axioms

Article

Nontrivial Solutions for a Class of Quasilinear Schrödinger Systems

Xue Zhang [1] **and Jing Zhang** [1,2,3,*]

1 College of Mathematics Science, Inner Mongolia Normal University, Hohhot 010011, China; 20214015009@mails.imnu.edu.cn
2 Key Laboratory of Infinite-Dimensional Hamiltonian System and Its Algorithm Application, Ministry of Education, Inner Mongolia Normal University, Hohhot 010011, China
3 Center for Applied Mathematics Inner Mongolia, Inner Mongolia Normal University, Hohhot 010011, China
* Correspondence: zhangjing@imnu.edu.cn

Abstract: In this thesis, we research quasilinear Schrödinger system as follows in which $3 < N \in \mathbb{R}$, $2 < p < N, 2 < q < N$, $V_1(x), V_2(x)$ are continuous functions, k, ι are parameters with $k, \iota > 0$, and nonlinear terms $f, h \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. We find a nontrivial solution (u, v) for all $\iota > \iota_1(k)$ by means of the mountain-pass theorem and change of variable theorem. Our main novelty of the thesis is that we extend Δ to Δ_p and Δ_q to find the existence of a nontrivial solution.

Keywords: change of variable; nontrivial solution; mountain-pass theorem

MSC: 35A01; 35J10; 35J50

Citation: Zhang, X.; Zhang, J. Nontrivial Solutions for a Class of Quasilinear Schrödinger Systems. *Axioms* **2024**, *13*, 182. https://doi.org/10.3390/axioms13030182

Academic Editor: Hsien-Chung Wu

Received: 4 February 2024
Revised: 4 March 2024
Accepted: 5 March 2024
Published: 11 March 2024

1. Introduction

We concerned the following quasilinear Schrödinger system for this paper

$$\begin{cases} -\Delta_p u + V_1(x)|u|^{p-2}u + \frac{k}{2}[\Delta_p |u|^2]u = \iota f(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta_q v + V_2(x)|v|^{q-2}v + \frac{k}{2}[\Delta_q |v|^2]v = \iota h(x,u,v), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

in which $3 < N \in \mathbb{R}$, $2 < p < N$, $2 < q < N$, $V_1(x), V_2(x)$ are continuous positive functions, k is a sufficiently large positive parameter, ι is a positive parameter, and $f, h \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$.

For a quasilinear Schrödinger system (1), by the symmetric mountain-pass theorem, ref. [1] found infinite solutions, for given nonlinear terms f, h. When $k = -2$, ref. [2] proved that it had nontrivial solutions.

The above quasilinear Schrödinger system for $p = q = 2$ is inspired by the quasilinear Schrödinger equation as below

$$i\epsilon\partial z = -\epsilon\Delta z + V(x)z - k(|z|^2)z - l\epsilon\Delta h(|z|^2)h'(|z|^2)z, \quad \text{for } x \in \mathbb{R}^N, \tag{2}$$

in which $V(x)$ is fixed potential, l is a constant, and k and h are real functions. In [3–5], Equation (2) is used to study several physical phenomenon with different h.

For $h(t) = t, l(t) = \mu t^{\frac{p-1}{2}}$ and $k > 0$, let $z(s,x) = exp(-iFs)u(x)$, an equivalent elliptic equation with variational structure is obtained

$$-\epsilon\Delta u + E(x)u - \epsilon k(\Delta(|u|^2))u = \mu|u|^{p-1}u, \quad u > 0 \ x \in \mathbb{R}^N, N > 2, \tag{3}$$

in which $E(x) = V(x) - F$ is also the potential function. There is a lot of research for problems similar to problem (3). Ref. [6] studied a problem that had multiple solutions by dual-approach techniques and variational methods when $k > 0$ is small enough. Ref. [7] used a minimization argument established on the ground states of soliton solutions. The symmetric critical principle and the mountain-pass theorem were used for finding solutions in [8].

In [9], for a type of quasilinear Schrödinger equation like (3), the author used the method developed by [10,11] to study ground state solutions. In addition, refs. [12–14] have also conducted research on equations of this type.

There is a large amount of research on system (1) for $p = q = 2$. In [15], Pohožaev manifold and Moser iteration were used for obtaining a ground state solution. By a suitable Nehari–Pohožaev-type constraint set and analyzing relational minimization issues, Wang and Huang found the ground state solutions for the same class system in [16]. In the Orlicz space, the concentration compactness principle and Nehari manifold method were used for finding a ground state solution in [17]. Ref. [18] used the monotonicity trick and the Moser iteration to obtain the result of positive solutions. In [19], Chen and Zhang found ground state solutions through minimization principle. By applying innovative application of variable transformation and the mountain-pass theorem, ref. [20] proved that quasilinear Schrödinger systems have a nontrivial solution.

Many papers mention replacing Δ with Δ_p to study the properties of the equation or system after changes, such as [1,13,21]. In fact, Δ is a special case of Δ_p, that is $\Delta_p = \Delta$ if $p = 2$. What we are interested in is the nontrivial solution to system (1) when $k > 0$ is large enough.

Throughout this paper, we need some assumptions. Firstly, we make $V_1(x)$, $V_2(x) \in C(\mathbb{R}^N, \mathbb{R})$ and encounter the ensuing properties.

$(\mathcal{V}_1)$ $\tilde{V} := \min\{\inf_{x\in\mathbb{R}^N} V_1(x), \inf_{x\in\mathbb{R}^N} V_2(x)\}$ and $0 < \tilde{V}$;

$(\mathcal{V}_2)$ $\exists V_0 > 0, \forall V \geq V_0, m\big(\{x \in \mathbb{R}^N : V_i(X) \leq V\}\big)$ is bounded, where $i = 1, 2$, m is defined as the Lebesgue measure in $\mathbb{R}^N$.

Meanwhile, assume that the terms f, h conform to the properties as follows:

(f_1) $\frac{f(x,s,t)}{|(s,t)|} \to 0$, $\frac{h(x,s,t)}{|(s,t)|} \to 0$, as $(s,t) \to (0,0)$;

(f_2) $\exists C_0 > 0$, which makes $\langle \nabla\zeta(x,s,t),(s,t)\rangle \leq C_0\Big(|(s,t)|^{(p-1,q-1)} + |(s,t)|^{(l_p-1,l_q-1)}\Big)$, $\forall s,t \in \mathbb{R}$, $p < l_p < p^*$, $q < l_q < q^*$, where $|(s,t)|^{(l_p,l_q)} = s^{l_p} + t^{l_q}$, $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$;

(f_3) $\exists\theta > 0$ satisfying $0 < \theta\zeta(x,s,t) \leq (s,t)\nabla\zeta(x,s,t)$, and $uf(x,u,v) \geq 0$, $vh(x,u,v) \geq 0$, in which $\nabla\zeta(x,s,t) = \Big(f(x,s,t), h(x,s,t)\Big)$.

The paper's core result is given below.

Theorem 1. *For given $k > 0$, there is $\iota_1(k) > 0$ for all $\iota > \iota_1(k)$, when $(\mathcal{V}_1)$, $(\mathcal{V}_2)$, and $(h_1) - (h_3)$ are true, in that system (1) has a nontrivial solution $(u,v) \in H$ and* $\max_{x\in\mathbb{R}^N} |(u(x),v(x))| \leq \Big(\big(\frac{1}{2^{p-3}k}\big)^{\frac{1}{p}}, \big(\frac{1}{2^{q-3}k}\big)^{\frac{1}{q}}\Big)$.

Let me introduce the basic framework of this paper. Preparation work was completed in Section 2. In Section 3, we consider issues related to the solution of the modified system. We acquire the solution for the first system (1) by use of the Morse iteration technique in Section 4. Section 5 makes a conclusion.

In this article, we use C to denote dissimilar positive constants, and B_R0 stands for a ball with its radius $R > 0$ and center at the origin. The operation $(x_1,x_2)(y_1,y_2) = x_1y_1 + x_2y_2$, and the operation $(x_1,x_2)^{(y_1,y_2)} = x_1^{y_1} + x_2^{y_2}$.

2. Preliminary Work

The corresponding Euler–Lagrange functional for (1) is as follows:

$$\begin{aligned} J_k(u,v) = &\frac{1}{p}\int_{\mathbb{R}^N}(1 - k2^{p-2}|u|^p)|\nabla u|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N} V_1(x)|u|^p\,dx \\ &+ \frac{1}{q}\int_{\mathbb{R}^N}(1 - k2^{q-2}|v|^q)|\nabla v|^q\,dx + \frac{1}{q}\int_{\mathbb{R}^N} V_2(x)|v|^q\,dx - \iota\int_{\mathbb{R}^N}\zeta(x,u,v)\,dx. \end{aligned}$$

The functional J_k has quasilinear terms, and it is difficult to consider the critical points in the Sobolev spaces.

We stipulate that $D = D_1 \times D_2$ and

$$\|(u,v)\| = \|u\| + \|v\|,$$

in which

$$D_1 = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_1(x)|u|^p\,dx < +\infty\}$$

given the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^p + V_1(x)|u|^p)\,dx\Big)^{\frac{1}{p}}$$

and

$$D_2 = \{v \in W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_2(x)|v|^q\,dx < +\infty\}$$

given the norm

$$\|v\| = \Big(\int_{\mathbb{R}^N} (|\nabla v|^q + V_2(x)|v|^q)\,dx\Big)^{\frac{1}{q}},$$

$W^{1,p}(\mathbb{R}^N)$ and $W^{1,q}(\mathbb{R}^N)$ are the Sobolev space.

To make (u,v) a solution for (1), if $\forall \varphi, \psi \in C_0^\infty(\mathbb{R}^N)$, $(u,v) \in H$ satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big((1 - 2^{(p-2)}k|u|^p)|\nabla u|^{p-1}\nabla\varphi - 2^{(p-2)}k|\nabla u|^p|u|^{p-1}\varphi\Big)\,dx + \int_{\mathbb{R}^N} V_1(x)|u|^{p-2}u\varphi\,dx \\
&+ \int_{\mathbb{R}^N} \Big((1 - 2^{(q-2)}k|v|^q)|\nabla v|^{q-1}\nabla\psi - 2^{(q-2)}k|\nabla v|^q|v|^{q-1}\psi\Big)\,dx + \int_{\mathbb{R}^N} V_2(x)|v|^{q-2}v\psi\,dx \\
&= \iota \int_{\mathbb{R}^N} \Big(f(x,u,v)\varphi + h(x,u,v)\psi\Big)\,dx.
\end{aligned} \tag{4}$$

Let $1 - 2^{p-2}k|u|^p > 0$, we define the functions as follows:

$$y_u(u) = \begin{cases} y_u(-u), \quad t < 0, \\ (1 - 2^{(p-2)}k|u|^p)^{\frac{1}{p}}, \quad 0 \le u < (\frac{1}{2^{(p-3)}k})^{\frac{1}{p}}, \\ \frac{1}{2^{(\frac{1}{p}+p-2)}ku^p} + 2^{(-\frac{1}{p}-1)}, \quad u \ge (\frac{1}{2^{(p-3)}k})^{\frac{1}{p}}, \end{cases}$$

and

$$y_v(v) = \begin{cases} y_v(-v), \quad t < 0, \\ (1 - 2^{(q-2)}k|v|^q)^{\frac{1}{q}}, \quad 0 \le v < (\frac{1}{2^{(q-3)}k})^{\frac{1}{q}}, \\ \frac{1}{2^{(\frac{1}{q}+q-2)}kv^q} + 2^{(-\frac{1}{q}-1)}, \quad v \ge (\frac{1}{2^{(q-3)}k})^{\frac{1}{q}}. \end{cases}$$

Then, $y_i(i) \in C^1(\mathbb{R}, (2^{(-\frac{1}{q}-1)}, 1])$, $i = u, v$, y_i is even and a convex function.

Affected by [22], we handle the following modified quasilinear Schrödinger system,

$$\begin{cases} -div(y_u^p(u)|\nabla u|^{p-1}) + y_u^{p-1}(u)y_u'(u)|\nabla u|^p + V_1(x)|u|^{p-1} = \iota f(x,u,v), \quad x \in \mathbb{R}^N, \\ -div(y_v^q(v)|\nabla v|^{q-1}) + y_v^{q-1}(v)y_v'(v)|\nabla v|^q + V_2(x)|v|^{q-1} = \iota h(x,u,v), \quad x \in \mathbb{R}^N. \end{cases} \tag{5}$$

Clearly, $\forall \varphi, \psi \in C_0^\infty(\mathbb{R}^N)$ and $(u,v) \in H$, (u,v) is a weak solution for (5), if it holds

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(y_u^p(u)|\nabla u|^{p-1}\nabla\varphi + y_u^{p-1}(u)y_u'(u)|\nabla u|^p\varphi + V_1(x)|u|^{p-1}\varphi\Big)\,dx \\
&+ \int_{\mathbb{R}^N} \Big(y_v^q(v)|\nabla v|^{q-1}\nabla\psi + y_v^{q-1}(v)y_v'(v)|\nabla v|^q\psi + V_2(x)|v|^{q-1}\psi\Big)\,dx \\
&= \iota \int_{\mathbb{R}^N} \Big(f(x,u,v)\varphi + h(x,u,v)\psi\Big)\,dx.
\end{aligned} \tag{6}$$

Obviously, if $\|(u,v)\|_\infty \le \left(\left(\frac{1}{2^{(p-3)}k}\right)^{\frac{1}{p}}, \left(\frac{1}{2^{(q-3)}k}\right)^{\frac{1}{q}} \right)$ and (u,v) is a solution for (5), so this particular solution (u,v) also satisfies system (1). Utilize the change of variable as follows:

$$z = Y(u) = \int_0^u y_u(t)\,dt, \quad w = Y(w) = \int_0^w y_v(t)\,dt;$$

then, the issue (5) can be simplified as:

$$\begin{cases} -\Delta_p z + \frac{V_1(x)(Y^{-1}(z))^{p-1}}{y_u(Y^{-1}(z))} = \iota \frac{f(x,Y^{-1}(z),Y^{-1}(w))}{Y^{-1}(z)}, & x \in \mathbb{R}^N, \\ -\Delta_q w + \frac{V_2(x)(Y^{-1}(w))^{q-1}}{y_v(Y^{-1}(w))} = \iota \frac{h(x,Y^{-1}(z),Y^{-1}(w))}{Y^{-1}(w)}, & x \in \mathbb{R}^N, \end{cases} \tag{7}$$

among them Y^{-1} and Y are inverse functions of each other, respectively. The corresponding function about (7) is

$$\begin{aligned} I_k(z,w) = &\frac{1}{p}\int_{\mathbb{R}^N} |\nabla z|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N} V_1(x)|Y^{-1}(z)|^p\,dx \\ &+ \frac{1}{q}\int_{\mathbb{R}^N} |\nabla w|^q\,dx + \frac{1}{q}\int_{\mathbb{R}^N} V_2(x)|Y^{-1}(w)|^q\,dx - \iota\int_{\mathbb{R}^N} \zeta(x,Y^{-1}(z),Y^{-1}(w))\,dx. \end{aligned} \tag{8}$$

Obviously, I_k has a good definition in D, and we can obtain the following lemma that are similar to [23].

Lemma 1. *The functions $y_u(u), y_v(v), Y(u), Y(v), Y^{-1}(z), Y^{-1}(w)$ satisfy these conditions as follows:*

(i) *$Y(i)$ and its inverse function $Y^{-1}(z), Y^{-1}(w)$ are odd, where $i = u, v$;*

(ii) $-1 \le \frac{u}{y_u(u)} y_u'(u) \le 0$, $-1 \le \frac{v}{y_v(v)} y_v'(v) \le 0$ *for all $u, v \in \mathbb{R}$;*

(iii) $|z| \le |Y^{-1}(z)| \le 2^{(-\frac{1}{p}-1)}|z|$, $|w| \le |Y^{-1}(w)| \le 2^{(-\frac{1}{q}-1)}|w|$ *for every $z, w \in \mathbb{R}$;*

(iv) $\lim_{i\to 0} \frac{Y^{-1}(i)}{i} = 1$, $\lim_{z\to\infty} \frac{Y^{-1}(z)}{z} = 2^{(-\frac{1}{p}-1)}$, $\lim_{w\to\infty} \frac{Y^{-1}(w)}{w} = 2^{(-\frac{1}{q}-1)}$, *where $i = z, w$;*

(v) $y_u(Y^{-1}(z)) \le \frac{z}{Y^{-1}(z)}$, $y_v(Y^{-1}(w)) \le \frac{w}{Y^{-1}(w)}$ *for all $z, w \in \mathbb{R}$.*

Proof. Clearly, (i) is established. The definition of y_u and y_v include

$$\lim_{z\to 0} \frac{Y^{-1}(z)}{z} = \lim_{z\to 0} \frac{1}{y_u(Y^{-1}(z))} = \frac{1}{y(0)} = 1,$$

$$\lim_{w\to 0} \frac{Y^{-1}(w)}{w} = \lim_{w\to 0} \frac{1}{y_v(Y^{-1}(w))} = \frac{1}{y(0)} = 1,$$

$$\lim_{z\to\infty} \frac{Y^{-1}(z)}{z} = \lim_{i\to\infty} \frac{1}{y_u(Y^{-1}(z))} = 2^{(-\frac{1}{p}-1)},$$

$$\lim_{w\to\infty} \frac{Y^{-1}(w)}{w} = \lim_{i\to\infty} \frac{1}{y_v(Y^{-1}(w))} = 2^{(-\frac{1}{q}-1)}.$$

Thus, (iv) is proven. Since y_u, y_v are decreasing in $|u|, |v|$, we obtain

$$Y(u) \ge u y_u(u) \ge 0, \quad u \ge 0 \text{ and } Y(u) \le u y_u(u) < 0, \quad u < 0,$$

$$Y(v) \ge v y_v(v) \ge 0, \quad v \ge 0 \text{ and } Y(v) \le v y_v(v) < 0, \quad v < 0,$$

and (v) has also been proven. From

$$\frac{d}{dz}\left[\frac{Y^{-1}(z)}{z}\right] = \frac{z - Y^{-1}(z) y_u(Y^{-1}(z))}{y_u(Y^{-1}(z)) z^2} \begin{cases} \ge 0, & z \ge 0, \\ < 0, & z < 0, \end{cases}$$

$$\frac{d}{dw}\Big[\frac{Y^{-1}(w)}{w}\Big] = \frac{w - Y^{-1}(w)y_v(Y^{-1}(w))}{y_v(Y^{-1}(w))w^2}\begin{cases} \geq 0, & w \geq 0, \\ < 0, & w < 0, \end{cases}$$

and (iv), we obtain (iii). Next, we prove (ii). We consider $u, v \geq 0$. $\frac{u}{y_u(u)}y'_u(u) \leq 0$ and $\frac{v}{y_v(v)}y'_v(v) \leq 0$ are clear. For $0 \leq u < (\frac{1}{2^{(p-3)}k})^{\frac{1}{p}}$ and $0 \leq v < (\frac{1}{2^{(q-3)}k})^{\frac{1}{q}}$, we have

$$\frac{u}{y_u(u)}y'_u(u) = \frac{-2^{(p-2)}k}{u^{-p} - 2^{(p-2)}k} \geq -1,$$

and

$$\frac{v}{y_v(v)}y'_v(v) = \frac{-2^{(q-2)}k}{v^{-q} - 2^{(q-2)}k} \geq -1.$$

For $u \geq (\frac{1}{2^{(p-3)}k})^{\frac{1}{p}}$ and $v \geq (\frac{1}{2^{(q-3)}k})^{\frac{1}{q}}$, we have

$$\frac{u}{y_u(u)}y'_u(u) = \frac{-2^{(-\frac{1}{p}-p+2)}k^{-1}p}{2^{(2-p-\frac{1}{p})}k^{-1} + 2^{(-\frac{1}{p}-1)}u^p} \geq -1,$$

and

$$\frac{v}{y_v(v)}y'_v(v) = \frac{-2^{(-\frac{1}{q}-q+2)}k^{-1}q}{2^{(2-q-\frac{1}{q})}k^{-1} + 2^{(-\frac{1}{q}-1)}v^q} \geq -1.$$

When $u, v < 0$, the proof method is similar to this. □

Lemma 2. *Let $(\mathcal{V}_1), (\mathcal{V}_2)$ and $(f_1) - (f_3)$ be true. To make $(u, v) = (Y^{-1}(z), Y^{-1}(w))$ a solution for (5), it is required that $(z, w) \in D$ is a critical point of J_k.*

Proof. Because $(z, w) \in D$ is a critical point of I_k, $\forall(\varphi, \psi) \in D$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z\nabla\varphi\, dx + \int_{\mathbb{R}^N} V_1(x)\frac{|Y^{-1}(z)|^{p-1}}{y_u(Y^{-1}(z))}\varphi\, dx \\ &+ \int_{\mathbb{R}^N} |\nabla w|^{q-2}\nabla w\nabla\psi\, dx + \int_{\mathbb{R}^N} V_2(x)\frac{|Y^{-1}(w)|^{q-1}}{y_v(Y^{-1}(w))}\psi\, dx \\ &= \iota\int_{\mathbb{R}^N}\Big(\frac{f(x, Y^{-1}(z), Y^{-1}(w))}{y_u(Y^{-1}(z))}\varphi + \frac{h(x, Y^{-1}(z), Y^{-1}(w))}{y_v(Y^{-1}(w))}\psi\Big)\, dx. \end{aligned} \tag{9}$$

By Lemma 1, we know $(u, v) := (Y^{-1}(z), Y^{-1}(w)) \in D$. Arbitrary to $\varphi_0, \psi_0 \in C_0^\infty(\mathbb{R}^N)$, let $(\varphi, \psi) := (y_u(u)\varphi_0, y_v(v)\psi_0) \in D$ in (9) simplify to

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z|^{p-2}\nabla z(y'_u(u)\varphi_0\nabla u + y_u(u)\nabla\varphi_0)\, dx + \int_{\mathbb{R}^N} V_1(x)\frac{|u|^{p-1}}{y_u(u)}g_u(u)\varphi_0\, dx \\ &+ \int_{\mathbb{R}^N} |\nabla w|^{q-2}\nabla w(y'_v(v)\psi_0\nabla v + y_v(v)\nabla\psi_0)\, dx + \int_{\mathbb{R}^N} V_2(x)\frac{|v|^{q-1}}{y_v(v)}y_v(v)\psi_0\, dx \\ &= \iota\int_{\mathbb{R}^N}\Big(\frac{f(x, u, v)}{y_u(u)}y_u(u)\varphi_0 + \frac{h(x, u, v)}{y_v(v)}y_v(v)\psi_0\Big)\, dx. \end{aligned}$$

Applying the fact that $z = Y(u)$, $w = Y(v)$, $\nabla z = y_u(u)\nabla u$ and $\nabla w = y_v(v)\nabla v$, after calculation, obtaining (6), thus, (u, v) is a weak solution of (5). □

Lemma 3. *Make $(\mathcal{V}_1), (\mathcal{V}_2)$ real. In D_1, D_2, $\{z_n\}, \{w_n\}$ are bounded, then, there is $z \in D_1 \cap L^{r_1}$ and $w \in D_2 \cap L^{r_2}$, up to a subsequence, $z_n \to z$ in L^{r_1}, $r_1 \in [p, p^*)$, $w_n \to w$ in L^{r_2}, $r_2 \in [q, q^*)$.*

Proof. The proof process is as shown in reference [1]. □

3. The Solution of the Modified System

Now, we study the modified system (5) and find its solution.

Lemma 4. *If $(f_1) - (f_3)$ are accurate, in that way*

(i) *there are $\rho, \pi > 0$ makes $I_k(z,w) \geq \pi$ valid for every (z,w) with $\|(z,w)\| = \rho$;*
(ii) *the existence of $(z,w) \in D \setminus \{(0,0)\}$ makes $I_k(z,w) \leq 0$ vaild.*

Proof. (i) By (f_2), $\forall \epsilon > 0$, $\exists C_0 > 0$ settle for

$$\langle \nabla \zeta(x,s,t),(s,t)\rangle \leq \epsilon|(s,t)|^{(p-1,q-1)} + C_0|(s,t)|^{(l_p-1,l_q-1)}, \tag{10}$$

where $p < l_p < p^*$ and $q < l_q < q^*$. Then,

$$\zeta(x,s,t) \leq \epsilon(\frac{1}{p},\frac{1}{q})|(s,t)|^{(p,q)} + C(\frac{1}{l_p},\frac{1}{l_q})|(s,t)|^{(l_p,l_q)}. \tag{11}$$

Let $\epsilon = \min\{\frac{2^{\frac{1}{p}}V_1(x)}{\iota}, \frac{2^{\frac{1}{q}}V_2(x)}{\iota}\}$, by Sobolev inequality, the Lemma $1(iii)$ and (11), assuming that $\frac{1}{2p}\|z\|^p > \frac{1}{2q}\|w\|^q$, $\frac{2^{(-\frac{l_p}{p}-l_p)}C^{l_p}}{l_p}\|z\|^{l_p} > \frac{2^{(-\frac{l_q}{q}-l_q)}C^{l_q}}{l_q}\|w\|^{l_q}$, we have

$$\begin{aligned}
I_k(z,w) \geq{}& \frac{1}{p}\int_{\mathbb{R}^N}|\nabla z|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N}V_1(x)|z|^p\,dx \\
&+ \frac{1}{q}\int_{\mathbb{R}^N}|\nabla w|^q\,dx + \frac{1}{q}\int_{\mathbb{R}^N}V_2(x)|w|^q\,dx \\
&- \iota\int_{\mathbb{R}^N}\Big(\epsilon(\frac{1}{p},\frac{1}{q})|(\Upsilon^{-1}(z),\Upsilon^{-1}(w))|^{(p,q)} + C(\frac{1}{l_p},\frac{1}{l_q})|(\Upsilon^{-1}(z),\Upsilon^{-1}(w))|^{(l_p,l_q)}\Big)\,dx \\
\geq{}& \frac{1}{p}\int_{\mathbb{R}^N}|\nabla z|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N}V_1(x)|z|^p\,dx \\
&+ \frac{1}{q}\int_{\mathbb{R}^N}|\nabla w|^q\,dx + \frac{1}{q}\int_{\mathbb{R}^N}V_2(x)|w|^q\,dx \\
&- \iota\int_{\mathbb{R}^N}\Big(\frac{\epsilon}{p}2^{(-1-p)}|z|^p + \frac{\epsilon}{q}2^{(-1-q)}|w|^q + \frac{C}{l_p}2^{(-\frac{l_p}{p}-l_p)}|z|^{l_p} + \frac{C}{l_q}2^{(-\frac{l_q}{q}-l_q)}|w|^{l_q}\Big)\,dx.
\end{aligned}$$

For $\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q$, when $\|z\| \geq 1, \|w\| \geq 1, p > q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2p}\frac{1}{2^q}\|(z,w)\|^q,$$

when $\|z\| \geq 1, \|w\| \geq 1, p < q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2q}\frac{1}{2^p}\|(z,w)\|^p,$$

when $\|z\| \geq 1, \|w\| < 1, p > q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2p}\frac{1}{2^q}\|(z,w)\|^q,$$

when $\|z\| \geq 1, \|w\| < 1, 2 < p < q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2p}\|z\|^p \geq \frac{1}{2p},$$

when $\|z\| < 1, \|w\| \geq 1, p > q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2q}\|w\|^q \geq \frac{1}{2q},$$

when $\|z\| < 1, \|w\| \geq 1, p < q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2q}\frac{1}{2^q}\|(z,w)\|^q,$$

when $\|z\| < 1, \|w\| < 1, p > q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2p}\frac{1}{2^q}\|(z,w)\|^q,$$

when $\|z\| < 1, \|w\| < 1, p < q$,

$$\frac{1}{2p}\|z\|^p + \frac{1}{2q}\|w\|^q \geq \frac{1}{2q}\frac{1}{2^p}\|(z,w)\|^p.$$

For $\|z\|^{l_p} + \|w\|^{l_q}$, when $\|z\| \geq 1, \|w\| \geq 1, l_p > l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|(z,w)\|^{l_p},$$

when $\|z\| \geq 1, \|w\| \geq 1, l_p < l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|(z,w)\|^{l_q},$$

when $\|z\| \geq 1, \|w\| < 1, l_p > l_q > q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|z\|^{l_p} + \|w\| \leq \|(z,w)\|^{l_p} + \|(z,w)\|,$$

when $\|z\| \geq 1, \|w\| < 1, l_p < l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|(z,w)\|^{l_q},$$

when $\|z\| < 1, \|w\| \geq 1, p < l_p < l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|z\| + \|w\|^{l_q} \leq \|(z,w)\|^{l_q} + \|(z,w)\|,$$

when $\|z\| < 1, \|w\| \geq 1, l_p > l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|(z,w)\|^{l_p},$$

when $\|z\| < 1, \|w\| < 1, l_p < l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|(z,w)\|^{l_p},$$

when $\|z\| < 1, \|w\| < 1, l_p > l_q$,

$$\|z\|^{l_p} + \|w\|^{l_q} \leq \|(z,w)\|^{l_q}.$$

Hence,

$$I_k(z,w) \geq \min\{\frac{1}{2p}\frac{1}{2^q}\|(z,w)\|^q, \frac{1}{2p}\frac{1}{2^p}\|(z,w)\|^p, \frac{1}{2q}\frac{1}{2^p}\|(z,w)\|^p, \frac{1}{2q}\frac{1}{2^q}\|(z,w)\|^q, \frac{1}{2p}, \frac{1}{2q}\}$$
$$- \max\{\|(z,w)\|^{l_p} l_s + \|(z,w)\| l_s, \|(z,w)\|^{l_q} l_s + \|(z,w)\| l_s\},$$

where $l_s := \max\{\iota\frac{C}{l_p}2^{(-\frac{l_p}{p}-l_p)}C^{l_p}, \iota\frac{C}{l_q}2^{(-\frac{l_q}{q}-l_q)}C^{l_q}\}$. Take $\|(z,w)\| = \rho$ small enough to satisfy

$$I_{k,\iota}(z,w) \geq \pi := \min\{\frac{1}{2p}\frac{1}{2^q}\rho^q, \frac{1}{2p}\frac{1}{2^p}\rho^p, \frac{1}{2q}\frac{1}{2^p}\rho^p, \frac{1}{2q}\frac{1}{2^q}\rho^q, \frac{1}{2p}, \frac{1}{2q}\} - \max\{\rho^{l_p}l_s + \rho l_s, \rho^{l_q}l_s + \rho l_s\}.$$

(ii) Choose $(\tau_1, \tau_2) \in D$ with $\tau_1, \tau_2 > 0$, from Lemma 1(iii), we obtain

$$|\tau_1|^p \leq \frac{|\Upsilon^{-1}(t\tau_1)|^p}{t^p} \leq 2^{(-p-1)}|\tau_1|^p, |\tau_2|^q \leq \frac{|\Upsilon^{-1}(t\tau_2)|^q}{t^q} \leq 2^{(-q-1)}|\tau_2|^q.$$

By (f_3), we know $\lim_{|(s_1,s_2)|\to+\infty} \frac{\zeta(x,s_1,s_2)}{|(s_1,s_2)|^p} = +\infty$. Therefore, for $p > q$, we have

$$\begin{aligned}\frac{I_k(t\tau_1, t\tau_2)}{t^p} \leq & \frac{1}{p}\int_{\mathbb{R}^N} |\nabla\tau_1|^p\,dx + \frac{2^{(-p-1)}}{p}\int_{\mathbb{R}^N} V_1(x)|\tau_1|^p\,dx \\ & + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla\tau_2|^q\,dx + \frac{2^{(-q-1)}}{q}\int_{\mathbb{R}^N} V_2(x)|\tau_2|^q\,dx \\ & - \iota\int_{\mathbb{R}^N} \frac{\zeta(x, \Upsilon^{-1}(t\tau_1), \Upsilon^{-1}(t\tau_2))}{(\Upsilon^{-1}(t\tau_1), \Upsilon^{-1}(t\tau_2))^p}\frac{(\Upsilon^{-1}(t\tau_1), \Upsilon^{-1}(t\tau_2))^p}{t^p}\,dx \to -\infty, \quad \text{as } t \to +\infty.\end{aligned} \tag{12}$$

In the same way, for $p < q$, as $t \to +\infty$, we have

$$\frac{I_k(t\tau_1, t\tau_2)}{t^q} \to -\infty.$$

Hence, $\exists t_0 > 0$ large enough, such that $(z,w) = (t_0\tau_1, t_0\tau_2)$ with $I_k(z,w) \leq 0$. □

To sum up, the $(PS)_c$ sequence exists and is denoted as $(z_n, w_n) \subset D$, therefore, as $n \to \infty$, we obtain

$$I_k(z_n, w_n) \to c, \quad I_k'(z_n, w_n) \to 0 \tag{13}$$

and

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_k(z_t, w_t),$$

$$\Gamma = \{(z_t, w_t) \in C([0,1]\times[0,1], D) : (z_0, w_0) = (0,0), (z_1, w_1) \neq (0,0), I_k(z_1, w_1) < 0\}.$$

Lemma 5. *If (f_3) are accurate, in that way for all $(PS)_c$ sequence (z_n, w_n) is bounded in D.*

Proof. For $p < q$, combining (13) and Lemma 1(ii), (iii) with (f_3), there is

$$
\begin{aligned}
& c+1+o_n(1)\|(z_n,w_n)\| \\
& \geq I_k(z_n,w_n)-\frac{1}{\theta}\langle I_k'(z_n,w_n),(Y^{-1}(z_n)y_u(Y^{-1}(z_n)),Y^{-1}(w_n)y_v(Y^{-1}(w_n)))\rangle \\
& =(\frac{1}{p}-\frac{1}{\theta})\int_{\mathbb{R}^N}|\nabla z_n|^p\,dx-\frac{1}{\theta}\int_{\mathbb{R}^N}|\nabla z_n|^p\frac{Y^{-1}(z_n)}{y_u(Y^{-1}(z_n))}y_u'(Y^{-1}(z_n))\,dx \\
& +(\frac{1}{p}-\frac{1}{\theta})\int_{\mathbb{R}^N}V_1(x)|Y^{-1}(z_n)|^p\,dx \\
& +(\frac{1}{q}-\frac{1}{\theta})\int_{\mathbb{R}^N}|\nabla w_n|^q\,dx-\frac{1}{\theta}\int_{\mathbb{R}^N}|\nabla w_n|^q\frac{Y^{-1}(w_n)}{y_v(Y^{-1}(w_n))}y_v'(Y^{-1}(w_n))\,dx \\
& +(\frac{1}{q}-\frac{1}{\theta})\int_{\mathbb{R}^N}V_2(x)|Y^{-1}(w_n)|^q\,dx \\
& +\int_{\mathbb{R}^N}\Big(\frac{1}{\theta}\langle\nabla\zeta(x,Y^{-1}(z_n),Y^{-1}(w_n)),(Y^{-1}(z_n)y_u'(Y^{-1}(z_n)),Y^{-1}(w_n)y_v'(Y^{-1}(w_n)))\rangle \\
& -\zeta(x,Y^{-1}(z_n),Y^{-1}(w_n))\Big)\,dx \\
& \geq(\frac{1}{p}-\frac{1}{\theta})\int_{\mathbb{R}^N}\Big(|\nabla z_n|^q+V_1(x)|z_n|^p\Big)\,dx+(\frac{1}{q}-\frac{1}{\theta})\int_{\mathbb{R}^N}\Big(|\nabla w_n|^q+V_2(x)|w_n|^q\Big)\,dx \\
& =(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^p+\|w_n\|^q),
\end{aligned}
\tag{14}
$$

when $\|z_n\|\geq 1,\|w_n\|\geq 1$,

$$
(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^p+\|w_n\|^q)\geq(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^p+\|w_n\|^p)\geq\frac{1}{2^p}\|(z_n,w_n)\|^p,
$$

when $\|z_n\|\geq 1,\|w_n\|<1$,

$$
(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^p+\|w_n\|^q)\geq(\frac{1}{p}-\frac{1}{\theta})\|z_n\|,
$$

when $\|z_n\|<1,\|w_n\|\geq 1$,

$$
(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^p+\|w_n\|^q)\geq(\frac{1}{p}-\frac{1}{\theta})\|w_n\|,
$$

when $\|z_n\|<1,\|w_n\|<1$,

$$
(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^p+\|w_n\|^q)\geq(\frac{1}{p}-\frac{1}{\theta})(\|z_n\|^q+\|w_n\|^q)\geq\frac{1}{2^q}\|(z_n,w_n)\|^q.
$$

Overall, for $p<q$, $(z_n,w_n)\subset D$ is bounded; similarly, for $p>q$, $(z_n,w_n)\subset D$ is also bounded. □

Since, $(PS)_c$ sequence $(z_n,w_n)\subset D$ is bounded, there is (z,w), and (z_n,w_n) have a subsequence recorded as (z_n,w_n) meet

$$
\begin{aligned}
&(z_n,w_n)\rightharpoonup(z,w)\quad\text{in } D,\\
&(z_n(x),w_n(x))\rightharpoonup(z(x),w(x))\quad\text{a.e. in }\mathbb{R}^N\times\mathbb{R}^N,\\
&(z_n,w_n)\to(z,w)\quad\text{in } L^r, L^r=L^{r_1}\times L^{r_2}.
\end{aligned}
\tag{15}
$$

I_k of (8) also is defined as

$$I_k(z_n, w_n) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla z_n|^p \, dx + \frac{1}{p}\int_{\mathbb{R}^N} V_1(x)|z_n|^p \, dx \\ + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \, dx + \frac{1}{q}\int_{\mathbb{R}^N} V_2(x)|w_n|^q \, dx - \iota \int_{\mathbb{R}^N} \eta(x, z_n, w_n) \, dx, \tag{16}$$

and

$$\eta(x, z_n, w_n) = \frac{1}{p}V_1(x)\Big(|z_n|^p - |Y^{-1}(z_n)|^p\Big) + \frac{1}{q}V_2(x)\Big(|w_n|^q - |Y^{-1}(w_n)|^q\Big) \\ + \iota\zeta\Big(x, Y^{-1}(z_n), Y^{-1}(w_n)\Big),$$

in the same

$$\langle \nabla\eta(x, z_n, w_n), (z_n, w_n)\rangle = V_1(x)(|z_n|^p - \frac{|Y^{-1}(z_n)|^{p-1}}{y_u(Y^{-1}(z_n))}) + \iota\frac{f(x, Y^{-1}(z_n), Y^{-1}(w_n))}{y_u(Y^{-1}(z_n))}z_n \\ + V_2(x)(|w_n|^q - \frac{|Y^{-1}(w_n)|^{q-1}}{y_v(Y^{-1}(w_n))}) + \iota\frac{h(x, Y^{-1}(w_n), Y^{-1}(z_n))}{y_v(y^{-1}(w_n))}w_n.$$

Lemma 6. *If (f_1), (f_2), $(\mathcal{V}_1)$, and $(\mathcal{V}_2)$ are accurate, (z_n, w_n) is a $(PS)_c$ sequence, and $(z_n, w_n) \rightharpoonup (z, w)$ in D, as $n \to \infty$, in that way*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \langle \nabla\eta(x, z_n, w_n), (z_n, w_n)\rangle \, dx = \int_{\mathbb{R}^N} \langle \nabla\eta(x, z, w), (z, w)\rangle \, dx, \tag{17}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \langle \nabla\eta(x, z_n, w_n), (z, w)\rangle \, dx = \int_{\mathbb{R}^N} \langle \nabla\eta(x, z, w), (z, w)\rangle \, dx. \tag{18}$$

Proof. From Lemma 3, since $z_n \to z$ in L^{r_1}, $w_n \to w$ in L^{r_2}, $r_1 \in [p, p^*)$, $r_2 \in [q, q^*)$, for $\forall\varepsilon > 0$, there is $R_1 > 0$ satisfied

$$\int_{B^c_{R_1}} |z_n|^p \, dx \le \varepsilon, \quad \int_{B^c_{R_1}} |z|^p \, dx \le \varepsilon, \\ \int_{B^c_{R_1}} |w_n|^q \, dx \le \varepsilon, \quad \int_{B^c_{R_1}} |w|^q \, dx \le \varepsilon. \tag{19}$$

Then,

$$\int_{B^c_{R_1}} V_1(x)|z_n|^p \, dx \le C\varepsilon, \quad \int_{B^c_{R_1}} V_1(x)|z|^p \, dx \le C\varepsilon, \\ \int_{B^c_{R_1}} V_2(x)|w_n|^q \, dx \le C\varepsilon, \quad \int_{B^c_{R_1}} V_2(x)|w|^q \, dx \le C\varepsilon. \tag{20}$$

It is from (15) that

$$\lim_{n\to\infty}\int_{B_{R_1}} V_1(x)|z_n|^p \, dx = \int_{B_{R_1}} V_1(x)|z|^p \, dx, \\ \lim_{n\to\infty}\int_{B_{R_1}} V_2(x)|w_n|^q \, dx = \int_{B_{R_1}} V_2(x)|w|^q \, dx. \tag{21}$$

By (20) and (21), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)|z_n|^p \, dx = \int_{\mathbb{R}^N} V_1(x)|z|^p \, dx, \\ \lim_{n\to\infty}\int_{\mathbb{R}^N} V_2(x)|w_n|^q \, dx = \int_{\mathbb{R}^N} V_2(x)|w|^q \, dx. \tag{22}$$

Deriving from Lemma 1 (ii) and (iii) that

$$\frac{|Y^{-1}(z_n)|^{p-1}}{y_u(Y^{-1}(z_n))}z_n \le 2^{-\frac{(p-1)(p+1)}{p}}|z_n|^{p-1}, \quad \frac{|Y^{-1}(w_n)|^{q-1}}{y_v(Y^{-1}(w_n))}w_n \le 2^{-\frac{(q-1)(q+1)}{q}}|w_n|^{q-1},$$

it follows from (22), (20) and (21) that

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)\frac{|Y^{-1}(z_n)|^{p-1}}{y_u(Y^{-1}(z_n))}z_n\,dx &= \int_{\mathbb{R}^N} V_1(x)\frac{|Y^{-1}(z)|^{p-1}}{y_u(Y^{-1}(z))}z\,dx, \\ \lim_{n\to\infty}\int_{\mathbb{R}^N} V_2(x)\frac{|Y^{-1}(w_n)|^{q-1}}{y_v(Y^{-1}(w_n))}w_n\,dx &= \int_{\mathbb{R}^N} V_2(x)\frac{|Y^{-1}(w)|^{q-1}}{y_v(Y^{-1}(w))}w\,dx. \end{aligned} \tag{23}$$

By (10), Lemma 1 (iii) and Hölder inequality,

$$\begin{aligned} &|\frac{f(x,Y^{-1}(z_n),Y^{-1}(w_n))}{y_u(Y^{-1}(z_n))}z_n + \frac{h(x,Y^{-1}(w_n),Y^{-1}(z_n))}{y_v(Y^{-1}(w_n))}w_n| \\ &= |\langle\nabla\zeta(x,z_n,w_n),(z_n,w_n)\rangle| \\ &\le 2^{(-\frac{(p+1)(p-1)}{p})}\epsilon|z_n|^p + 2^{(-\frac{(q+1)(q-1)}{q})}\epsilon|w_n|^q + 2^{(-\frac{(p+1)(l_p-1)}{p})}C|z_n|^{l_p} + 2^{(-\frac{(q+1)(l_q-1)}{q})}C|w_n|^{l_q}. \end{aligned}$$

By (19), we obtain

$$\begin{aligned} \int_{B^c_{R_1}} |z_n|^{l_p}\,dx \le C\varepsilon, \quad \int_{B^c_{R_1}} |z|^{l_p}\,dx \le C\varepsilon, \\ \int_{B^c_{R_1}} |w_n|^{l_q}\,dx \le C\varepsilon, \quad \int_{B^c_{R_1}} |w|^{l_q}\,dx \le C\varepsilon. \end{aligned} \tag{24}$$

Thus,

$$\begin{aligned} &\int_{B^c_{R_1}} |\langle\nabla\zeta(x,Y^{-1}(z_n),Y^{-1}(w_n)),(z_n,w_n)\rangle|\,dx \\ &\le 2^{(-\frac{(p+1)(p-1)}{p})}\epsilon\int_{B^c_{R_1}}|z_n|^p\,dx + 2^{(-\frac{(q+1)(q-1)}{q})}\epsilon\int_{B^c_{R_1}}|w_n|^q\,dx \\ &+ 2^{(-\frac{(p+1)(l_p-1)}{p})}C\int_{B^c_{R_1}}|z_n|^{l_p}\,dx + 2^{(-\frac{(q+1)(l_q-1)}{q})}C\int_{B^c_{R_1}}|w_n|^{l_q}\,dx \\ &= (2^{(-\frac{(p+1)(p-1)}{p})}\epsilon + 2^{(-\frac{(q+1)(q-1)}{q})}\epsilon + 2^{(-\frac{(p+1)(l_p-1)}{p})}C + 2^{(-\frac{(q+1)(l_q-1)}{q})}C)\varepsilon. \end{aligned} \tag{25}$$

By (15),

$$\begin{aligned} &\lim_{n\to\infty}\int_{B_{R_1}} |\langle\nabla\zeta(x,Y^{-1}(z_n),Y^{-1}(w_n)),(z_n,w_n)\rangle|\,dx \\ &= \int_{B_{R_1}} |\langle\nabla\zeta(x,Y^{-1}(z),Y^{-1}(w)),(z,w)\rangle|\,dx. \end{aligned} \tag{26}$$

By (25) and (26),

$$\begin{aligned} &\lim_{n\to\infty}\int_{\mathbb{R}^N}\langle\nabla\zeta(x,Y^{-1}(z_n),Y^{-1}(w_n)),(z_n,w_n)\rangle\,dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\Big(\frac{f(x,Y^{-1}(z_n),Y^{-1}(w_n))}{y_u(Y^{-1}(z_n))}z_n + \frac{h(x,Y^{-1}(w_n),Y^{-1}(z_n))}{y_v(Y^{-1}(w_n))}w_n\Big)\,dx \\ &= \int_{\mathbb{R}^N}\Big(\frac{f(x,Y^{-1}(z),Y^{-1}(w))}{y_u(Y^{-1}(z))}z + \frac{h(x,Y^{-1}(w),Y^{-1}(z))}{y_v(Y^{-1}(w))}w\Big)\,dx. \end{aligned} \tag{27}$$

Combining (22), (23) and (27), it is easy to obtain (17). Similarly, (18) can also be obtained. □

Lemma 7. *If $(f_1),(f_2),(\mathcal{V}_1)$ and $(\mathcal{V}_2)$ are satisfied, then, in D, any $(PS)_c$ sequence (z_n,w_n) received in (13) exhibits a robust subsequence of convergence.*

Proof. It follows from Lemma 5 that (z_n,w_n) is bounded in D and its subsequences (z_n,w_n), as $n\to\infty$ satisfy $(z_n,w_n)\rightharpoonup(z,w)\in D$ and $\langle I_k'(z_n,w_n),(z_n,w_n)\rangle=o_n(1)$, adding Lemma 6 to reveal

$$\lim_{n\to\infty}(\|z_n\|^p+\|w_n\|^q)=\int_{\mathbb{R}^N}\langle\nabla\eta(x,z,w),(z,w)\rangle\,dx.$$

From $\langle I_k'(z_n,w_n),(z,w)\rangle=o_n(1)$,

$$\begin{aligned}&\lim_{n\to\infty}\Big(\int_{\mathbb{R}^N}|\nabla z_n|^{p-1}\nabla z\,dx+\int_{\mathbb{R}^N}V_1(x)|z_n|^{p-1}z\,dx+\int_{\mathbb{R}^N}|\nabla w_n|^{q-1}\nabla w\,dx+\int_{\mathbb{R}^N}V_2(x)|w_n|^{q-1}w\,dx\Big)\\&=\int_{\mathbb{R}^N}\nabla\eta(x,z,w)\,dx+o_n(1),\end{aligned}$$

equivalent to

$$\lim_{n\to\infty}(\|z_n\|^p+\|w_n\|^q)=\|z\|^p+\|w\|^q.$$

Therefore, $(z_n,w_n)\to(z,w)$ in D. □

From Lemmas 4–7, similar to [20], Theorem 2 can be concluded.

Theorem 2. *(z,w) is nontrivial solution of the problem (7), if $(\mathcal{V}_1),(\mathcal{V}_2)$ and $(f_1)-(f_3)$ is true.*

4. Proof of Main Result

Now, we try to prove that the solution $(u,v)=(Y^{-1}(z),Y^{-1}(w))$ is solution of (1).

Lemma 8. *(z,w) is a nontrivial critical point of I_k, the critical value is c, in that, $\exists K\in\mathbb{R}$ with $K>0$ unrelated to ι make*

$$\|z\|^p+\|w\|^q\le Kc. \tag{28}$$

Proof. Add (14) from Lemma 1 (ii), (iii) and (f_3), there is

$$\begin{aligned}\theta c&=\theta I_k(z,w)-\langle I_k'(z,w),(Y^{-1}(z)y_u(Y^{-1}(z)),Y^{-1}(w)y_v(Y^{-1}(w)))\rangle\\&\le(\frac{1}{p}-\frac{1}{\theta})(\|z\|^p+\|w\|^q).\end{aligned}$$

Hence,

$$\|z\|^p+\|w\|^q\le\frac{\theta pc}{\theta-p}=Kc.$$

The proof is completed. □

Lemma 9. *(z,w) is the critical point of the function $I_k(z,w)$, in that, $\exists C\in\mathbb{R}$ with $C>0$, and C is unrelated to ι makes*

$$\|z\|_\infty\le C\iota^{\frac{1}{p^*-l_p}}\|z\|_{p^*},\quad\|w\|_\infty\le C\iota^{\frac{1}{q^*-l_q}}\|w\|_{q^*}. \tag{29}$$

Proof. For every $n_0\in\mathbb{N}$, taking $\beta>1$ be the given constant, let

$$A_{n_0}=\{x\in\mathbb{R}^N:|z|^{\beta-1}\le n_0,|w|^{\beta-1}\le n_0\},\quad B_{n_0}=\mathbb{R}^N\setminus A_{n_0},$$

$$(u_{n_0},v_{n_0})=\begin{cases}(z|z|^{p(\beta-1)},w|w|^{q(\beta-1)}),&x\in A_{n_0},\\ {n_0}^2(z,w),&x\in B_{n_0}\end{cases}$$

as well as

$$(z_{n_0}, w_{n_0}) = \begin{cases} (z|z|^{\beta-1}, w|w|^{\beta-1}), & x \in A_{n_0}, \\ ({n_0}^{\frac{2}{p}}, {n_0}^{\frac{2}{q}})(z, w), & x \in B_{n_0}. \end{cases}$$

Apparently, $(u_{n_0}, v_{n_0}), (z_{n_0}, w_{n_0}) \in D$. For (z, w) is a nontrivial solution of (7), in that

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big(|\nabla z|^{p-2} \nabla z \nabla u_{n_0} + V_1(x) \frac{|Y^{-1}(z)|^{p-1}}{y_u(Y^{-1}(z))} u_{n_0} \Big)\, dx \\ &+ \int_{\mathbb{R}^N} \Big(|\nabla w|^{q-2} \nabla w \nabla v_{n_0} + V_2(x) \frac{|Y^{-1}(w)|^{q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big)\, dx \\ &= \iota \int_{\mathbb{R}^N} \Big(\frac{f(x, Y^{-1}(z), Y^{-1}(w))}{y_u(Y^{-1}(z))} u_{n_0} + \frac{h(x, Y^{-1}(z), Y^{-1}(w))}{y_v(Y^{-1}(w))} v_{n_0} \Big)\, dx. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z|^{p-1} \nabla u_{n_0}\, dx + \int_{\mathbb{R}^N} |\nabla w|^{q-1} \nabla v_{n_0}\, dx \\ =&(p\beta - p + 1) \int_{A_{n_0}} |\nabla z|^p |z|^{p(\beta-1)}\, dx + (q\beta - q + 1) \int_{A_{n_0}} |\nabla w|^q |w|^{q(\beta-1)}\, dx \\ &+ {n_0}^2 \int_{B_{n_0}} |\nabla z|^p\, dx + {n_0}^2 \int_{B_{n_0}} |\nabla w|^q\, dx, \end{aligned} \tag{30}$$

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z_{n_0}|^p\, dx + \int_{\mathbb{R}^N} |\nabla w_{n_0}|^q\, dx \\ =&\beta^p \int_{A_{n_0}} |\nabla z|^p |z|^{p(\beta-1)}\, dx + \beta^q \int_{A_{n_0}} |\nabla w|^q |w|^{q(\beta-1)}\, dx + {n_0}^2 \int_{B_{n_0}} |\nabla z|^p\, dx + {n_0}^2 \int_{B_{n_0}} |\nabla w|^q\, dx. \end{aligned} \tag{31}$$

Therefore, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z|^{p-1} \nabla u_{n_0}\, dx + \int_{\mathbb{R}^N} |\nabla w|^{q-1} \nabla v_{n_0}\, dx - {n_0}^2 \int_{B_{n_0}} |\nabla z|^p\, dx - {n_0}^2 \int_{B_{n_0}} |\nabla w|^q\, dx \\ \geq &\int_{A_{n_0}} |\nabla z|^p |z|^{p(\beta-1)}\, dx + \int_{A_{n_0}} |\nabla w|^q |w|^{q(\beta-1)}\, dx. \end{aligned} \tag{32}$$

From (31) and (32), let $\beta^c = \max\{\beta^p, \beta^q\}$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla z_{n_0}|^p\, dx + \int_{\mathbb{R}^N} |\nabla w_{n_0}|^q\, dx \\ \leq&\beta^c \int_{\mathbb{R}^N} |\nabla z|^{p-1} \nabla u_{n_0}\, dx + \beta^c \int_{\mathbb{R}^N} |\nabla w|^{q-1} \nabla v_{n_0}\, dx \\ &- {n_0}^2 \beta^c \int_{B_{n_0}} |\nabla z|^p\, dx - {n_0}^2 \beta^c \int_{B_{n_0}} |\nabla w|^q\, dx + {n_0}^2 \int_{B_{n_0}} |\nabla z|^p\, dx + {n_0}^2 \int_{B_{n_0}} |\nabla w|^q\, dx \\ \leq&\beta^c \int_{\mathbb{R}^N} |\nabla z|^{p-1} \nabla u_{n_0}\, dx + \beta^c \int_{\mathbb{R}^N} |\nabla w|^{q-1} \nabla v_{n_0}\, dx. \end{aligned} \tag{33}$$

It follows from (33) and $\beta > 1$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla z_{n_0}|^p \, dx + \int_{\mathbb{R}^N} |\nabla w_{n_0}|^q \, dx + \beta^c \int_{\mathbb{R}^N} \Big(V_1(x) \frac{|Y^{-1}(z)|^{p-1}}{y_u(Y^{-1}(z))} u_{n_0} + V_2(x) \frac{|Y^{-1}(w)|^{q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
\leq& \beta^c \int_{\mathbb{R}^N} |\nabla z|^{p-1} \nabla u_{n_0} \, dx + \beta^c \int_{\mathbb{R}^N} |\nabla w|^{q-1} \nabla v_{n_0} \, dx \\
&+ \beta^c \int_{\mathbb{R}^N} \Big(V_1(x) \frac{|Y^{-1}(z)|^{p-1}}{y_u(Y^{-1}(z))} u_{n_0} + V_2(x) \frac{|Y^{-1}(w)|^{q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
=& \beta^c \iota \int_{\mathbb{R}^N} \Big(\frac{f(x, Y^{-1}(z), Y^{-1}(w))}{y_u(Y^{-1}(z))} u_{n_0} + \frac{h(x, Y^{-1}(z), Y^{-1}(w))}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
\leq& \beta^c \iota \int_{\mathbb{R}^N} \epsilon \Big(\frac{(Y^{-1}(z))^{p-1}}{y_u(Y^{-1}(z))} u_{n_0} + \frac{(Y^{-1}(w))^{q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
&+ \beta^c \iota \int_{\mathbb{R}^N} C \Big(\frac{(Y^{-1}(z))^{l_p-1}}{y_u(Y^{-1}(z))} u_{n_0} + \frac{(Y^{-1}(w))^{l_q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
\leq& \beta^c \int_{\mathbb{R}^N} \Big(V_1(x) \frac{(Y^{-1}(z))^{p-1}}{y_u(Y^{-1}(z))} u_{n_0} + V_2(x) \frac{(Y^{-1}(w))^{q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
&+ \beta^c \iota \int_{\mathbb{R}^N} C \Big(\frac{(Y^{-1}(z))^{l_p-1}}{y_u(Y^{-1}(z))} u_{n_0} + \frac{(Y^{-1}(w))^{l_q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx,
\end{aligned} \tag{34}
$$

where $0 < \epsilon < \min\{\frac{V_1(x)}{\iota}, \frac{V_1(x)}{\iota}\}$. By Lemma (iii) and the fact of $z^{p-1}u_{n_0} = z_{n_0}^p$ and $w^{q-1}v_{n_0} = w_{n_0}^q$, we can obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla z_{n_0}|^p \, dx + \int_{\mathbb{R}^N} |\nabla w_{n_0}|^q \, dx \\
\leq& \beta^c \iota C \int_{\mathbb{R}^N} \Big(\frac{(Y^{-1}(z))^{l_p-1}}{y_u(Y^{-1}(z))} u_{n_0} + \frac{(Y^{-1}(w))^{l_q-1}}{y_v(Y^{-1}(w))} v_{n_0} \Big) \, dx \\
\leq& \beta^c \iota C \int_{\mathbb{R}^N} \Big(2^{(-\frac{l_p(p+1)}{p})} |z|^{l_p-p} z_{n_0}^p + 2^{(-\frac{l_q(q+1)}{q})} |w|^{l_q-q} w_{n_0}^q \Big) \, dx \\
\leq& \beta^c \iota C \int_{\mathbb{R}^N} \Big(|z|^{l_p-p} z_{n_0}^p + |w|^{l_q-q} w_{n_0}^q \Big) \, dx.
\end{aligned} \tag{35}
$$

If $J(a) + J(b) \leq L(a) + L(b)$, we have $J(a) \leq L(a)$, $J(b) \leq L(b)$. It follows from (35) that

$$\int_{\mathbb{R}^N} |\nabla z_{n_0}|^p \, dx \leq \beta^c \iota C \int_{\mathbb{R}^N} |z|^{l_p-p} z_{n_0}^p \, dx, \tag{36}$$

$$\int_{\mathbb{R}^N} |\nabla w_{n_0}|^q \, dx \leq \beta^c \iota C \int_{\mathbb{R}^N} |w|^{l_q-q} w_{n_0}^q \, dx. \tag{37}$$

From the Sobolev inequality, when $S > 0$, we have

$$(\int_{A_{n_0}} |\nabla z_{n_0}|^{p^*} \, dx)^{\frac{N-p}{N}} \leq S \int_{\mathbb{R}^N} |\nabla z_{n_0}|^p \, dx,$$

combining (36) and Hölder inequality, we can obtain

$$(\int_{A_{n_0}} |\nabla z_{n_0}|^{p^*} \, dx)^{\frac{N-p}{N}} \leq \beta^c \iota S C \|z\|_{p_2}^{l_p-p} \|z\|_{p_1}^{p},$$

where $\frac{p}{p_1} + \frac{l_p-p}{p^*} = 1$. Note that $|z_{n_0}| = |z|^\beta$ in A_{n_0} and $|z_{n_0}| \leq |z|^\beta$, thus

$$(\int_{A_{n_0}} |\nabla z_{n_0}|^{p^*\beta} \, dx)^{\frac{N-p}{N}} \leq \beta^c \iota S C \|z\|_{p_2}^{l_p-p} \|z\|_{p_1\beta}^{p\beta}.$$

Action $n_0 \to \infty$ on the above equation, with

$$\|z\|_{\beta p^*} \leq \beta^{\frac{c}{p\beta}} \iota^{\frac{1}{p\beta}} S^{\frac{1}{p\beta}} C^{\frac{1}{p\beta}} \|z\|_{p_2}^{(l_p-p)\frac{1}{p\beta}} \|z\|_{p_1\beta}. \tag{38}$$

Denoting $\sigma = \frac{p^*}{p_1}$ and let $\beta = \sigma$ in (38), we can obtain

$$\|z\|_{\sigma p^*} \leq \sigma^{\frac{c}{p\sigma}} \iota^{\frac{1}{p\sigma}} S^{\frac{1}{p\sigma}} C^{\frac{1}{p\sigma}} \|z\|_{p_2}^{(l_p-p)\frac{1}{p\sigma}} \|z\|_{p^*}. \tag{39}$$

Taking $\beta = \sigma^2$, we see that

$$\|z\|_{\sigma^2 p^*} \leq \sigma^{\frac{2c}{p\sigma^2}} \iota^{\frac{1}{p\sigma^2}} S^{\frac{1}{p\sigma^2}} C^{\frac{1}{p\sigma^2}} \|z\|_{p_2}^{(l_p-p)\frac{1}{p\sigma^2}} \|z\|_{p^*\sigma}. \tag{40}$$

From (39) and (40), we have

$$\|z\|_{\sigma^2 p^*} \leq \sigma^{\frac{c}{p}(\frac{1}{\sigma}+\frac{2}{\sigma^2})} \iota^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2})} S^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2})} C^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2})} \|z\|_{p_2}^{(l_p-p)\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2})} \|z\|_{p^*}.$$

For (38), continuing this approch by taking $\beta = \sigma^j (j = 1, 2, \dots)$, then

$$\|z\|_{\sigma^i p^*} \leq \sigma^{\frac{c}{p}\sum_{j=1}^{i}\frac{i}{\sigma^j}} (\iota^{\frac{1}{p}} S^{\frac{1}{p}} C^{\frac{1}{p}} \|z\|_{p_2}^{(l_p-p)\frac{1}{p}})^{\sum_{j=1}^{i}\frac{1}{\sigma^j}} \|z\|_{p^*}.$$

Setting $i \to +\infty$ and using the Sobolev inequality, then

$$\begin{aligned} \|z\|_{\infty} &\leq \sigma^{\frac{c}{p(\sigma-1)^2}} (\iota^{\frac{1}{p}} S^{\frac{1}{p}} C^{\frac{1}{p}} C)^{\frac{1}{(\sigma-1)}} \|z\|_{p^*} \\ &= C\iota^{\frac{1}{p(\sigma-1)}} \|z\|_{p^*} \\ &= C\iota^{\frac{1}{p^*-l_p}} \|z\|_{p^*}, \end{aligned} \tag{41}$$

in which C is not related to ι. Similarly, we have

$$\|w\|_{\infty} \leq C\iota^{\frac{1}{q^*-l_q}} \|w\|_{q^*},$$

where C is not related to ι. □

Proof of Theorem 1. Let $\chi > 0$ and

$$T = \{x \in \mathbb{R}^N : \phi_1(x) \geq \chi\} \cap \{x \in \mathbb{R}^N : \phi_2(x) \geq \chi\}$$

be a nonempty set. From (f_2) and (f_3), for $x \in T$, there is $C > 0$, which makes

$$\zeta(x, s, t) \geq C|(s, t)|^{(l_p, l_q)}. \tag{42}$$

Supposing that (z, w) be a critical point of I_k with the critical value c. By Theorem 2 and (42), we obtain

$$
\begin{aligned}
c &\leq \max_{t>0} I_k(t\phi_1, t\phi_2) \\
&\leq \max_{t>0} \Big(\frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \phi_1|^p \, dx + \frac{t^p}{p} \int_{\mathbb{R}^N} 2^{(-1-p)} V_1(x) |\phi_1|^p \, dx \\
&\quad + \frac{t^q}{q} \int_{\mathbb{R}^N} |\nabla \phi_2|^q \, dx + \frac{t^q}{q} \int_{\mathbb{R}^N} 2^{(-1-q)} V_2(x) |\phi_2|^q \, dx \\
&\quad - \iota t^{l_p} C 2^{-\frac{(p+1)l_p}{p}} \int_{\mathbb{R}^N} |\phi_1|^{l_p} \, dx - \iota t^{l_q} C 2^{-\frac{(q+1)l_q}{q}} \int_{\mathbb{R}^N} |\phi_2|^{l_q} \, dx \Big) \\
&\leq C \iota^{-\frac{p}{l_p - p}}.
\end{aligned}
$$

From (28), (29) and the continuous embedding $D_1 \hookrightarrow L^{r_1}, D_2 \hookrightarrow L^{r_2}$, we obtain

$$
\begin{aligned}
\|z\|_\infty &\leq C \iota^{\frac{1}{p^* - l_p}} \|z\|_{p^*} \leq C \iota^{\frac{1}{p^* - l_p}} \|z\| \leq C \iota^{\frac{1}{p^* - l_p}} (Kc)^{\frac{1}{p}} \\
&\leq C \iota^{\frac{1}{p^* - l_p}} (K \iota^{-\frac{p}{l_p - p}})^{\frac{1}{p}} \leq C_1 \iota^{\frac{2l_p - p^* - p}{(p^* - l_p)(l_p - p)}},
\end{aligned} \tag{43}
$$

where C_1 is a constant. Since $p < l_p < p^*$, for given $k > 0$, there is $\iota_1(k) = (2^3 k C_1^p)^{\frac{(p^* - l_p)(l_p - p)}{p(p^* - 2l_p + p)}}$, which makes for each $\iota > \iota_1(k)$, it satisfies

$$
\|u\|_\infty = \left\| Y^{-1}(z) \right\|_\infty \leq 2^{-\frac{1}{p} - 1} (\|z\|_\infty \leq 2^{-\frac{1}{p} - 1} C_1 \iota^{\frac{2l_p - p^* - p}{(p^* - l_p)(l_p - p)}} \leq (\frac{1}{2^{p-3} k})^{\frac{1}{p}}.
$$

Similarly, we may obtain $\|v\|_\infty \leq (\frac{1}{2^{q-3} k})^{\frac{1}{q}}$. Hence, the system (1) has a nontrivial solution $(u, v) = (Y^{-1}(z), Y^{-1}(w))$.

5. Conclusions

We study the related problem of the quasilinear Schrödinger system containing the operator Δ_p and Δ_q. By using the variable transformation to process quasilinear terms, combined with the mountain-pass theorem, we received a nontrivial solution of the system. It is worth considering whether the variable exponent has an impact on the above conclusion, and trying to extend p and q to $p(x)$ and $q(x)$ is also a meaningful issue.

Author Contributions: Writing—original draft, X.Z.; Writing—review & editing, J.Z. The authors declare that they have contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: Jing Zhang was supported by the Natural Science Foundation of Inner Mongolia Autonomous Region (No. 2022MS01001), Key Laboratory of Infinite-dimensional Hamiltonian System and Its Algorithm Application (Inner Mongolia Normal University), Ministry of Education (No. 2023KFZD01), Research Program of science and technology at Universities of Inner Mongolia Autonomous Region (No. NJYT23100), Mathematics First-class Disciplines Cultivation Fund of Inner Mongolia Normal University (No. 2024YLKY14) and the Fundamental Research Funds for the Inner Mongolia Normal University (No. 2022JBQN072). Xue Zhang was supported by the Fundamental Research Funds for the Inner Mongolia Normal University (2022JBXC03) and Graduate students' research Innovation fund of Inner Mongolia Normal University (CXJJS22100).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Chen, C.; Yang, H. Multiple Solutions for a Class of Quasilinear Schrödinger Systems in $\mathbb{R}^N$. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 611–636. [CrossRef]
2. Severo, U.; Silva, E. On the existence of standing wave solutions for a class of quasilinear Schr?dinger systems. *J. Math. Anal. Appl.* **2014**, *412*, 763–775. [CrossRef]
3. Laedke, E.W.; Spatschek, K.H.; Stenflo, L. Evolution theorem for a class of perturbed envelope soliton solutions. *J. Math. Phys.* **1983**, *24*, 2764–2769. [CrossRef]
4. Lange, H.; Toomire, B.; Zweifel, P.F. Time-dependent dissipation in nonlinear Schrödinger systems. *J. Math. Phys.* **1995**, *36*, 1274–1283. [CrossRef]
5. Ritchie, B. Relativistic self-focusing and channel formation in laser-plasma interactions. *Phys. Rev. E* **1994**, *50*, 687–689. [CrossRef]
6. Chen, J.; Huang, X.; Cheng, B.; Zhu, C. Some results on standing wave solutions for a class of quasilinear Schrödinger equations. *J. Math. Phys.* **2019**, *60*, 091506. [CrossRef]
7. Liu, J.Q.; Wang, Z.Q. Soliton solutions for quasilinear Schrödinger equations I. *Proc. Am. Math. Soc.* **2002**, *131*, 441–448. [CrossRef]
8. Severo, U.B. Symmetric and nonsymmetric solutions for a class of quasilinear Schrödinger equations. *Adv. Nonlinear Stud.* **2008**, *8*, 375–389. [CrossRef]
9. Chen, J.; Chen, B.; Huang, X. Ground state solutions for a class of quasilinear Schrödinger equations with Choquard type nonlinearity. *Appl. Math. Lett.* **2020**, *102*, 106141. [CrossRef]
10. Jeanjean, L. On the existence of bounded Palais–Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$. *Proc. R. Soc. Edinb. Sect. A* **1999**, *129*, 787–809. [CrossRef]
11. Yang, X.; Zhang, W.; Zhao, F. Existence and multiplicity of solutions for a quasilinear Choquard equation via perturbation method. *J. Math. Phys.* **2018**, *59*, 081503. [CrossRef]
12. Alves, C.O.; de Morais Filho, D.C. Existence of concentration of positive solutions for a Schrödinger logarithmic equation. *Z. Angew. Math. Phys.* **2018**, *69*, 144. [CrossRef]
13. Duan, S.Z.; Wu, X. An existence result for a class of p-Laplacian elliptic systems involving homogeneous nonlinearities in $\mathbb{R}^N$. *Nonlinear Anal.* **2011**, *74*, 4723–4737. [CrossRef]
14. Moameni, A. Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$. *Differ. Equ.* **2006**, *229*, 570–587. [CrossRef]
15. Chen, J.; Zhang, Q. Existence of positive ground state solutions for quasilinear Schrödinger system with positive parameter. *Appl. Anal.* **2022**, *102*, 2676–2691. [CrossRef]
16. Wang, Y.; Huang, X. Ground states of Nehari-Pohožaev type for a quasilinear Schrödinger system with superlinear reaction. *Electron. Res. Arch.* **2023**, *31*, 2071–2094. [CrossRef]
17. Guo, Y.; Tang, Z. Ground state solutions for quasilinear Schrödinger systems. *J. Math. Anal. Appl.* **2012**, *389*, 322–339. [CrossRef]
18. Chen, J.; Zhang, Q. Positive solutions for quasilinear Schrödinger system with positive parameter. *Z. Angew. Math. Phys.* **2022**, *73*, 144. [CrossRef]
19. Chen, J.; Zhang, Q. Ground state solution of Nehari-Pohožaev type for periodic quasilinear Schrödinger system. *J. Math. Phys.* **2020**, *61*, 101510. [CrossRef]
20. Li, G. On the existence of nontrivial solutions for quasilinear Schrödinger systems. *Bound. Value Probl.* **2022**, *2022*, 40. [CrossRef]
21. Szulkin, A.; Weth, T. The method of Nehari manifold. In *Handbook of Nonconvex Analysis and Applications*; Gao, D.Y., Motreanu, D., Eds.; International Press: Boston, MA, USA, 2010; pp. 2314–2351.
22. Alves, C.O.; Wang, Y.; Shen, Y. Soliton solutions for a class of quasilinear Schrödinger equations with a parameter. *J. Differ. Equ.* **2015**, *259*, 318–343. [CrossRef]
23. Wang, Y.; Li, Z. Existence of solutions to quasilinear Schrödinger equations involving critical Sobolev exponent. *Taiwan. J. Math.* **2018**, *22*, 401–420. [CrossRef]