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Mittag-Leffler-Type Stability of BAM Neural Networks Modeled by the Generalized Proportional Riemann–Liouville Fractional Derivative

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Abstract: The main goal of the paper is to use a generalized proportional Riemann–Liouville fractional derivative (GPRLFD) to model BAM neural networks and to study some stability properties of the equilibrium. Initially, several properties of the GPRLFD are proved, such as the fractional derivative of a squared function. Additionally, some comparison results for GPRLFD are provided. Two types of equilibrium of the BAM model with GPRLFD are defined. In connection with the applied fractional derivative and its singularity at the initial time, the Mittag-Leffler exponential stability in time of the equilibrium is introduced and studied. An example is given, illustrating the meaning of the equilibrium as well as its stability properties.

Keywords: BAM neural networks; Mittag-Leffler-type stability; fractional differential equations; generalized proportional Riemann–Liouville fractional derivative

MSC: 34A34; 34A08; 34D20

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1. Introduction

One of the main qualitative properties of the solutions of differential equations is stability. There are various types of stability defined, studied and applied to different types of differential equations, especially to fractional differential equations. The stability of Hadamard fractional differential equations is studied in [1]. The stability of Caputo-type fractional derivatives are studied by many authors, and many sufficient conditions are obtained (for example, see Mittag-Leffler stability in [2], and the application of Lyapunov functions in [3]). Concerning fractional differential equations with Riemann–Liouville fractional derivatives, the stability of linear systems is studied in [4], nonlinear systems in [5,6], Lyapunov functions are applied and comparison results are established in [7], practical stability is studied in [8], and existence and Ulam stability in [9]. Note that the initial condition for fractional differential equations with the Riemann–Liouville-type fractional derivative is totally different from the initial condition for ordinary differential equations or for fractional differential equations with Caputo-type derivatives. Some authors did not take this into account, and consequently, a gap exists in the study of stability. Concerning the basic concepts of the stability for Riemann–Liouville fractional differential equations, we note [10], in which several up-to-date types of fractional derivatives are defined, studied and applied to differential equations. Recently, the so-called generalized proportional fractional integrals and derivatives were defined (see [11,12]). Similar to classical fractional derivatives, there are two main types of generalized proportional fractional derivatives: Caputo-type and Riemann–Liouville-type. Several results concerning the existence (see, for example, [13,14]), integral presentation of the solutions (see, for example, [15]), stability properties (see, for example, [16,17]) and applications to some models (see, for example, [16]) are considered with the Caputo type of generalized proportional fractional derivatives. Additionally, there are some results concerning the Riemann–Liouville type. Some existence results are obtained in [18]. In [19,20], the oscillation properties of fractional differential equations with a generalized proportional Riemann–Liouville fractional derivative are studied. The existence and uniqueness of a coupled system is studied in [21] in the case of three-point generalized fractional integral boundary conditions. In this paper, initially, we prove some comparison results for generalized proportional Riemann–Liouville fractional derivatives. Additionally, we discuss the behavior of the solutions on small enough intervals about the initial time. Some examples are given, illustrating the necessity of excluding the initial time when the stability is studied. The obtained results are a basis for studying a stability property of the equilibrium of a model of neural networks. The models of neural networks are important issues due to their successful application in pattern recognition, artificial intelligence, automatic control, signal processing, optimization, etc. In the past decades, several types of fractional derivatives were applied to the models of neural networks to describe the dynamics of the neurons more adequately. Many qualitative properties of their equilibrium have been studied. In this paper, we apply the generalized proportional Riemann–Liouville fractional derivative to the BAM model of neutral networks. Recently, bi-directional associative memory (BAM) neural networks were extensively investigated and successfully applied to signal processing, pattern recognition, associative memory and optimization problems. For more adequate modeling of the dynamics of the state of neurons, several types of derivatives are applied, including various types of fractional derivatives. We refer the reader, for example, to the study of existence and stability for models with ordinary derivatives and discontinuous neuron activations [22], the delay model [23], and the study of stability for a model with the Caputo fractional derivative [24]. Reviews of the application of fractional derivatives to the neural networks are given in [25,26].

One of the main properties of the applied fractional derivative is its singularity at the initial time. In connection with this, we define in an appropriate way an exponential Mittag-Leffler stability in time, excluding the initial time. Additionally, two types of equilibrium, deeply connected with the applied fractional derivative, are defined. Sufficient conditions based on the new comparison results are obtained and illustrated with examples. The rest of this paper is organized as follows. In Section 2, some notes on fractional calculus are provided; the basic definitions of the generalized proportional fractional integrals and derivatives are given in the case when the order of fractional derivative is in the interval $(0, 1)$ and the parameter is in $(0, 1]$. The connection with the tempered fractional integrals and the derivatives is discussed. In Section 3, we prove some comparison results for generalized Riemann–Liouville fractional derivatives. In Section 4, the model of BAM neural networks with GPRLFD is set up and studied. Two types of equilibriums are defined. These definitions are deeply connected with the applied GPRLFD and its properties, which are totally different from those of ordinary derivatives and Caputo-type fractional derivatives. The Mittag-Leffler exponential stability in time of both types of equilibriums is defined and studied. Finally, an example is given to illustrate the theoretical results and statements.

2. Some Notes on Fractional Calculus

The main goal in this paper is to apply a partial case of fractional derivatives to a model and to investigate the stability behavior of the model. In connection with this, we will give a brief discussion about fractional derivatives known in the literature. The main idea of fractional calculus is the generalization of the differential operator to an operator with any real or complex number order. The most standard of these operators are the Riemann–Liouville fractional integral and derivatives (for basic definitions and properties, see, for example, the classical books [27–30]). In the last few decades, many different definitions have been proposed. As a comprehensive definition appealing to general principles of mathematics, the fractional derivative is a fractional power of the infinitesimal generator of a strictly continuous semigroup of contractions. We mention the Marchaud operator, generated by a semigroup, which is well described and compared with the existing ones in the paper [31], and its detailed presentation, together with the constructions with the exponential multiplier, is given in the classical book [29]. Additionally, a differential operator with a fractional integro-differential operator composition in final terms is presented and studied in [32]. Another way to generalize the classical definitions is the approach whereby some multipliers can be added to make a new construction with some similar properties. For more information about the definitions of fractional integrals and derivatives with exponential kernel, called tempered fractional integrals and derivatives, and some applications to stochastic process, Brownian motion, etc., we refer the reader to [33]. Recently, refs. [11,12] generalized fractional integrals and derivatives by considering exponential functions with a fraction in the power, and these were called generalized proportional fractional ones. The used parameter in the exponential kernel gives us more detailed information.

We recall some basic definitions and properties relevant to the generalized proportional fractional derivative and integral. The terms and notations are adopted from [11,12].

Definition 1. *Ref. [11] (The generalized proportional fractional integral) (GPFI) Let $v : [a,b] \to \mathbb{R}$, $b \leq \infty$, and $\rho \in (0,1]$, $q \geq 0$. We define the GPFI of the function v by $({}_a\mathcal{I}^{0,\rho}v)(t) = v(t)$ and*

$$({}_a\mathcal{I}^{q,\rho}v)(t) = \frac{1}{\rho^q\Gamma(q)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{q-1}v(s)\,ds, \quad t \in (a,b]. \tag{1}$$

Definition 2. *Ref. [11] (The generalized proportional Riemann–Liouville fractional derivative) (GPRLFD) Let $v : [a,b] \to \mathbb{R}$, $b \leq \infty$, and $\rho \in (0,1]$, $q \in (0,1)$. Define the GPRLFD of the function v by*

$$\begin{aligned}({}^R_a\mathcal{D}^{q,\rho}v)(t) = \frac{1}{\rho^{1-q}\Gamma(1-q)}\Big((1-\rho)\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}v(s)\,ds \\ + \rho\frac{d}{dt}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}v(s)\,ds\Big), \quad t \in (a,b].\end{aligned} \tag{2}$$

Remark 1. *The constructions with the exponential multiplier were considered also in the monograph [29].*

Remark 2. *The parameter q in Definitions 1 and 2 is interpreted as an order of integration and differentiation, respectively. The parameter ρ is connected with the power of the exponential function. In the case $\rho = 1$, the given fractional integral and derivative reduce to the classical Riemann–Liouville fractional integral*

$${}_aI^q_t v(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}v(s)\,ds, \tag{3}$$

and the Riemann–Liouville fractional derivative

$${}^{RL}_{a}D^q_t v(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_a^t (t-s)^{-q}v(s)\,ds, \tag{4}$$

The relation between the GPRLFD and the Riemann–Liouville fractional derivative is given in the following Lemma.

Lemma 1. *Let $\rho \in (0,1], q \in (0,1)$, and $v \in C([a,b])$, $b \leq \infty$. Then,*

$$({}_a\mathcal{D}^{q,\rho}v)(t) = \rho^q e^{\frac{\rho-1}{\rho}t}\left({}^{RL}_{a}D^q_t\left(e^{\frac{1-\rho}{\rho}t}v(t)\right)\right), \quad t \in (a,b]. \tag{5}$$

Proof. From Equations (2) and (4), we have

$$\begin{aligned}({}_a^R\mathcal{D}^{q,\rho}v)(t) &= \frac{1}{\rho^{1-q}\Gamma(1-q)}\Big((1-\rho)\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}v(s)\,ds \\ &\quad + \rho\frac{d}{dt}e^{\frac{\rho-1}{\rho}t}\int_a^t e^{\frac{1-\rho}{\rho}s}(t-s)^{-q}v(s)\,ds\Big) \\ &= \frac{1}{\rho^{-q}\Gamma(1-q)}e^{\frac{\rho-1}{\rho}t}\frac{d}{dt}\int_a^t e^{\frac{1-\rho}{\rho}s}(t-s)^{-q}v(s)\,ds \\ &= \rho^q e^{\frac{\rho-1}{\rho}t}({}_a^{RL}D_t^q e^{\frac{1-\rho}{\rho}t}v(t)).\end{aligned}$$

□

Remark 3. *The equality (5) gives us an opportunity to apply some of the properties known in the literature for Riemann–Liouville fractional derivatives to GPRLFD. However, it does not allow us to directly apply properties of the solutions of fractional differential equations with Riemann–Liouville fractional derivatives to those with GPRLFD. That is why it is absolutely necessary to study independently differential equations with GPRLFD and to obtain sufficient conditions for some qualitative properties of their solutions, such as various types of stability.*

Define the set

$$C_{q,\rho}([a,b],\mathbb{R}^n) = \{v : [a,b] \to \mathbb{R}^n : \text{ for any } t \in (a,b] \text{ there exists } ({}_a^{RL}\mathcal{D}^{q,\rho}v)(t) < \infty\}.$$

We will provide some results which are partial cases of the obtained ones in [12] and which will be used in our further considerations.

Lemma 2. *(*semigroup property*) (Theorem 3.8, Corollary 3.10, Theorem 3.11, Lemma 3.12 [12]) If $\rho \in (0,1]$, $Re(q) > 0$, $Re(\beta) > 0$, and $v \in C([a,b])$, $b \le \infty$, we have the following:*

$$\begin{aligned}&{}_a\mathcal{I}^{q,\rho}\left({}_a\mathcal{I}^{\beta,\rho}v\right)(t) = {}_a\mathcal{I}^{\beta,\rho}({}_a\mathcal{I}^{q,\rho}v)(t) = \left({}_a\mathcal{I}^{q+\beta,\rho}v\right)(t) \\ &({}_a^R\mathcal{D}^{\beta,\rho}{}_a\mathcal{I}^{q,\rho}v)(t) = {}_a\mathcal{I}^{q-\beta,\rho}v)(t), \qquad 0 < \beta < q, \\ &({}_a^R\mathcal{D}^{q,\rho}{}_a\mathcal{I}^{q,\rho}v)(t) = v(t) \\ &{}_a\mathcal{I}^{q,\rho}({}_a^R\mathcal{D}^{q,\rho}v)(t) = v(t) - \frac{({}_a\mathcal{I}^{1-q,\rho}v)(a)}{\rho^{q-1}\Gamma(q)}e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{q-1}.\end{aligned} \tag{6}$$

Lemma 3. *(Lemma 2 [15]) Let $\rho \in (0,1]$, $q \in (0,1)$, and $y \in C([a,b],\mathbb{R})$.*

(i) Let there exist a limit $\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-q}y(t)\right) = c < \infty$. Then, $({}_a\mathcal{I}^{1-q,\rho}y)(a) = c\frac{\Gamma(q)}{\rho^{1-q}}e^{\frac{\rho-1}{\rho}a}$.

(ii) Let $({}_a\mathcal{I}^{1-q,\rho}y)(a+) = b < \infty$. If there exists the limit $\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-q}y(t)\right)$, then $\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-q}y(t)\right) = \frac{b\rho^{1-q}e^{\frac{1-\rho}{\rho}a}}{\Gamma(q)}$.

Lemma 4. *Example 4.4 in [11] The solution of the initial value problem (IVP) for the scalar linear GPRLFDE*

$$({}_a^{RL}\mathcal{D}^{q,\rho}u)(t) = \rho^q\lambda u(t) + f(t),\ \ ({}_a\mathcal{I}^{1-q,\rho}u)(a+) = u_0,\ \ q \in (0,1),\ \rho \in (0,1]$$

has a solution $v \in C_{q,\rho}([a,\infty))$ *given by*

$$\begin{aligned} u(t) &= u_0 \rho^{1-q} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{q-1} E_{q,q}(\lambda(t-a)^q) \\ &\quad + \rho^{-q} \int_a^t E_{q,q}(\lambda(t-s)^q) e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{q-1} f(s) ds, \end{aligned}$$

where $E_{q,q}(t)$ *is the Mittag-Leffler function of two parameters,* $\lambda \in \mathbb{R}$.

Corollary 1. ${}_a^{RL}\mathcal{D}^{q,\rho}(e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{q-1}) = 0, \ \ t > a.$

The proof of Corollary 1 follows from Lemma 4 with $\lambda = 0, f(t) \equiv 0$ and the equality $E_{q,q}(0) = \frac{1}{\Gamma(q)}$.

Proposition 1. *(Proposition 3.7 in [11]).* ${}_a^{RL}\mathcal{D}^{q,\rho}(e^{\frac{\rho-1}{\rho}(t-a)}) = \frac{1}{\rho^q \Gamma(1-q)} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{-q}$, $t > a$.

Remark 4. *In Theorem 2.1 [34], it is proved that tempered fractional integrals and derivatives could be theoretically expressed as an infinite series of classical Riemann–Liouville fractional integrals and derivatives. The same is true for GPFI and GPRLFD. However, the practical application of infinite series is very difficult. It requires independent study of differential equations with GPRLFD and finding applicable sufficient conditions for the properties of their solutions.*

3. Comparison Results for GPRLFD

Lemma 5. *Let* $v \in C([a,b],\mathbb{R})$, $a < b < \infty$ *be Lipschitz, and let there exist a point* $T \in (a,b]$ *such that* $v(T) = 0$, *and* $v(t) < 0$, *for* $a \le t < T$. *Then, if the GPRLFD of* v *exists for* $t = T$ *with* $q \in (0,1)$, $\rho \in (0,1]$, *then the inequality* $({}_a^{RL}\mathcal{D}^{q,\rho} v)(t)|_{t=T} \ge 0$ *holds.*

Proof. Let $H(t) = \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q} v(s)\, ds$ for $t \in [a,b]$. According to (2), we have

$$\begin{aligned} ({}_a^R\mathcal{D}^{q,\rho} v)(T) &= \frac{1}{\rho^{1-q}\Gamma(1-q)} \Big((1-\rho)H(T) + \rho \lim_{h\to 0+} \frac{H(T-h) - H(T)}{h} \Big) \\ &= \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h\to 0+} \Big((1-\rho)H(T) + \rho \frac{H(T-h) - H(T)}{h} \Big). \end{aligned} \tag{7}$$

There exists a constant $K > 0$ such that $0 > v(s) = v(s) - v(T) \ge K(s-T)$ for $s \in [T-h, T)$, $h > 0$, and

$$\begin{aligned} \int_{T-h}^T e^{\frac{1-\rho}{\rho}s}(T-s)^{-q} v(s)\, ds &\ge -K \int_{T-h}^T e^{\frac{1-\rho}{\rho}s}(T-s)^{1-q}\, ds \\ &= \frac{K e^{\frac{1-\rho}{\rho}T}}{(\frac{1-\rho}{\rho})^{2-q}} \Big(\Gamma(2-q, h\frac{1-\rho}{\rho}) - \Gamma(2-q) \Big) \equiv M(h), \end{aligned} \tag{8}$$

where $\Gamma(.,.)$ is the incomplete Gamma function and

$$\lim_{h\to 0+} \frac{\Gamma(2-q, h\frac{1-\rho}{\rho}) - \Gamma(2-q)}{h} = 0. \tag{9}$$

Thus, using $e^{\frac{\rho-1}{\rho}h}(T-h-s)^q < (T-s)^q$ for $s \in [T-h,T), h > 0, \rho \in (0,1]$, and $v(s) < 0$ on $[a,T)$ we get

$$
\begin{aligned}
&H(T-h) - H(T) = \int_a^T \Big(e^{\frac{\rho-1}{\rho}(T-h-s)}(T-h-s)^{-q} - e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\Big) v(s)\, ds \\
&- \int_{T-h}^T e^{\frac{\rho-1}{\rho}(T-s)}\Big(e^{\frac{1-\rho}{\rho}h}(T-h-s)^{-q} - (T-s)^{-q}\Big) v(s)\, ds \\
&+ \int_{T-h}^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q} v(s)\, ds \\
&\geq \int_a^T \Big(e^{\frac{\rho-1}{\rho}(T-h-s)}(T-h-s)^{-q} - e^{\frac{\rho-1}{\rho}(T-s)}(t-s)^{-q}\Big) v(s)\, ds + M(h) e^{\frac{\rho-1}{\rho}T}.
\end{aligned} \tag{10}
$$

Using (8)–(10), we obtain

$$
\begin{aligned}
&\lim_{h\to 0+} \Big((1-\rho)H(T) + \frac{\rho}{h}(H(t) - H(T-h)) \Big) \\
&\geq (1-\rho)\int_a^T e^{\frac{\rho-1}{\rho}(T-s)}(t-s)^{-q} v(s)\, ds \\
&+ \rho \int_a^T \lim_{h\to 0+} \frac{e^{\frac{\rho-1}{\rho}(T-h-s)}(T-h-s)^{-q} - e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}}{h} v(s)\, ds \\
&+ \lim_{h\to 0+} \frac{M(h)}{h}\rho e^{\frac{\rho-1}{\rho}T} \\
&= (1-\rho)\int_a^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q} v(s)\, ds + \rho\int_a^T \frac{d}{dT}\Big(e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q}\Big) v(s)\, ds \\
&= (1-\rho)\int_a^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q} v(s)\, ds \\
&+ \int_a^T \Big((\rho-1)e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-q} - q\rho e^{\frac{\rho-1}{\rho}(T-s)}(t-s)^{-1-q}\Big) v(s)\, ds \\
&= -q\rho \int_a^T e^{\frac{\rho-1}{\rho}(T-s)}(T-s)^{-1-q} v(s)\, ds > 0.
\end{aligned} \tag{11}
$$

□

Example 1. *Consider $v(t) = e^{\frac{\rho-1}{\rho}t}(t-2)$ for $t \in [0,2]$, $\rho = 0.5$. Note that $v(t) < 0$ for $t \in [0,2)$, $v(2) = 0$ and for any $q \in (0,1)$ we have*

$$
\begin{aligned}
({}_0^R\mathcal{D}^{q,\rho} v)(t)|_{t=2} &= \frac{1}{0.5^{1-q}\Gamma(1-q)}\Big(0.5\int_0^2 e^{-(2-s)}(s-2)^{-q} e^{-s}(2-s)\, ds \\
&\quad + 0.5\frac{d}{dt}\int_0^t e^{-(t-s)}(t-s)^{-q} e^{-s}(s-2)\, ds|_{t=2}\Big) \\
&= \frac{1}{0.5^{-q}\Gamma(1-q)}\Big(-e^{-2}\int_0^2 (2-s)^{1-q}\, ds + \frac{d}{dt}e^{-t}\int_0^t (t-s)^{-q}(s-2)\, ds|_{t=2}\Big) \\
&= \frac{1}{0.5^{-q}\Gamma(1-q)}\Big(-\frac{2^{2-q}}{(2-q)e^2} + \frac{d}{dt}(\frac{e^{-t}t^{1-q}(t+2q-4)}{2-3q+q^2})|_{t=2}\Big) \\
&= \frac{1}{0.5^{-q}\Gamma(1-q)}\Big(-\frac{2^{2-q}}{(2-q)e^2} + 2^{-q}\frac{4-2q^2}{(2-3q+q^2)e^2}\Big) > 0.
\end{aligned} \tag{12}
$$

Remark 5. *A similar claim to Lemma 5, but for the Riemann–Liouville fractional derivatives, is proved in [7].*

Lemma 6. *Let $g \in C([t_0, b] \times \mathbb{R}, \mathbb{R})$, the functions $\mu, \nu \in C_{q,\rho}([t_0, b], \mathbb{R})$ be Lipschitz and satisfy the inequalities*

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\mu)(t) < g(t, \mu(t)), \ \ t \in (t_0, b], \quad \lim_{t \to t_0+} \Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\mu(t) \Big) = \mu_0 \frac{\rho^{q-1}}{\Gamma(q)}, \tag{13}$$

and

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(t) \geq g(t, \nu(t)), \ \ t \in (t_0, b], \quad \lim_{t \to t_0+} \Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\nu(t) \Big) = \nu_0 \frac{\rho^{q-1}}{\Gamma(q)}. \tag{14}$$

Then, if $\mu_0 < \nu_0$, the inequality $\mu(t) < \nu(t), \ t \in (t_0, b]$ holds.

Proof. Suppose the contrary. Because $\mu_0 < \nu_0$, and the functions $e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\mu(t)$ and $e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\nu(t)$ are continuous, there exists a point $\tau \in (t_0, b]$ such that $\mu(t) < \nu(t), \ t \in [t_0, \tau)$ and $\mu(\tau) = \nu(\tau)$. According to Lemma 5, for $v = \mu - \nu, \ a = t_0$ we obtain $0 = g(\tau, \mu(\tau)) - g(\tau, \nu(\tau)) > ({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\mu)(t)|_{t=\tau} - ({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(t)|_{t=\tau} = ({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\mu - \nu)(t)|_{t=\tau} \geq 0$.

The obtained contradiction proves the claim. □

In the case when the initial condition contains the generalized proportional fractional integral, we obtain the following result.

Corollary 2. *Let $g \in C([t_0, b] \times \mathbb{R}, \mathbb{R})$, the functions $\mu, \nu \in C_{q,\rho}([t_0, b], \mathbb{R})$ be Lipschitz and satisfy the inequalities*

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\mu)(t) < g(t, \mu(t)), \ \ t \in (t_0, b], \quad ({}_{t_0}\mathcal{I}^{1-q,\rho}\mu)(t)|_{t=t_0} = \mu_0, \tag{15}$$

and

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(t) \geq g(t, \nu(t)), \ \ t \in (t_0, b], \quad ({}_{t_0}\mathcal{I}^{1-q,\rho}\nu)(t)|_{t=t_0} = \nu_0. \tag{16}$$

Then, if $\mu_0 < \nu_0$, the inequality $\mu(t) < \nu(t), \ t \in (t_0, b]$ holds.

Corollary 3. *Let the functions $\mu, \nu \in C_{q,\rho}([t_0, b], \mathbb{R})$ be Lipschitz and satisfy the inequalities*

$$\begin{aligned} &({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\mu)(t) < ({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(t), \ \ t \in (t_0, b], \\ &\lim_{t \to t_0+} \Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\mu(t) \Big) < \lim_{t \to t_0+} \Big(e^{\frac{1-\rho}{\rho}(t-t_0)}(t-t_0)^{1-q}\nu(t) \Big). \end{aligned} \tag{17}$$

Then, the inequality $\mu(t) < \nu(t), \ t \in (t_0, b]$ holds.

Lemma 7. *Let the function $\nu \in C_{q,\rho}([t_0, b], \mathbb{R})$ and $\nu^2 \in C_{q,\rho}([t_0, b], \mathbb{R})$. Then, the inequality*

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu^2)(t) \leq 2\nu(t)({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(t), \quad t \in (t_0, b] \tag{18}$$

holds.

Proof. Fix a point $T \in (t_0, b]$ and define the function $\mu(s) = (\nu(T) - \nu(s))^2$ for all $s \in [t_0, T]$. The function $(-\mu(s))$ satisfies all the conditions of Lemma 5 for $v = -\nu, \ a = t_0$, and we obtain $({}_{t_0}^{RL}\mathcal{D}^{q,\rho}(-\mu)(t)|_{t=T} \geq 0$, i.e., applying Definition 2, we get

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}(\mu)(t)|_{t=T} = \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h \to 0+} \Big((1-\rho)H(T) + \rho \frac{H(T-h) - H(T)}{h} \Big) \leq 0, \tag{19}$$

where $H(t) = \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-q}\mu(\sigma)\, d\sigma, \ t \in [t_0, b]$.

Define the functions

$$P(t) = \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-q} \nu(s)\, ds,\ t \in [t_0, b]$$

and

$$W(t) = \int_{t_0}^{t} e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-q} \nu^2(s)\, ds,\ \ t \in [t_0, b].$$

According to Definition 2, we have

$$\begin{aligned}({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(t) &= \frac{1}{\rho^{1-q}\Gamma(1-q)}\Big((1-\rho)P(t) + \rho \lim_{h\to 0+} \frac{P(t-h)-P(t)}{h}\Big) \\ &= \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h\to 0+} \Big((1-\rho)P(t) + \rho \frac{P(t-h)-P(t)}{h}\Big)\end{aligned} \tag{20}$$

and

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu^2)(t) = \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h\to 0+} \Big((1-\rho)W(t) + \rho \frac{W(t-h)-W(t)}{h}\Big). \tag{21}$$

Note

$$\nu^2(s) - 2\nu(T)\nu(s) = (\nu(T)-\nu(s))^2 - \nu^2(s) = \mu(s) - \nu^2(s) \le \mu(s),\ s \in [t_0, T], \tag{22}$$

and

$$\begin{aligned}&W(T) - 2\nu(T)P(T) = \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q}\Big(\nu^2(\sigma) - 2\nu(T)\nu(\sigma)\Big)\, d\sigma \\ &\le \int_{t_0}^{T} e^{\frac{\rho-1}{\rho}(T-s)} (T-s)^{-q}\mu(\sigma)\, d\sigma = H(T), \\ &W(T-h) - 2\nu(T)P(T-h) = \int_{t_0}^{T-h} e^{\frac{\rho-1}{\rho}(T-h-s)} (T-h-s)^{-q}\Big(\nu^2(\sigma) - 2\nu(T)\nu(\sigma)\Big)\, d\sigma \\ &\le \int_{t_0}^{T-h} e^{\frac{\rho-1}{\rho}(T-h-s)} (T-h-s)^{-q}\mu(\sigma)\, d\sigma = H(T-h).\end{aligned} \tag{23}$$

Then,

$$\begin{aligned}&({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu^2)(T) - 2\nu(T)({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\nu)(T) \\ &= \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h\to 0+} \Big((1-\rho)(W(T) - 2\nu(T)P(T)) \\ &+ \rho \frac{(W(T-h) - \nu(T)P(T-h) - (W(T) - \nu(T)P(T))}{h}\Big) \\ &= \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h\to 0+} \Big((1-\rho)(W(T) - 2\nu(T)P(T)) \\ &+ \rho \frac{(W(T-h) - \nu(T)P(T-h)) - (W(T) - \nu(T)P(T))}{h}\Big) \\ &\le \frac{1}{\rho^{1-q}\Gamma(1-q)} \lim_{h\to 0+} \Big((1-\rho)H(T) + \rho \frac{H(T-h) - H(T)}{h}\Big) \\ &= ({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\mu)(T) \le 0.\end{aligned} \tag{24}$$

Because $T \in (t_0, b]$ is an arbitrary point, the claim is proved. □

Corollary 4. *Let the functions* $v_i \in C_{q,\rho}([t_0,b],\mathbb{R})$ *and* $v_i^2 \in C_{q,\rho}([t_0,b],\mathbb{R})$, $i = 1,2,\dots,n$. *Then, the inequality*

$$({}_{t_0}^{RL}\mathcal{D}^{q,\rho}\sum_{i=1}^{n} v_i^2(\cdot))(t) \le 2\sum_{i=1}^{n} v_i(t)({}_{t_0}^{RL}\mathcal{D}^{q,\rho}v_i(\cdot))(t), \quad t \in (t_0,b] \tag{25}$$

holds.

Remark 6. *Note that several authors ([35]) used the inequality (25) for the Riemann–Liouville fractional derivative to prove the main results, citing the results from [3,36], which concern the Caputo fractional derivative.*

Remark 7. *Fractional differential operators in a variety of settings under general assumptions regarding the weighted factor were considered by Kukushkin [37], and we refer the reader to that paper for a nice overview.*

4. BAM Neural Networks Modeled by GPRLFD

The general model of the fractional-order BAM neural networks with the GPRLFD is described by the following state equations:

$$\begin{aligned}
({}_{0}^{RL}\mathcal{D}^{q,\rho}x_i)(t) &= -a_i(t)x_i(t)) + \sum_{k=1}^{m} b_{i,k}(t)f_k(y_k(t)) + I_i(t), \ \ t > 0, \ i = 1,2,\dots,n, \\
({}_{0}^{RL}\mathcal{D}^{q,\rho}y_j)(t) &= -c_j(t)y_j(t)) + \sum_{k=1}^{n} d_{j,k}(t)g_k(y_k(t)) + J_j(t), \ \ t > 0, \ j = 1,2,\dots,m,
\end{aligned} \tag{26}$$

where $x_i(t)$ and $y_j(t)$ are the state variables of the i-th neuron in the first layer at time t and the state variables of the j-th neuron in the second layer at time t, respectively, n and m are the numbers of units in the first and second layers in the neural network, ${}_{0}^{RL}\mathcal{D}^{q,\rho}$ denotes the GPRLFD of order $q \in (0,1)$, $\rho \in (0,1]$, $f_i(u)$ and $g_j(u)$ denote the activation functions, $b_{i,k}(t), d_{i,k}(t) : [0,\infty) \to \mathbb{R}$ denote the connection weight coefficients of the neurons, $a_i(t), c_j(t) : [0,\infty) \to (0,\infty)$ represent the decay coefficients of signals at time t, and $I_i(t), J_j(t)$ denotes the external inputs of the first and second layers, respectively, at time t.

The initial conditions associated with the model (26) can be written in the form

$$({}_{0}\mathcal{I}^{1-q,\rho}x_i)(t)|_{t=0} = x_i^0, \quad ({}_{0}\mathcal{I}^{1-q,\rho}y_j)(t)|_{t=0} = y_j^0, \quad i = 1,2\dots,n, \ j = 1,2,\dots,m. \tag{27}$$

Remark 8. *According to Lemma 3, the initial conditions (27) could be replaced by initial conditions of the type*

$$\lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-q}x_i(t)\right) = x_i^0\frac{\rho^{q-1}}{\Gamma(q)}, \qquad \lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-q}y_i(t)\right) = y_i^0\frac{\rho^{q-1}}{\Gamma(p)}. \tag{28}$$

The goal of this paper is to study a special type of stability of the model (26) with initial conditions (27) or their equivalent (28).

Initially, we will consider an example to discuss some properties of the solutions of equations with the generalized proportional Riemann–Liouville fractional derivative.

Example 2. *Consider the initial value problem for the scalar differential equation with GPRLFD*

$$({}_{0}^{RL}\mathcal{D}^{q,\rho}u)(t) = -u(t), \quad ({}_{0}\mathcal{I}^{1-q,\rho}u)(0+) = u_0,$$

where $q \in (0,1)$, $\rho \in (0,1]$. According to Lemma 4 with $\lambda = -\frac{1}{\rho^q}$, $f(t,u) \equiv 0$, the solution is given by

$$u(t;u_0) = u_0\rho^{1-q}e^{\frac{\rho-1}{\rho}t}t^{q-1}E_{q,q}(-(\frac{t}{\rho})^q).$$

For any nonzero initial value, we have $\lim_{t\to 0+} u(t;u_0) = \infty$ and $\lim_{t\to\infty} u(t;u_0) = 0$. Then, for any $\epsilon > 0$ there exists $T = T(\epsilon, u_0)$ such that $|u(t;u_0)| < \epsilon$ for $t > T$, but we could not find a nonzero initial value u_0 such that $|u(t;u_0)| < \epsilon$ for $t \geq 0$.

The above example illustrates that any type of stability for differential equations with GPRLFD has to be defined in a different way than those for ordinary differential equations or differential equations with the Caputo-type fractional derivative. The initial time has to be excluded. Some authors do not exclude the initial time (it is usually 0), and they do not note that order $q \in (0,1)$ of the Riemann–Liouville fractional derivative of a constant depends on the expressions t^{-q} and t^{q-1},which are not bounded for points close enough to the initial time 0 (see, for example, [38–40]). Note that the main concepts of stability of the Riemann–Liouville fractional derivative are discussed and studied in [10].

We now introduce the class Λ of Lyapunov-like functions, which will be used to investigate the stability of the model (26).

Definition 3. *Let $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. We will say that the function $V(x) : \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda(\Delta)$ if $V(x) \in C(\Delta)$ and it is locally Lipschitzian.*

Remark 9. *Lyapunov functions could be applied with the quadratic function $V(x) = \sum_{i=1}^n x_i^2$, $x = (x_1, x_2, \ldots, x_n)$ for which Corollary 4 could be applied.*

Note that some authors, when applying Lyapunov functions to fractional differential equations, use the equality $({}_{t_0}^{RL}D^q|\nu|)(t) = sign(\nu(t))({}_{t_0}^{RL}D^q\nu)(t)$ (see, for example, (31)). However, this equality is not true for all continuous functions ν.

Example 3. *Let $\nu(t) = t-1$, $t \in [0,2]$, $q = 0.3$, $t_0 = 0$. Then, for $t \in (1,2)$, we get*

$$\begin{aligned}
{}_0^{RL}D_t^{0.3}|t-1| &= \frac{1}{\Gamma(0.7)}\frac{d}{dt}\int_0^t (t-s)^{-0.3}|s-1|ds \\
&= \frac{1}{\Gamma(0.7)}\frac{d}{dt}\int_0^t (t-s)^{-0.3}sign(s-1)\ (s-1)ds \\
&= \frac{1}{\Gamma(0.7)}\frac{d}{dt}\Big(-\int_0^1 (t-s)^{-0.3}\ (s-1)ds + \int_1^t (t-s)^{-0.3}\ (s-1)ds\Big) \\
&\neq \frac{1}{\Gamma(0.7)}\frac{d}{dt}\int_0^t (t-s)^{-0.3}(s-1)ds = sign(t-1)\Big({}_0^{RL}D_t^{0.3}(t-1)\Big).
\end{aligned} \tag{29}$$

In connection with the above remark and example, we will use the quadratic function as a Lyapunov function.

We will define the equilibrium of the neural networks (26) and (27). Usually, the equilibrium is a point whose derivative is zero, and satisfies an appropriate algebraic equation. In the case where the generalized proportional derivative (Caputo or Riemann–Liouville type) is taken for a nonzero constant, then the result is not equal to zero (which is true for the ordinary derivative and the Caputo derivative). For the generalized proportional Caputo fractional derivative, the equilibrium is defined by $Ce^{\frac{\rho-1}{\rho}t}$ and studied for some types of stability in [16]. In the case of the Riemann–Liouville fractional derivative, the equilibrium is defined as a constant in [39], but because ${}_0^{RL}D_t^q 1 = \frac{t^{-q}}{\Gamma(1-q)}$, the algebraic system (12) [39] could not be satisfied for all $t \geq 0$ because the right-hand side part does not depend on t but the left-hand side part depends on the variable t^{-q}, which has no bound as $t \to 0+$.

A similar situation occurs with the GPRLFD. We will study the stability behavior of the model (26) in several cases.

4.1. General Case of the Model

Consider the model (26) in the general case, when at least one of the coefficients and the external inputs in both layers are variable in time.

4.1.1. Variable in Time Equilibrium

Applying Corollary 1 with $a = 0$, we will define the equilibrium of (26):

Definition 4. *The function $U^*(t) = (x^*(t), y^*(t)) : (0, \infty) \to \mathbb{R}^{n+m}$, where $x^*(t) = Ce^{\frac{\rho-1}{\rho}t}t^{q-1}$ and $y^*(t) = Ke^{\frac{\rho-1}{\rho}t}t^{q-1}$ with $C = (C_1, C_2, \ldots, C_n)$, $K = (K_1, K_2, \ldots, K_m)$, $C_i = const, i = 1, 2, \ldots, n$, $K_j = const, j = 1, 2, \ldots, m$, is called an equilibrium of the model of fractional order BAM neural networks (26) if the equalities*

$$
\begin{aligned}
a_i(t)C_i e^{\frac{\rho-1}{\rho}t}t^{q-1} &= \sum_{k=1}^{m} b_{i,k}(t) f_k(K_k e^{\frac{\rho-1}{\rho}t}t^{q-1}) + I_i(t),\ t \geq 0,\ \ i = 1, 2\ldots, n \\
b_j(t)K_j e^{\frac{\rho-1}{\rho}t}t^{q-1} &= \sum_{k=1}^{n} d_{j,k}(t) g_k(C_k e^{\frac{\rho-1}{\rho}t}t^{q-1}) + J_j(t),\ t \geq 0,\ \ j = 1, 2\ldots, m
\end{aligned}
\tag{30}
$$

hold.

Note that $\lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-q}U^*(t)\right) = U^0$ where $U^0 = (C, K)$, and therefore, the equilibrium $U^*(t)$ is a solution of the model (26) and (27) with $x_0 = C\frac{\Gamma(q)}{\rho^{1-q}}$ and $y_0 = K\frac{\Gamma(q)}{\rho^{1-q}}$.

Let $U^*(t)$ be an equilibrium of (26) defined by Definition 4. Consider the change of variables $u(t) = x(t) - x^*(t), v(t) = y(t) - y^*(t),\ t \geq 0$, in system (26). Then, we obtain

$$
\begin{aligned}
({}_0^{RL}\mathcal{D}^{q,\rho}u_i)(t) &= -a_i(t)u_i(t) + \sum_{k=1}^{m} b_{i,k}(t)F_k(t, v_k(t)),\ \ t > 0,\ i = 1, 2, \ldots, n, \\
({}_0^{RL}\mathcal{D}^{q,\rho}v_j)(t) &= -b_j(t)v_j(t) + \sum_{k=1}^{n} d_{j,k}(t)G_k(t, u_k(t)),\ \ t > 0,\ j = 1, 2, \ldots, m,
\end{aligned}
\tag{31}
$$

where $F_j(t, u) = f_j(u + y_j^*(t)) - f_j(y_j^*(t))$, $G_i(t, u) = g_i(u + x_i^*(t)) - g_i(x_i^*(t))$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$ for $t > 0,\ u \in \mathbb{R}$.

The initial conditions associated with the revised model (31) can be written in the form

$$
\begin{aligned}
({}_0\mathcal{I}^{1-q,\rho}u_i)(t)|_{t=0} &= x_i^0 - C_i\frac{\Gamma(q)}{\rho^{1-q}},\ \ \ i = 1, 2\ldots, n, \\
({}_0\mathcal{I}^{1-q,\rho}v_j)(t)|_{t=0} &= y_j^0 - K_j\frac{\Gamma(q)}{\rho^{1-q}},\ j = 1, 2, \ldots, m.
\end{aligned}
\tag{32}
$$

Note that the system (31) has a zero solution (with zero initial values).

Definition 5. *Let $\alpha \in (0, 1)$ and $\rho \in (0, 1]$. The equilibrium $U^*(t)$ of (26) is called* Mittag-Leffler exponentially stable in time *if there exists $T > 0$ such that, for any solution $U(t) = (x(t), y(t))$ of (26) and (27), the inequality*

$$
\|U(t) - U^*(t)\| \leq \Xi\left(\left\|v^0 - U^0\frac{\Gamma(q)}{\rho^{1-q}}\right\|\right)e^{\lambda\frac{\rho-1}{\rho}t}E_{q,q}(-\lambda t^q),\ t \geq T,
$$

holds, where $v^0 = (x^0, y^0)$, $\lambda > 0$ is a constant, and $\Xi \in C([0, \infty), [0, \infty))$, $\Xi(0) = 0$, is a given locally Lipschitz function.

Remark 10. *The Mittag-Leffler exponential stability in time of the equilibrium* $(x^*(t), y^*(t))$ *of* (26) *implies that every solution* $(x(t), y(t))$ *of the model* (26) *satisfies* $\lim_{t\to\infty} \|x(t) - x^*(t)\| = 0$, $\lim_{t\to\infty} \|y(t) - y^*(t)\| = 0$ *for any initial values.*

Theorem 1. *Let the following assumptions hold:*

1. $q \in (0,1)$ *and* $\rho \in (0,1]$.
2. *The functions* $a_i, c_j \in C(\mathbb{R}_+, (0,\infty)), b_{i,j}, d_{j,i}, I_i, J_j \in C(\mathbb{R}_+, \mathbb{R})$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$.
3. *The activation functions* $f_i, g_j \in C(\mathbb{R}, \mathbb{R})$, *and there exist positive constants* μ_i, η_j $i = 1,2,\dots,n$, *such that* $|f_i(v) - f_i(w)| \le \mu_i |v - w|$ *and* $|g_j(v) - g_j(w)| \le \eta_j |v - w|$ *for* $v, w \in \mathbb{R}$, $i = 1,2,\dots,n$, $j = 1,2,\dots,m$.
4. *There exist constants* $C_i, K_j, i = 1,2,\dots,n, \ j = 1,2,\dots,m$, *such that the algebraic system* (30) *is satisfied for all* $t \ge 0$.
5. *There exist constants* $\lambda_i, \mu_j > 0, \ i = 1,2,\dots,n, \ j = 1,2,\dots,m$, *such that the inequalities*

$$2a_i(t) - \sum_{k=1}^{m} |b_{i,k}(t)| - \eta_i^2 \sum_{j=1}^{m} |d_{j,i}(t)| \ge \lambda_i, \quad t \ge 0, \quad i = 1,2,\dots,n$$

$$2c_j(t) - \sum_{k=1}^{n} |d_{j,k}(t)| - \mu_j^2 \sum_{i=1}^{n} |b_{i,j}(t)| \ge \mu_j, \quad t \ge 0, \quad j = 1,2,\dots,m.$$

hold.

Then, the equilibrium $U^*(t) = (C_1, C_2, \dots, C_n, K_1, K_2, \dots, K_m) e^{\frac{\rho-1}{\rho} t} t^{q-1}$ *of model* (26) *is Mittag-Leffler exponentially stable.*

Remark 11. *Condition 4 of Theorem 1 guarantees the existence of the equilibrium* $U^*(t)$ *of* (26).

Proof. Consider the Lyapunov function $V(x,y) = 0.5\sum_{i=1}^{n} x_i^2 + 0.5\sum_{j=1}^{m} y_j^2$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

Let $U(\cdot) = (x(\cdot), y(\cdot)) \in \mathbb{R}^{n+m}$ be a solution of (26) and (27), and let $X(t) = x(t) - x^*(t), Y(t) = y(t) - y^*(t),\ t \ge 0$ where $U^*(\cdot) = (x^*(\cdot), y^*(\cdot))$.

Then, according to Corollary 4, we get

$$\begin{aligned}
&({}_0^{RL}\mathcal{D}^{q,\rho} V(X(\cdot), Y(\cdot)))(t) = 0.5 \sum_{i=1}^{n} ({}_0^{RL}\mathcal{D}^{q,\rho} X_i^2(\cdot))(t) + 0.5 \sum_{j=1}^{m} ({}_0^{RL}\mathcal{D}^{q,\rho} Y_j^2(\cdot))(t) \\
&\le \sum_{i=1}^{n} X_i(t) ({}_0^{RL}\mathcal{D}^{q,\rho} X_i(\cdot))(t) + \sum_{j=1}^{m} Y_j(t) ({}_0^{RL}\mathcal{D}^{q,\rho} Y_j(\cdot))(t) \\
&= \sum_{i=1}^{n} \left(-a_i(t) X_i^2(t) + \sum_{k=1}^{m} b_{i,k}(t) X_i(t) F_k(t, Y_k(t)) \right) \\
&\quad + \sum_{j=1}^{n} \left(-c_j(t) Y_j^2(t) + \sum_{k=1}^{n} dj,k(t) Y_j(t) G_k(t, X_k(t)) \right) \\
&\le \sum_{i=1}^{n} \left(-a_i(t) X_i^2(t) + \sum_{k=1}^{m} |b_{i,k}(t)| 0.5 (X_i^2(t) + F_k^2(t, Y_k(t)) \right) \\
&\quad + \sum_{j=1}^{m} \left(-c_i(t) Y_j^2(t) + \sum_{k=1}^{n} |d_{j,k}(t)| 0.5 (Y_j^2(t) + G_k^2(t, X_k(t)) \right) \\
&\le \sum_{i=1}^{n} \left(-a_i(t) + 0.5 \sum_{k=1}^{m} |b_{i,k}(t)| + 0.5 \eta_i^2 \sum_{j=1}^{m} |d_{j,i}(t)| \right) X_i^2(t) \\
&\quad + \sum_{j=1}^{m} \left(-c_j(t) + 0.5 \sum_{k=1}^{n} |d_{j,k}(t)| + 0.5 \mu_j^2 \sum_{i=1}^{n} |b_{i,j}(t)| \right) Y_j^2(t) \\
&\le -\gamma V(X(t), Y(t)),
\end{aligned} \tag{33}$$

where $\gamma = \min_{i=1,2,\dots,n,\ j=1,2,\dots,m}\{\lambda_i, \mu_j\}$.

Additionally, we have

$$\begin{aligned}
&\lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-q}V(X(t),Y(t))\right) = 0.5\lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-q}\left(\sum_{i=1}^{n}X_i^2+\sum_{j=1}^{m}Y_j^2\right)\right)\\
&= 0.5\sum_{i=1}^{n}\left(x_i^0\frac{\rho^{1-q}}{\Gamma(q)}-C_i\right)^2+0.5\sum_{j=1}^{m}\left(y_j^0\frac{\rho^{1-q}}{\Gamma(q)}-K_j\right)^2 \qquad (34)\\
&= 0.5\left(\frac{\rho^{1-q}}{\Gamma(q)}\right)^2\left(\left\|v^0-U^0\frac{\Gamma(q)}{\rho^{1-q}}\right\|\right)^2 < u_0\frac{\rho^{1-q}}{\Gamma(q)},
\end{aligned}$$

where $u_0 = \frac{\rho^{1-q}}{\Gamma(q)}\left(\left\|v^0-U^0\frac{\Gamma(q)}{\rho^{1-q}}\right\|\right)^2$, $v^0=(x^0,y^0)$, $U^0=(C,K)$.

Consider the scalar equation $({}_0^{RL}\mathcal{D}^{q,\rho}u(\cdot))(t) = -\gamma u(t)$ with the initial condition $({}_0\mathcal{I}^{1-q,\rho}u)(t)|_{t=0} = u_0$. According to Lemma 4, it has a solution

$$u(t) = u_0\rho^{1-q}e^{\frac{1-\rho}{\rho}t}t^{q-1}E_{q,q}(-\gamma(\frac{t}{\rho})^q).$$

Because $\lim_{t\to\infty}t^{q-1}=0$, there exists $T=T(q)>0$ such that $t^{q-1}\le 1$ for $t\ge T$. According to Corollary 3, we obtain for $t\ge T$

$$V(X(t),Y(t)) < u(t) \le \frac{\rho^{2-2q}}{\Gamma(q)}\left(\left\|v^0-U^0\frac{\Gamma(q)}{\rho^{1-q}}\right\|\right)^2 e^{\frac{1-\rho}{\rho}t}E_{q,q}(-\gamma(\frac{t}{\rho})^q).$$

Thus, the equilibrium $U^*(\cdot)$ is Mittag-Leffler exponentially stable with $\Xi(u) = \frac{\rho^{2-2q}}{\Gamma(q)}u^2$. □

4.1.2. Constant Equilibrium

We define the equilibrium of the model (26) as a constant vector in the form $V^* = (C_1, C_2, \dots, C_{n+m})$.

From Equation (5), using CAS Wolfram Mathematica, we obtain

$$({}_a\mathcal{D}^{q,\rho}1)(t) = \rho^q e^{\frac{\rho-1}{\rho}t}\left({}_a^{RL}D_t^q\left(e^{\frac{1-\rho}{\rho}t}\right)\right) = (1-\rho)^q\left(1-\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma(-q)}\right) \qquad (35)$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the upper incomplete gamma function. It is clear that $\lim_{t\to 0}\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma(-q)} = \infty$ and $\lim_{t\to\infty}\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma(-q)} = 0$ for $q\in(0,1)$ and $\rho\in(0,1]$.

Based on (35), we will define the constant equilibrium of (26):

Definition 6. *The constant vector $V^* = (C_1, C_2, \dots, C_{n+m})$ is called a constant equilibrium of the model of fractional order BAM neural networks (26) if the equalities*

$$\begin{aligned}
&C_i\left((1-\rho)^q\left(1-\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma(-q)}\right)+a_i(t)\right) = \sum_{k=1}^{m}b_{i,k}(t)f_k(C_{n+k})+I_i(t),\ t\ge 0,\ \ i=1,2\dots,n\\
&C_{n+j}\left((1-\rho)^q\left(1-\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma(-q)}\right)+b_j(t)\right) = \sum_{k=1}^{n}d_{j,k}(t)g_k(C_k)+J_j(t),\ t\ge 0,\ \ j=1,2\dots,m
\end{aligned} \qquad (36)$$

hold.

Note that $\lim_{t\to 0+}\left(e^{\frac{1-\rho}{\rho}t}t^{1-q}V^*\right)=0$, and therefore, the equilibrium V^* is a solution of the model (26) and (27) with $x_0=y_0=0$.

Let V^* be a constant equilibrium of (26) defined by Definition 6. Consider the change of variables $u_i(t)=x_i(t)-C_i, v_j(t)=y_j(t)-C_{n+j},\ t\geq 0$, in system (26). Then, applying (35) and (36), we obtain

$$\begin{aligned}
({}_0^{RL}\mathcal{D}^{q,\rho}u_i)(t)&=-a_i(t)u_i(t)+\sum_{k=1}^{m}b_{i,k}(t)F_k(v_k(t)),\ \ t>0,\ i=1,2,\dots,n,\\
({}_0^{RL}\mathcal{D}^{q,\rho}v_j)(t)&=-b_j(t)v_j(t)+\sum_{k=1}^{n}d_{j,k}(t)G_k(u_k(t)),\ \ t>0,\ j=1,2,\dots,m,
\end{aligned}\tag{37}$$

where $F_j(u)=f_j(u+C_{n+j})-f_j(C_{n+j}), G_i(u)=g_i(u+C_i)-g_i(C_i), u\in\mathbb{R}, i=1,2,\dots,n, j=1,2,\dots,m, u\in\mathbb{R}$.

Note that the system (31) has a zero solution (with zero initial values).

Definition 7. *Let $\alpha\in(0,1)$ and $\rho\in(0,1]$. The constant equilibrium V^* of (26) is called* Mittag-Leffler exponentially stable in time *if there exists $T>0$ such that, for any solution $U(t)=(x(t),y(t))$ of (26) and (27), the inequality*

$$\|U(t)-V^*\|\leq\Xi\Big(\Big\|v^0\Big\|\Big)e^{\lambda\frac{\rho-1}{\rho}t}E_{q,q}(-\lambda t^q),\ t\geq T,$$

holds, where $v^0=(x^0,y^0)$, $\lambda>0$ is a constant, and $\Xi\in C([0,\infty),[0,\infty))$, $\Xi(0)=0$, is a given locally Lipschitz function.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then, the constant equilibrium $V^*=(C_1,C_2,\dots,C_{n+m})$ of model (26) is Mittag-Leffler exponentially stable.*

The proof is similar to the one in Theorem 1, so we omit it.

4.2. Partial Case—Constant Coefficient and Constant Inputs in the Model

Let all coefficients in both layers, as well as the external inputs, be constants, i.e., $a_i(t)\equiv a_i, c_j(t)\equiv c_j, b_{i,k}(t)\equiv b_{i,k}, d_{j,k}(t)\equiv d_{j,k}, I_i(t)\equiv I_i,\ J_j(t)\equiv J_j,\ i=1,2,\dots,n, j=1,2,\dots,m$.

Then, for a variable in time equilibrium, the algebraic system (30) reduces to

$$\begin{aligned}
a_iC_ie^{\frac{\rho-1}{\rho}t}t^{q-1}&=\sum_{k=1}^{m}b_{i,k}f_k(K_ke^{\frac{\rho-1}{\rho}t}t^{q-1})+I_i,\ t\geq 0,\ \ i=1,2\dots,n,\\
b_jK_je^{\frac{\rho-1}{\rho}t}t^{q-1}&=\sum_{k=1}^{n}d_{j,k}g_k(C_ke^{\frac{\rho-1}{\rho}t}t^{q-1})+J_j,\ t\geq 0,\ \ j=1,2\dots,m.
\end{aligned}\tag{38}$$

The system (38) could have a solution $(C_1,C_2,\dots,C_n,K_1,\dots,K_m)$, i.e., the model (26) could have a variable in time equilibrium.

For a constant equilibrium, the algebraic system (36) reduces to

$$\begin{aligned}
C_i(1-\rho)^q\left(1-\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma((-q)}\right)&=-a_iC_i+\sum_{k=1}^{m}b_{i,k}f_k(C_{n+k})+I_i,\ t\geq 0,\ \ i=1,2\dots,n\\
C_{n+j}(1-\rho)^q\left(1-\frac{\Gamma(-q,\frac{1-\rho}{\rho}t)}{\Gamma(-q)}\right)&=-b_jC_{n+j}+\sum_{k=1}^{n}d_{j,k}g_k(C_k)+J_j,\ t\geq 0,\ \ j=1,2\dots,m
\end{aligned}\tag{39}$$

If there is no external input, i.e., $I_i = 0$, $J_j = 0$ and $f_i(0) = 0, g_j(0) = 0$, $i - 1, 2, \dots, n$, $j = 1, 2, \dots, m$, then the system (39) has a zero solution $C_k = 0$, $k = 1, 2, \dots, n + m$, i.e., the model (26) has a zero equilibrium.

If there is external input, i.e., at least one of I_i, J_j are nonzero, then the system (39) has no solution; thus, the model has no constant equilibrium.

5. Examples

Example 4. *Consider the following BAM neural networks of two layers with two neurons with the GPRLFD:*

$$\begin{aligned}
({}_0^{RL}\mathcal{D}^{\alpha,\rho}x_1)(t) &= -x_1(t) + \frac{0.1}{1+e^{-y_1(t)}} - 0.05,\\
({}_0^{RL}\mathcal{D}^{\alpha,\rho}x_2)(t) &= -\left(1 + e^{\frac{\rho-1}{\rho}t}\right)x_2(t) - e^{\frac{\rho-1}{\rho}t}\frac{1}{1+e^{-y_2(t)}} + e^{\frac{\rho-1}{\rho}t},\\
({}_0^{RL}\mathcal{D}^{\alpha,\rho}y_1)(t) &= -\left(1 + 0.5e^{\frac{\rho-1}{\rho}t}\right)y_1(t) - e^{\frac{\rho-1}{\rho}t}\frac{1}{1+e^{-x_1(t)}} + \frac{1}{1+e^{-x_2(t)}} + 0.5(e^{\frac{\rho-1}{\rho}t} - 1)\\
({}_0^{RL}\mathcal{D}^{\alpha,\rho}y_2)(t) &= -\left(1.5 + e^{\frac{\rho-1}{\rho}t}\right)y_2(t) - \frac{1}{1+e^{-y_2(t)}} + 0.5,
\end{aligned} \tag{40}$$

with coefficients $a_1(t) = 1$, $a_2(t) = 1 + e^{\frac{\rho-1}{\rho}t}$, $c_1(t) = 1 + 0.5e^{\frac{\rho-1}{\rho}t}$, $c_2(t) = 1.5 + e^{\frac{\rho-1}{\rho}t}$, *the activation functions* $f_k(u), g_k(u) = \frac{1}{1+e^{-u}} > 0$, $k = 1, 2, u \in \mathbb{R}$, *are equal to the sigmoid function with* $\mu_k = \eta_k = 0.25$, *the external inputs are given by*

$$I_1(t) = -0.05,\ I_2(t) = e^{\frac{\rho-1}{\rho}t},\ J_1(t) = 0.5(e^{\frac{\rho-1}{\rho}t} - 1),\ J_2(t) = 0.5,$$

and

$$B = \{b_{i,k}(t)\} = \begin{bmatrix} 0.1 & 0 \\ 0 & -e^{\frac{\rho-1}{\rho}t} \end{bmatrix}, \quad D = \{d_{i,k}(t)\} = \begin{bmatrix} -e^{\frac{\rho-1}{\rho}t} & 1 \\ 0 & -1 \end{bmatrix}.$$

Then, the algebraic system (30) *reduces to*

$$\begin{aligned}
a_1(t)C_1e^{\frac{\rho-1}{\rho}t}t^{q-1} &= \frac{b_{1,1}}{1+e^{-K_1e^{\frac{\rho-1}{\rho}t}t^{q-1}}} + I_1(t),\ t \geq 0,\\
a_2(t)C_2e^{\frac{\rho-1}{\rho}t}t^{q-1} &= \frac{b_{2,2}}{1+e^{-K_2e^{\frac{\rho-1}{\rho}t}t^{q-1}}} + I_2(t),\ t \geq 0,\\
c_1(t)K_1e^{\frac{\rho-1}{\rho}t}t^{q-1} &= d_{1,1}(t)\frac{1}{1+e^{-C_1e^{\frac{\rho-1}{\rho}t}t^{q-1}}} + d_{1,2}(t)\frac{1}{1+e^{-C_2e^{\frac{\rho-1}{\rho}t}t^{q-1}}} + J_1(t),\\
c_2(t)K_2e^{\frac{\rho-1}{\rho}t}t^{q-1} &= d_{2,1}(t)\frac{1}{1+e^{-C_1e^{\frac{\rho-1}{\rho}t}t^{q-1}}} + d_{2,2}(t)\frac{1}{1+e^{-C_2e^{\frac{\rho-1}{\rho}t}t^{q-1}}} + J_2(t),\ t \geq 0.
\end{aligned} \tag{41}$$

The system (41) *has a zero solution* $C_1 = C_2 = K_1 = K_2 = 0$.

Then, for $\rho \in (0, 1]$, $q \in (0, 1)$, *system* (40) *has the equilibrium* $U^*(t) = (0, 0, 0, 0)$.

Additionally, Condition 5 of Theorem 1 is satisfied because of the inequalities

$$2a_1(t) - |b_{1,1}(t)| - |b_{1,2}(t)| - \eta_1^2|d_{1,1}(t)| - \eta_2^2|d_{2,1}(t)| \geq \lambda_1 = 1.8375, \quad t \geq 0,$$

$$2a_2(t) - |b_{2,1}(t)| - |b_{2,2}(t)| - \eta_1^2|d_{1,2}(t)| - \eta_2^2|d_{2,2}(t)| \geq \lambda_2 = 1.875, \quad t \geq 0,$$

$$2c_1(t) - |d_{1,1}(t)| - |d_{1,2}(t)| - \mu_1^2|b_{1,1}(t)| + \mu_2^2|b_{2,1}(t)| \geq \mu_1 = 0.99375, \quad t \geq 0,$$

$$2c_2(t) - |d_{2,1}(t)| - |d_{2,2}(t)| - \mu_2^2|b_{1,2}(t)| + \mu_2^2|b_{2,2}(t)| \geq \mu_2 = 1, \quad t \geq 0,$$

According to Theorem 2, the zero equilibrium of (40) is Mittag-Leffler exponentially stable, i.e., every solution $(x_1(\cdot), y_2(\cdot), y_1(\cdot), y_2(\cdot))$ *of* (40) *with the initial condition*

$$({}_0\mathcal{I}^{1-q,\rho}x_i)(t)|_{t=0} = x_i^0, \quad ({}_0\mathcal{I}^{1-q,\rho}y_j)(t)|_{t=0} = y_j^0, \quad i,j = 1,2,$$

satisfies the inequality

$$\sqrt{x_1^2(t) + x_2^2(t) + y_1^2(t) + y_2^2(t)} \leq \frac{\rho^{2-2q}}{\Gamma(q)}\Big((x_1^0)^2 + (x_2^0)^2 + (y_1^0)^2 + (y_2^0)^2\Big)E_{q,q}(-\frac{0.99375}{\rho^q}t^q)$$

with $\gamma = min(1.8375, 1.875, 0.99375, 1)$.

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