int_{\mathbb{R}^N} |\Delta\psi(t_n,x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^N} |\psi(t_n,x)|^{p+2} dx\right| \\ \le& \rho_n^4\left|E(\psi_0) + \frac{|\mu|}{2}\int_{\mathbb{R}^N} |\nabla\psi(t_n,x)|^2 dx\right|. \end{aligned} \tag{26}$$

Thus, applying the inequality (8), we deduce that $E_0(v_n) \to 0$, as $n \to \infty$. This implies $\int_{\mathbb{R}^N} |v_n(x)|^{p+2}dx \to \frac{p+2}{2}\|\Delta R\|_{L^2}^2$.

Set $m^{p+2} = \frac{p+2}{2}\|\Delta R\|_{L^2}^2$ and $M = \|\Delta R\|_{L^2}$. Then, it follows from Lemma 7 that there exist $V \in H^2$ and $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, such that, up to a subsequence,

$$v_n(\cdot + x_n) = \rho_n^{N/2}\psi(t_n, \rho_n(\cdot + x_n)) \rightharpoonup V \ \ \textit{weakly in } H^2 \tag{27}$$

with

$$\|V\|_{L^2} \geq G_0. \tag{28}$$

Note that

$$\frac{a(t_n)}{\rho_n} = \frac{a(t_n)\|\Delta\psi(t_n)\|_{L^2}^{1/2}}{\|\Delta R\|_{L^2}^{1/2}} \to \infty, \ \ as\ n \to \infty.$$

Then, for every $r > 0$, there exists $n_0 > 0$, such that for every $n > n_0$, $r\rho_n < a(t_n)$. Therefore, using (27), we obtain

$$\begin{aligned}
\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq a(t_n)} |\psi(t_n,x)|^2 dx &\geq \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq r\rho_n} |\psi(t_n,x)|^2 dx \\
&\geq \liminf_{n\to\infty} \int_{|x-x_n|\leq r\rho_n} |\psi(t_n,x)|^2 dx \\
&= \liminf_{n\to\infty} \int_{|x|\leq r} \rho_n^N |\psi(t_n, \rho_n(x+x_n))|^2 dx \\
&= \liminf_{n\to\infty} \int_{|x|\leq r} |v(t_n, x+x_n)|^2 dx \\
&\geq \liminf_{n\to\infty} \int_{|x|\leq r} |V(x)|^2 dx, \ \ \textit{for every } r > 0,
\end{aligned}$$

which means that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq a(t_n)} |\psi(t_n,x)|^2 dx \geq \int_{\mathbb{R}^N} |V(x)|^2 dx.$$

Since the sequence $\{t_n\}_{n=1}^{\infty}$ is arbitrary, we obtain

$$\liminf_{t\to T^*} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq a(t)} |\psi(t,x)|^2 dx \geq \int_{\mathbb{R}^N} |R(x)|^2 dx. \tag{29}$$

Observe that for every $t \in [0, T^*)$, the function $g(y) := \int_{|x-y|\leq a(t)} |\psi(t,x)|^2 dx$ is continuous on $y \in \mathbb{R}^N$ and $g(y) \to 0$ as $|y| \to \infty$. So, there exists a function $x(t) \in \mathbb{R}^N$, such that for every $t \in [0, T^*)$

$$\sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq a(t)} |\psi(t,x)|^2 dx = \int_{|x-x(t)|\leq a(t)} |\psi(t,x)|^2 dx.$$

This and (29) yield (24). □

Next, we study the limiting profile of the blow-up H^2-solutions with critical norms. To do so, we recall the following characterization of the ground states:

Lemma 9 (Characterization of ground states [28])**.** *Let $p = \frac{8}{N}$. If $u \in H^2$ is such that $\|u\|_{L^2} = G_0$ and*

$$E_0(u) := \frac{1}{2}\|u\|_{\dot{H}^2}^2 - \frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2} = 0,$$

then there exists $R \in \mathcal{G}_0$, such that u is of the form

$$u(x) = e^{i\theta}\lambda^{\frac{N}{2}}R(\lambda x + x_0),$$

for $\theta \in \mathbb{R}^N$, $\lambda > 0$ and $x_0 \in \mathbb{R}^N$.

Theorem 3. *Let $\psi_0 \in H^2$, $\mu < 0$, $p = \frac{8}{N}$. Assume $\|\psi_0\|_{L^2} = G_0$ and the corresponding solution ψ of (1) blows up in finite time $T^* > 0$, then there exist $R_1 \in \mathcal{G}_0$, $\rho(t) > 0$, $x(t) \in \mathbb{R}^N$ and $\theta(t) \in [0, 2\pi)$, such that*

$$\rho^{N/2}(t)\psi(t, \rho(t)(\cdot + x(t)))e^{i\theta(t)} \to R_1 \text{ strongly in } H^2, \text{ as } t \to T^*. \tag{30}$$

Proof. We use the notations in the proof of Theorem 2. Assume that $\|\psi_0\|_{L^2} = G_0$. Recall that we have verified that $\|V\|_{L^2} \geq G_0$ in the proof of Theorem 2. Whence

$$G_0 \leq \|V\|_{L^2} \leq \liminf_{n\to\infty} \|v_n\|_{L^2} = \liminf_{n\to\infty} \|\psi(t_n)\|_{L^2} = \|\psi_0\|_{L^2} = G_0,$$

and then,

$$\lim_{n\to\infty} \|v_n\|_{L^2} = \|V\|_{L^2} = G_0, \tag{31}$$

which implies

$$v_n(\cdot + x_n) \to V \text{ strongly in } L^2 \text{ as } n \to \infty.$$

We infer from the inequality (8) that

$$\|\nabla(v_n(\cdot + x_n) - V)\|_{L^2}^2 \leq C\|v_n(\cdot + x_n) - V\|_{L^2}\|\Delta(v_n(\cdot + x_n) - V)\|_{L^2}.$$

From $\|\Delta v_n(\cdot + x_n)\|_{L^2} \leq C$, we obtain

$$\nabla v_n(\cdot + x_n) \to \nabla V \text{ in } L^2 \text{ as } n \to \infty.$$

Next, we will prove that $v_n(\cdot + x_n)$ converges to V strongly in H^2. For this purpose, we estimate as follows:

$$\begin{aligned} 0 = \lim_{n\to\infty} |E_0(v_n)| &= \left|\frac{1}{2}\int_{\mathbb{R}^N} |\Delta R(x)|^2 dx - \frac{1}{p+2}\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n(x)|^{p+2}dx\right| \\ &= \left|\frac{1}{2}\int_{\mathbb{R}^N} |\Delta R(x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^N} |V(x)|^{p+2}dx\right|. \end{aligned} \tag{32}$$

Thus, we infer from the inequality (9) that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\Delta R(x)|^2 dx = \frac{1}{p+2}\int_{\mathbb{R}^N} |V(x)|^{p+2}dx \leq \frac{1}{2}\frac{\|V\|_{L^2}^p}{G_0^p}\|\Delta V\|_{L^2}^2 = \frac{1}{2}\|\Delta V\|_{L^2}^2. \tag{33}$$

On the other hand, we deduce from (25) that $\|\Delta V\|_{L^2} \leq \liminf_{n\to\infty} \|\Delta v_n(\cdot + x_n)\|_{L^2} = \|\Delta R\|_{L^2}$. Hence, we have $\|Q\|_{H^2} = \|V\|_{H^2}$ and

$$v_n(\cdot + x_n) \to V \text{ strongly in } H^2 \text{ as } n \to \infty. \tag{34}$$

This and (33) imply that

$$E_0(V) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta V(x)|^2 dx - \frac{1}{p+2}\int_{\mathbb{R}^N} |V(x)|^{p+2}dx = 0.$$

Up to now, we have verified that

$$\|V\|_{L^2} = G_0 \text{ and } E_0(V) = 0.$$

Applying Lemma 9, there exists $R_1 \in \mathcal{G}_0$, such that

$$V(x) = e^{i\theta} R_1(x + x_0) \; for\ some\ \theta \in [0, 2\pi),\ x_0 \in \mathbb{R}^N$$

and

$$\rho_n^{N/2} \psi(t_n, \rho_n(\cdot + x_0)) \to e^{i\theta} R_1(\cdot + x_0) \; strongly\ in\ H^2\ as\ n \to \infty.$$

Since the sequence $\{t_n\}_{n=1}^{\infty}$ is arbitrary, we infer that there are two functions $x(t) \in \mathbb{R}^N$ and $\theta(t) \in [0, 2\pi)$, such that

$$\rho^{N/2}(t) e^{i\theta(t)} \psi(t, \rho(t)(x + x(t))) \to R_1 \; strongly\ in\ H^2\ as\ t \to T^*.$$

□

3.3. The L^2-Supercritical Case

In this subsection, we investigate some dynamical properties of the blow-up solutions for (1) with $\psi_0 \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ in the L^2-supercritical case. The main difficulty in this consideration is the lack of conservation of mass.

Theorem 4. *Let $\mu \in \mathbb{R}$, $\frac{8}{N} < p < 4^*$, $\psi_0 \in \dot{H}^{\gamma} \cap \dot{H}^2$ with $\gamma = \min\{\gamma_c, 1\}$. If the solution $\psi(t)$ of (1) blows up in finite time $T^* > 0$ and satisfies*

$$\sup_{t \in [0, T^*)} \|\psi(t)\|_{\dot{H}^{\gamma_c}} < \infty \;\; if\ \gamma_c \le 1, \quad \sup_{t \in [0, T^*)} \|\psi(t)\|_{\dot{H}^{\gamma_c} \cap \dot{H}^1} < \infty \;\; if\ 1 < \gamma_c < 2. \tag{35}$$

Assume that $a(t) > 0$, such that

$$a(t) \|\Delta \psi(t)\|_{L^2}^{\frac{1}{2-\gamma_c}} \to \infty, \tag{36}$$

as $t \to T^*$. Then, there exist $x_1(t), x_2(t) \in \mathbb{R}^N$, such that

$$\liminf_{t \to T^*} \int_{|x - x_1(t)| \le a(t)} |(-\triangle)^{\frac{\gamma_c}{2}} \psi(t, x)|^2 dx \ge G_1^2, \tag{37}$$

and

$$\liminf_{t \to T^*} \int_{|x - x_2(t)| \le a(t)} |\psi(t, x)|^{p_c} dx \ge G_2^{p_c}. \tag{38}$$

Remark 4. *The assumption $\psi_0 \in \dot{H}^{\gamma} \cap \dot{H}^2$ with $\gamma = \min\{\gamma_c, 1\}$ guarantees that the energy $E(\psi)$ is well-defined.*

Proof. Let $Q \in \mathcal{G}$; we set

$$\rho(t) = \|\Delta Q\|_{L^2}^{\frac{1}{2-\gamma_c}} / \|\Delta \psi(t)\|_{L^2}^{\frac{1}{2-\gamma_c}} \;\; and \;\; v(t, x) = \rho^{\frac{4}{p}}(t) \psi(t, \rho(t) x).$$

Let $\{t_n\}_{n=1}^{\infty}$ be an any time sequence, such that $t_n \to T^*$, $\rho_n = \rho(t_n)$ and $v_n(x) = v(t_n, x)$. Then, it follows from assumption (35) that v_n satisfies $\|v_n\|_{\dot{H}^{\gamma_c}} = \|\psi(t_n)\|_{\dot{H}^{\gamma_c}} < \infty$ uniformly in n. Moreover, by some direct computations, we obtain

$$\|\Delta v_n\|_{L^2} = \rho_n^{2-\gamma_c} \|\Delta \psi(t_n)\|_{L^2} = \|\Delta Q\|_{L^2},$$

and

$$\begin{aligned}|E_0(v_n)| =&\left|\frac{1}{2}\int_{\mathbb{R}^N}|\Delta v_n(x)|^2dx-\frac{1}{p+2}\int_{\mathbb{R}^N}|v_n(x)|^{p+2}dx\right|\\ =&\rho_n^{2(2-\gamma_c)}\left|\frac{1}{2}\int_{\mathbb{R}^N}|\Delta\psi(t_n,x)|^2dx-\frac{1}{p+2}\int_{\mathbb{R}^N}|\psi(t_n,x)|^{p+2}dx\right|\\ =&\rho_n^{2(2-\gamma_c)}\left|E(\psi(t_n))-\frac{\mu}{2}\int_{\mathbb{R}^N}|\nabla\psi(t_n,x)|^2dx\right|\\ =&\frac{\|\Delta Q\|_{L^2}^2}{\|\Delta\psi(t_n)\|_{L^2}^2}\left|E(\psi_0)-\frac{\mu}{2}\int_{\mathbb{R}^N}|\nabla\psi(t_n,x)|^2dx\right|. \end{aligned}\tag{39}$$

When $0<\gamma_c<1$, applying the inequality (8), that is

$$\|\nabla\psi(t_n)\|_{L^2}^2\le\|\Delta\psi(t_n)\|_{L^2}^{\frac{2(1-\gamma_c)}{2-\gamma_c}}\|\psi(t_n)\|_{\dot H^{\gamma_c}}^{\frac{2}{2-\gamma_c}},\tag{40}$$

we have $E_0(v_n)\to0$ as $n\to\infty$. When $1\le\gamma_c<2$, it follows from (35) that $E_0(v_n)\to0$ as $n\to\infty$. These imply that $\|v_n\|_{L^{p+2}}^{p+2}\to\frac{p+2}{2}\|\Delta Q\|_{L^2}^2$ as $n\to\infty$.

Set $m^{p+2}=\frac{p+2}{2}\|\Delta Q\|_{L^2}^2$ and $M=\|\Delta Q\|_{L^2}$. Then, it follows from Lemma 8 that there exist $V\in\dot H^{\gamma_c}\cap\dot H^2$ and $\{x_n\}_{n=1}^{\infty}\subset\mathbb{R}^N$, such that up to a subsequence,

$$v_n(\cdot+x_n)=\rho_n^{\frac{4}{p}}\psi(t_n,\rho_n\cdot+x_n)\rightharpoonup V\ \ \textit{weakly in } \dot H^{\gamma_c}\cap\dot H^2$$

with

$$\|V\|_{\dot H^{\gamma_c}}\ge G_1.\tag{41}$$

By the definition of $\dot H^{\gamma_c}$, we have

$$(-\Delta)^{\frac{\gamma_c}{2}}\rho_n^{\frac{4}{p}}\psi(t_n,\rho_n\cdot+x_n)\rightharpoonup(-\Delta)^{\frac{\gamma_c}{2}}V\ \ \textit{weakly in } L^2.$$

Thus, for any $R>0$,

$$\int_{|x|\le R}|(-\Delta)^{\frac{\gamma_c}{2}}V(x)|^2dx\le\liminf_{n\to\infty}\int_{|x-x_n|\le\rho_nR}|(-\Delta)^{\frac{\gamma_c}{2}}\psi(t_n,x)|^2dx.$$

In view of the assumption $a(t_n)/\rho_n\to\infty$, this implies immediately

$$\int_{|x|\le R}|(-\Delta)^{\frac{\gamma_c}{2}}V|^2dx\le\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le a(t_n)}|(-\Delta)^{\frac{\gamma_c}{2}}\psi(t_n,x)|^2dx.$$

Then, we can prove this theorem by a similar argument as that in Theorem 3. The proof of (38) is similar, so we omit it. This completes the proof. □

Let us now study the limiting profile of the blow-up $\dot H^{\gamma_c}\cap\dot H^2$ solutions with critical norms. To do so, we recall the following characterization of the ground states.

Lemma 10 (Characterization of ground states [25]). *Let $\frac{8}{N}<p<4^*$.*

1. *If $u\in\dot H^{\gamma_c}\cap\dot H^2$ is such that $\|u\|_{\dot H^{\gamma_c}}=G_1$ and*

$$E_0(u):=\frac{1}{2}\|u\|_{\dot H^2}^2-\frac{1}{p+2}\|u\|_{L^{p+2}}^{p+2}=0,$$

then, u is of the form

$$u(x)=e^{i\theta}\lambda^{\frac{4}{p}}Q(\lambda x+x_0),$$

for some $Q\in\mathcal{G}$, $\theta\in\mathbb{R}^N$, $\lambda>0$ and $x_0\in\mathbb{R}^N$.

2. *If $u \in L^{p_c} \cap \dot{H}^2$ is such that $\|u\|_{L^{p_c}} = G_2$ and*

$$H(u) := \frac{1}{2}\|u\|^2_{\dot{H}^2} - \frac{1}{p+2}\|u\|^{p+2}_{L^{p+2}} = 0,$$

then, u is of the form

$$u(y) = e^{i\vartheta}\rho^{\frac{4}{p}} R(\rho y + y_0),$$

for some $R \in \mathcal{K}$, $\vartheta \in \mathbb{R}^N, \rho > 0$ and $y_0 \in \mathbb{R}^N$.

Proposition 1 (Limiting profile with critical norms)**.** *Let $\mu \in \mathbb{R}$, $\frac{8}{N} < p < 4^*$, $\psi_0 \in \dot{H}^\gamma \cap \dot{H}^2$ with $\gamma = \min\{\gamma_c, 1\}$, and the corresponding solution $\psi(t)$ of (1) blows up in the finite time $T^* > 0$.*

1. *Assume that*

$$\sup_{t\in[0,T^*)} \|\psi(t)\|_{\dot{H}^{\gamma_c}} = G_1. \tag{42}$$

If $1 < \gamma_c < 2$, assume further that $\sup_{t\in[0,T^*)} \|\psi(t)\|_{\dot{H}^1} < \infty$. Then, there exists $Q_1 \in \mathcal{G}$, $\theta(t) \in \mathbb{R}, \lambda(t) > 0$ and $y(t) \in \mathbb{R}^N$, such that

$$e^{i\theta(t)}\lambda^{\frac{4}{p}}(t)\psi(t, \lambda(t)\cdot + y(t)) \to Q_1 \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2,$$

as $t \uparrow T^*$.

2. *Assume that*

$$\sup_{t\in[0,T^*)} \|\psi(t)\|_{\dot{H}^{\gamma_c}} < \infty, \quad \sup_{t\in[0,T^*)} \|\psi(t)\|_{L^{p_c}} = G_2. \tag{43}$$

If $1 < \gamma_c < 2$, assume further that $\sup_{t\in[0,T^*)} \|\psi(t)\|_{\dot{H}^1} < \infty$. Then, there exist $Q_2 \in \mathcal{K}, \vartheta(t) \in \mathbb{R}, \rho(t) > 0$ and $z(t) \in \mathbb{R}^N$, such that

$$e^{i\vartheta(t)}\rho^{\frac{4}{p}}(t)\psi(t, \rho(t)\cdot + z(t)) \to Q_2 \text{ strongly in } L^{p_c} \cap \dot{H}^2,$$

as $t \uparrow T^*$.

Proof. We only treat the first term, the second one is similar. It is enough to show that for any $(t_n)_{n\geq 1}$ satisfying $t_n \uparrow T^*$, there exists a subsequence still denoted by $(t_n)_{n\geq 1}$, $Q_1 \in \mathcal{G}$, sequences $\theta_n \in \mathbb{R}, \lambda_n > 0$ and $y_n \in \mathbb{R}^N$, such that

$$e^{it\theta_n}\lambda_n^{\frac{4}{p}}\psi(t_n, \lambda_n \cdot + y_n) \to Q_1 \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2, \tag{44}$$

as $n \to \infty$. Using the notation given in the proof of Theorem 4, we have

$$v_n(\cdot + y_n) = \lambda_n^{\frac{4}{p}}\psi(t_n, \lambda_n \cdot + y_n) \rightharpoonup V \text{ weakly in } \dot{H}^{\gamma_c} \cap \dot{H}^2,$$

as $n \to \infty$ with $\|V\|_{\dot{H}^{\gamma_c}} \geq G_1$. By the semi-continuity of weak convergence, (41) and (42), we have

$$G_1 \leq \|V\|_{\dot{H}^{\gamma_c}} \leq \liminf_{n\to\infty} \|v_n\|_{\dot{H}^{\gamma_c}} = \liminf_{n\to\infty} \|\psi(t_n)\|_{\dot{H}^{\gamma_c}} \leq G_1.$$

We thus obtain

$$\lim_{n\to\infty} \|v_n\|_{\dot{H}^{\gamma_c}} = \|V\|_{\dot{H}^{\gamma_c}} = G_1. \tag{45}$$

This shows that $v_n(\cdot + y_n) \to V$ strongly in $\dot{H}^{\gamma_c}$ as $n \to \infty$. Using the sharp Gagliardo–Nirenberg inequality (12), we have

$$v_n(\cdot + y_n) \to V \text{ strongly in } L^{p+2},$$

as $n \to \infty$. Using (39) and (45), the sharp Gagliardo–Nirenberg inequality (12) yields

$$\|Q\|_{\dot{H}^2}^2 = \frac{2}{p+2} \lim_{n\to\infty} \|v_n\|_{L^{p+2}}^{p+2} = \frac{2}{p+2} \|V\|_{L^{p+2}}^{p+2} \le \left(\frac{\|V\|_{\dot{H}^{\gamma_c}}}{G_1} \right)^p \|V\|_{\dot{H}^2}^2 = \|V\|_{\dot{H}^2}^2.$$

This combined with

$$\|V\|_{\dot{H}^2} \le \liminf_{n\to\infty} \|v_n\|_{\dot{H}^2} = \|Q\|_{\dot{H}^2}$$

shows that

$$\lim_{n\to\infty} \|v_n\|_{\dot{H}^2} = \|V\|_{\dot{H}^2} = \|Q\|_{\dot{H}^2}. \tag{46}$$

Combining (45), (46) and the fact $v(\cdot + y_n) \rightharpoonup V$ weakly in $\dot{H}^{\gamma_c} \cap \dot{H}^2$, we conclude that

$$v_n(\cdot + y_n) \to V \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2,$$

as $n \to \infty$. In particular, we have

$$E_0(V) = \lim_{n\to\infty} E_0(v_n) = 0.$$

Therefore, we have proved that $V \in \dot{H}^{\gamma_c} \cap \dot{H}^2$ and satisfies

$$\|V\|_{\dot{H}^{\gamma_c}} = G_1, \quad E_0(V) = 0.$$

Applying Lemma 10, there exists $Q_1 \in \mathcal{G}$, such that $V(y) = e^{i\theta} \lambda^{\frac{4}{p}} Q_1(\lambda y + y_0)$ for some $\theta \in \mathbb{R}$, $\lambda > 0$ and $y_0 \in \mathbb{R}^N$. We thus obtain

$$v_n(\cdot + y_n) = \lambda_n^{\frac{4}{p}} \psi(t_n, \lambda_n \cdot + y_n) \to V = e^{i\theta} \lambda^{\frac{4}{p}} Q_1(\lambda \cdot + y_0) \text{ strongly in } \dot{H}^{\gamma_c} \cap \dot{H}^2,$$

as $n \to \infty$. Redefining variables, we prove (44). The proof is complete. □

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