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Initial Coefficients Estimates and Fekete–Szegö Inequality Problem for a General Subclass of Bi-Univalent Functions Defined by Subordination

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Abstract: In the present work, we aim to introduce and investigate a novel comprehensive subclass of normalized analytic bi-univalent functions involving Gegenbauer polynomials and the zero-truncated Poisson distribution. For functions in the aforementioned class, we find upper estimates of the second and third Taylor–Maclaurin coefficients, and then we solve the Fekete–Szegö functional problem. Moreover, by setting the values of the parameters included in our main results, we obtain several links to some of the earlier known findings.

Keywords: normalized analytic bi-univalent functions; Gegenbauer polynomials; Poisson distribution; Fekete–Szegö inequality problem

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1. Introduction

Let f be an analytic function defined on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = f'(0) - 1 = 0$. Thus, f can be written as the following series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

The class of all f functions given by (1) is denoted by $\mathcal{A}$ and the class of all f functions in $\mathcal{A}$, which are univalent, is denoted by $\mathcal{S}$ (for more details, see [1]; see also some of the recent studies [2–4]). It is well known that every f function in the class $\mathcal{S}$ has an inverse map f^{-1} given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{2}$$

Given a univalent function $f \in \mathcal{A}$. If the inverse map f^{-1} is also univalent, then f is called a bi-univalent function in $\mathbb{U}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{U}$ given by (1). For a characterization of the class Σ and some interesting examples of subclasses of the class Σ, see [5–11].

For any two analytic functions f and g in the class $\mathcal{A}$, we say $f(z) \prec g(z)$ in $\mathbb{U}$ (read f is subordinate to g) if there exists an analytic function $\omega(z)$, satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$. For more details, we refer the reader to [12–15].

The orthogonal polynomials play a central and important role in many applications in mathematics, physics, and engineering. The set of Gegenbauer polynomials is a general subclass of Jacobi polynomials. For fundamental definitions and some important properties, the readers are referred to [16–19], and for neoteric investigations that connect geometric function theory with the classical orthogonal polynomials, see [20–29].

Given $\alpha > -\frac{1}{2}$. The Gegenbauer polynomials $C_n^{\alpha}(x)$ for $n = 2, 3, \dots$ are constructed by the next recurrence relation.

$$\begin{aligned} C_0^{\alpha}(x) &= 1; \\ C_1^{\alpha}(x) &= 2\alpha x; \\ C_n^{\alpha}(x) &= \frac{1}{n}\left[2x(\alpha+n-1)C_{n-1}^{\alpha}(x) - (2\alpha+n-2)C_{n-2}^{\alpha}(x)\right]. \end{aligned} \tag{3}$$

Herein, we will use the following Gegenbauer polynomials:

$$\begin{aligned} C_0^{\alpha}(x) &= 1; \\ C_1^{\alpha}(x) &= 2\alpha x; \\ C_2^{\alpha}(x) &= 2\alpha(1+\alpha)x^2 - \alpha. \end{aligned} \tag{4}$$

Special cases of Gegenbauer polynomials are Legendre polynomials $P_n(x) = C_n^{\frac{1}{2}}(x)$ $(\alpha = \frac{1}{2})$ and Chebyshev polynomials of the second kind $U_n(x) = C_n^1(x)$ $(\alpha = 1)$.

Gegenbauer polynomials can be generated by

$$G_{\alpha}(x,z) = \frac{1}{(1-2xz+z^2)^{\alpha}},$$

where $x \in [-1,1]$ and $z \in \mathbb{U}$. Note that, when x is fixed, the generating function G_{α} is an analytic function in $\mathbb{U}$, and hence, it can be written in the form of the following Taylor–Maclaurin series:

$$G_{\alpha}(x,z) = \sum_{n=0}^{\infty} C_n^{\alpha}(x) z^n, \ z \in \mathbb{U}. \tag{5}$$

The zero-truncated Poisson distribution has found widespread use in modeling many real-life phenomena that deal only with positive enumeration. Let X be a discrete random variable that obeys the zero-truncated Poisson distribution. The probability density function of X can be written as

$$P_m(X=s) = \frac{m^s}{(e^m-1)s!}, \ s = 1,2,3,\dots,$$

where m is a positive real number representing the parameter mean.

Recently, Yousef et al. [30] introduced the following power series expansion:

$$\mathbb{F}(m,z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m-1)(n-1)!} z^n, \ z \in \mathbb{U}, m > 0.$$

Consider the analytic function f given by (1). The problem of finding the best upper estimate of the absolute value of the coefficient functional

$$\Delta_{\eta}(f) = a_3 - \eta a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\eta}{2}\left(f''(0)\right)^2\right) \tag{6}$$

is called the Fekete–Szegö problem [31]. The solution of this problem is of great interest in the geometric function theory. In the literature, there is a huge amount of results for several classes of functions that deal with the solution of the Fekete–Szegö problem (see, [32–39]).

2. The Class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$

The aim of this section is to introduce our new comprehensive subclass of normalized analytic bi-univalent functions. Recently, Yousef et al. [40] have introduced a comprehensive subclass $\mathfrak{M}_{\Sigma}^{\beta}(\mu,\lambda,\delta)$ of normalized analytic bi-univalent functions, which is defined as follows.

Definition 1. *For $\mu,\delta \geq 0, \lambda \geq 1$ and $0 \leq \beta < 1$, a function $f \in \Sigma$ given by (1) belongs to the class $\mathfrak{M}_{\Sigma}^{\beta}(\mu,\lambda,\delta)$ if the following inequalities hold true for all $z,w \in \mathbb{U}$.*

$$Re\left((1-\mu)\left(\frac{f(z)}{z}\right)^{\lambda} + \mu f'(z)\left(\frac{f(z)}{z}\right)^{\lambda-1} + \delta\zeta z f''(z)\right) > \beta$$

and

$$Re\left((1-\mu)\left(\frac{g(w)}{w}\right)^{\lambda} + \mu g'(w)\left(\frac{g(w)}{w}\right)^{\lambda-1} + \delta\zeta w g''(w)\right) > \beta,$$

where the function $g(w) = f^{-1}(w)$ is defined by (2) and $\zeta = \frac{2\mu+\lambda}{2\mu+1}$.

Consider the following linear operator

$$\Psi_m : \mathcal{A} \to \mathcal{A}$$

defined by

$$\Psi_m f(z) := \mathbb{F}(m,z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m-1)(n-1)!} a_n z^n, \ z \in \mathbb{U}, \tag{7}$$

where the character "$*$" stands for the Hadamard product of two series.

In the sequel, assume $f \in \Sigma$ given by (1), $g = f^{-1}$ given by (2), G_α given by (5), Ψ_m defined by (7), $x \in (\frac{1}{2},1]$, $\mu,\delta \geq 0, \lambda \geq 1$, and $\alpha, m > 0$.

Motivated essentially by the class in Definition 1, we aim in this work to define a novel comprehensive subclass of normalized analytic bi-univalent functions $\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$ governed by Gegenbauer polynomials and the zero-truncated Poisson distribution series.

Definition 2. *We say that $f \in \mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$, if the next conditions are verified.*

$$(1-\mu)\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda} + \mu(\Psi_m f(z))'\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda-1} + \delta\zeta z(\Psi_m f(z))'' \prec G_\alpha(x,z)$$

and

$$(1-\mu)\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda} + \mu(\Psi_m g(w))'\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda-1} + \delta\zeta w(\Psi_m g(w))'' \prec G_\alpha(x,w),$$

where $\zeta = \frac{2\mu+\lambda}{2\mu+1}$.

By setting the values of the parameters λ, μ and δ, we establish many new subclasses of the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$, as shown below.

Subclass 1. *We say that* $f \in {}^1\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda) := \mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,0)$, *if the next conditions are verified.*

$$(1-\mu)\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda} + \mu(\Psi_m f(z))'\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda-1} \prec G_{\alpha}(x,z)$$

and

$$(1-\mu)\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda} + \mu(\Psi_m g(w))'\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda-1} \prec G_{\alpha}(x,w),$$

where $\zeta = \frac{2\mu+\lambda}{2\mu+1}$.

Subclass 2. *We say that* $f \in {}^2\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\delta) := \mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,1,\delta)$, *if the next conditions are verified.*

$$(1-\mu)\frac{\Psi_m f(z)}{z} + \mu(\Psi_m f(z))' + \delta z(\Psi_m f(z))'' \prec G_{\alpha}(x,z)$$

and

$$(1-\mu)\frac{\Psi_m g(w)}{w} + \mu(\Psi_m g(w))' + \delta w(\Psi_m g(w))'' \prec G_{\alpha}(x,w).$$

The above subclass was introduced and studied by Yousef et al. [30].

Subclass 3. *We say that* $f \in {}^3\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu) := \mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,1,0)$, *if the next conditions are verified.*

$$(1-\mu)\frac{\Psi_m f(z)}{z} + \mu(\Psi_m f(z))' \prec G_{\alpha}(x,z)$$

and

$$(1-\mu)\frac{\Psi_m g(w)}{w} + \mu(\Psi_m g(w))' \prec G_{\alpha}(x,w).$$

The above subclass was introduced and studied by Amourah et al. [41].

Subclass 4. *We say that* $f \in {}^4\mathfrak{G}_{\Sigma}^{\alpha}(x) := \mathfrak{G}_{\Sigma}^{\alpha}(x,1,1,0)$, *if the next conditions are verified.*

$$(\Psi_m f(z))' \prec G_{\alpha}(x,z)$$

and

$$(\Psi_m g(w))' \prec G_{\alpha}(x,w).$$

This work is concerned with finding the upper estimates of the initial Taylor–Maclaurin coefficients ($|a_2|$ and $|a_3|$) and the absolute value of the coefficient functional $a_3 - \eta a_2^2$ of functions belonging to the subclass $\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$. To prove our results, we use the next lemma.

Lemma 1 ([42], p. 172)**.** *Given* $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$. *If for all* $z \in \mathbb{U}$ *we have* $|\omega(z)| < 1$, *then* $|\omega_1| \leq 1$ *and* $\quad |\omega_n| \leq 1 - |\omega_1|^2$, *for* $n = 2,3,\ldots$.

3. Main Results

Theorem 1. *If* $f \in \mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$, *then*

$$|a_2| \leq \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[2\alpha(2\mu+\lambda)(\lambda-1) + 2\alpha(2\mu+\lambda+6\delta\zeta)(e^m-1) - 2(1+\alpha)(\mu+\lambda+2\delta\zeta)^2\right]x^2 + (\mu+\lambda+2\delta\zeta)^2\right|}}, \quad (8)$$

and

$$|a_3| \leq \frac{4\alpha^2 x^2(e^m-1)^2}{m^2(\mu+\lambda+2\delta\zeta)^2} + \frac{4\alpha x(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)}.$$

Proof. If f belongs to the class $\mathfrak{G}_{\Sigma}^{\alpha}(x,\mu,\lambda,\delta)$, then Definition 2 asserts that we can find two analytic functions in $\mathbb{U}$, namely ω and v, satisfy $\omega(0)=0=v(0)$ and for all $z,w\in\mathbb{U}$: $|\omega(z)|<1$, $|v(w)|<1$, and

$$(1-\mu)\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda}+\mu(\Psi_m f(z))'\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda-1}+\delta\zeta z(\Psi_m f(z))''=G_{\alpha}(x,\omega(z)), \tag{9}$$

and

$$(1-\mu)\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda}+\mu(\Psi_m g(w))'\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda-1}+\delta\zeta w(\Psi_m g(w))''=G_{\alpha}(x,v(w)). \tag{10}$$

From the equalities (9) and (10), for $z,w\in\mathbb{U}$ we obtain

$$\begin{aligned}(1-\mu)\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda}+\mu(\Psi_m f(z))'\left(\frac{\Psi_m f(z)}{z}\right)^{\lambda-1}&+\delta\zeta z(\Psi_m f(z))''\\ &=1+C_1^{\alpha}(x)c_1z+\left[C_1^{\alpha}(x)c_2+C_2^{\alpha}(x)c_1^2\right]z^2+\ldots,\end{aligned} \tag{11}$$

and

$$\begin{aligned}(1-\mu)\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda}+\mu(\Psi_m g(w))'\left(\frac{\Psi_m g(w)}{w}\right)^{\lambda-1}&+\delta\zeta w(\Psi_m g(w))''\\ &=1+C_1^{\alpha}(x)d_1w+\left[C_1^{\alpha}(x)d_2+C_2^{\alpha}(x)d_1^2\right]w^2+\ldots,\end{aligned} \tag{12}$$

where

$$\omega(z)=\sum_{j=1}^{\infty}c_jz^j,\quad \text{and}\quad v(w)=\sum_{j=1}^{\infty}d_jw^j. \tag{13}$$

Referring to Lemma 1, we have

$$|c_j|\le 1\quad\text{and}\quad |d_j|\le 1\quad\text{for all}\quad j\in\mathbb{N}. \tag{14}$$

So, from Equations (11) and (12), we obtain

$$\frac{m(\mu+\lambda+2\delta\zeta)}{e^m-1}a_2=C_1^{\alpha}(x)c_1, \tag{15}$$

$$\frac{m^2(\lambda-1)(2\mu+\lambda)}{2(e^m-1)^2}a_2^2+\frac{m^2(2\mu+\lambda+6\delta\zeta)}{2(e^m-1)}a_3=C_1^{\alpha}(x)c_2+C_2^{\alpha}(x)c_1^2, \tag{16}$$

$$-\frac{m(\mu+\lambda+2\delta\zeta)}{e^m-1}a_2=C_1^{\alpha}(x)d_1, \tag{17}$$

and

$$\frac{m^2[(2\mu+\lambda)(2e^m+\lambda-3)+12\delta\zeta(e^m-1)]}{2(e^m-1)^2}a_2^2-\frac{m^2(2\mu+\lambda+6\delta\zeta)}{2(e^m-1)}a_3=C_1^{\alpha}(x)d_2+C_2^{\alpha}(x)d_1^2. \tag{18}$$

It follows from (15) and (17) that

$$c_1=-d_1, \tag{19}$$

and

$$\frac{2m^2(\mu+\lambda+2\delta\zeta)^2}{(e^m-1)^2}a_2^2=[C_1^{\alpha}(x)]^2\left(c_1^2+d_1^2\right). \tag{20}$$

Adding (16) and (18) yields

$$\frac{m^2[(2\mu+\lambda)(\lambda+e^m-2)+6\delta\zeta(e^m-1)]}{(e^m-1)^2}a_2^2 = C_1^\alpha(x)(c_2+d_2)+C_2^\alpha(x)\left(c_1^2+d_1^2\right). \tag{21}$$

Substituting the value of $\left(c_1^2+d_1^2\right)$ from (20) in the right hand side of (21), we deduce that

$$\left[\lambda-1+(e^m-1)\left(1+\frac{6\delta\zeta}{(2\mu+\lambda)}\right)-\frac{2(\mu+\lambda+2\delta\zeta)^2C_2^\alpha(x)}{(2\mu+\lambda)\left[C_1^\alpha(x)\right]^2}\right]\frac{m^2(2\mu+\lambda)}{(e^m-1)^2}a_2^2 = C_1^\alpha(x)(c_2+d_2). \tag{22}$$

Now, using (4), (14) and (22), we find that (8) holds.
Moreover, if we subtract (18) from (16), we have

$$\frac{m^2(2\mu+\lambda+6\delta\zeta)}{(e^m-1)}\left(a_3-a_2^2\right) = C_1^\alpha(x)(c_2-d_2)+C_2^\alpha(x)\left(c_1^2-d_1^2\right). \tag{23}$$

Then, in view of (19) and (20), the Equation (23) becomes

$$a_3 = \frac{(e^m-1)^2\left[C_1^\alpha(x)\right]^2}{2m^2(\mu+\lambda+2\delta\zeta)^2}\left(c_1^2+d_1^2\right)+\frac{(e^m-1)C_1^\alpha(x)}{m^2(2\mu+\lambda+6\delta\zeta)}(c_2-d_2).$$

Thus, applying (4), we conclude that

$$|a_3| \le \frac{4\alpha^2x^2(e^m-1)^2}{m^2(\mu+\lambda+2\delta\zeta)^2}+\frac{4\alpha x(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)},$$

and the proof of the theorem is complete. □

The next result regards the Fekete–Szegö functional problem for functions in the class $\mathfrak{G}_\Sigma^\alpha(x,\mu,\lambda,\delta)$.

Theorem 2. *If $f \in \mathfrak{G}_\Sigma^\alpha(x,\mu,\lambda,\delta)$, then*

$$\left|a_3-\eta a_2^2\right| \le \begin{cases} \dfrac{4\alpha x(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)}, & \text{if} \quad |\eta-1| \le M, \\[2ex] \dfrac{8\alpha^2x^3(e^m-1)^2|1-\eta|}{\left|m^2\left\{\left(2\alpha[(2\mu+\lambda)(\lambda-1)+(2\mu+\lambda+6\delta\zeta)(e^m-1)]-2(1+\alpha)(\mu+\lambda+2\delta\zeta)^2\right)x^2+(\mu+\lambda+2\delta\zeta)^2\right\}\right|}, & \text{if} \quad |\eta-1| \ge M, \end{cases}$$

where

$$M := \left|1+\frac{2\alpha x^2(2\mu+\lambda)(\lambda-1)-(2(1+\alpha)x^2-1)(\mu+\lambda+2\delta\zeta)^2}{2\alpha x^2(e^m-1)(2\mu+\lambda+6\delta\zeta)}\right|.$$

Proof. If f lies in the class $\mathfrak{G}_\Sigma^\alpha(x,\mu,\lambda,\delta)$, then from (22) and (23) we have

$$\begin{aligned} a_3-\eta a_2^2 &= (1-\eta)\frac{(e^m-1)^2\left[C_1^\alpha(x)\right]^3(c_2+d_2)}{m^2\left\{\left(C_1^\alpha(x)\right)^2[(2\mu+\lambda)(\lambda-1)+(2\mu+\lambda+6\delta\zeta)(e^m-1)]-2C_2^\alpha(x)(\mu+\lambda+2\delta\zeta)^2\right\}} \\ &\quad +\frac{(e^m-1)C_1^\alpha(x)}{m^2(2\mu+\lambda+6\delta\zeta)}(c_2-d_2) \\ &= C_1^\alpha(x)\left[\left(h(\eta)+\frac{(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)}\right)c_2+\left(h(\eta)-\frac{(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)}\right)d_2\right], \end{aligned}$$

and

$$h(\eta)=\frac{(e^m-1)^2\left[C_1^\alpha(x)\right]^2(1-\eta)}{m^2\left\{\left(C_1^\alpha(x)\right)^2[(2\mu+\lambda)(\lambda-1)+(2\mu+\lambda+6\delta\zeta)(e^m-1)]-2C_2^\alpha(x)(\mu+\lambda+2\delta\zeta)^2\right\}},$$

Then, in view of (4), we conclude that

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{4\alpha x(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)}, & \text{if} \quad 0\leq|h(\eta)|\leq\frac{(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)},\\ 4\alpha x|h(\eta)|, & \text{if} \quad |h(\eta)|\geq\frac{(e^m-1)}{m^2(2\mu+\lambda+6\delta\zeta)},\end{cases}$$

which completes the proof of Theorem 2. □

4. Consequences and Corollaries

By referring to the Subclass 1 (considering $\delta=0$), Subclass 2 (considering $\lambda=1$), Subclass 3 (considering $\lambda=1$ and $\delta=0$), and Subclass 4 (considering $\lambda=1,\delta=0$ and $\mu=1$), and from Theorems 1 and 2, we deduce the next consequences, respectively.

Setting $\delta=0$, we obtain the following corollary.

Corollary 1. *If $f\in {}^1\mathfrak{G}_\Sigma^\alpha(x,\mu,\lambda)$, then*

$$|a_2|\leq\frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[2\alpha(2\mu+\lambda)(\lambda-1)+2\alpha(2\mu+\lambda)(e^m-1)-2(1+\alpha)(\mu+\lambda)^2\right]x^2+(\mu+\lambda)^2\right|}},$$

$$|a_3|\leq\frac{4\alpha^2x^2(e^m-1)^2}{m^2(\mu+\lambda)^2}+\frac{4\alpha x(e^m-1)}{m^2(2\mu+\lambda)},$$

and

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{4\alpha x(e^m-1)}{m^2(2\mu+\lambda)}, & \textit{if} \quad |\eta-1|\leq K,\\ \frac{8\alpha^2x^3(e^m-1)^2|1-\eta|}{\left|m^2\left\{\left(2\alpha(2\mu+\lambda)[(\lambda-1)+(e^m-1)]-2(1+\alpha)(\mu+\lambda)^2\right)x^2+(\mu+\lambda)^2\right\}\right|}, & \textit{if} \quad |\eta-1|\geq K,\end{cases}$$

where

$$K:=\left|1+\frac{2\alpha x^2(2\mu+\lambda)(\lambda-1)-(2(1+\alpha)x^2-1)(\mu+\lambda)^2}{2\alpha x^2(e^m-1)(2\mu+\lambda)}\right|.$$

Next, setting $\lambda=1$ yields the following consequence.

Corollary 2 ([30]). *If $f\in {}^2\mathfrak{G}_\Sigma^\alpha(x,\mu,\delta)$, then*

$$|a_2|\leq\frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[2\alpha(1+2\mu+6\delta)(e^m-1)-2(1+\alpha)(1+\mu+2\delta)^2\right]x^2+(1+\mu+2\delta)^2\right|}},$$

$$|a_3|\leq\frac{4\alpha^2x^2(e^m-1)^2}{m^2(1+\mu+2\delta)^2}+\frac{4\alpha x(e^m-1)}{m^2(1+2\mu+6\delta)},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x(e^m-1)}{m^2(1+2\mu+6\delta)}, & \text{if} \quad |\eta - 1| \leq L, \\ \\ \frac{8\alpha^2 x^3 (e^m-1)^2 |1-\eta|}{\left|m^2\left\{\left[2\alpha(1+2\mu+6\delta)(e^m-1)-2(1+\alpha)(1+\mu+2\delta)^2\right]x^2+(1+\mu+2\delta)^2\right\}\right|}, & \text{if} \quad |\eta - 1| \geq L, \end{cases}$$

where

$$L := \left|1 - \frac{(1+\mu+2\delta)^2\left[2(1+\alpha)x^2-1\right]}{2\alpha x^2(e^m-1)(1+2\mu+6\delta)}\right|.$$

Now, setting $\lambda = 1$ and $\delta = 0$, we have the following consequence.

Corollary 3 ([41]). *If $f \in {}^3\mathfrak{G}_\Sigma^\alpha(x,\mu)$, then*

$$|a_2| \leq \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{\left|\left[2\alpha(1+2\mu)(e^m-1)-2(1+\alpha)(1+\mu)^2\right]x^2+(1+\mu)^2\right|}},$$

$$|a_3| \leq \frac{4\alpha^2 x^2(e^m-1)^2}{m^2(1+\mu)^2} + \frac{4\alpha x(e^m-1)}{m^2(1+2\mu)},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x(e^m-1)}{m^2(1+2\mu)}, & \text{if} \quad |\eta - 1| \leq M, \\ \\ \frac{8\alpha^2 x^3 (e^m-1)^2 |1-\eta|}{\left|m^2\left\{\left[2\alpha(1+2\mu)(e^m-1)-2(1+\alpha))(1+\mu)^2\right]x^2+(1+\mu)^2\right\}\right|}, & \text{if} \quad |\eta - 1| \geq M, \end{cases}$$

where

$$M := \left|1 - \frac{(1+\mu)^2\left[2(1+\alpha)x^2-1\right]}{2\alpha x^2(e^m-1)(1+2\mu)}\right|.$$

Finally, sitting $\lambda = 1$, $\delta = 0$, and $\mu = 1$, we obtain our last consequence.

Corollary 4. *If $f \in {}^4\mathfrak{G}_\Sigma^\alpha(x)$, then*

$$|a_2| \leq \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{|[6\alpha(e^m-1)-8(1+\alpha)]x^2+4|}},$$

$$|a_3| \leq \frac{\alpha^2 x^2(e^m-1)^2}{m^2} + \frac{4\alpha x(e^m-1)}{3m^2},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{4\alpha x(e^m-1)}{3m^2}, & \text{if} \quad |\eta - 1| \leq N, \\ \\ \frac{8\alpha^2 x^3 (e^m-1)^2 |1-\eta|}{\left|m^2\left\{[6\alpha(e^m-1)-8(1+\alpha)]x^2+4\right\}\right|}, & \text{if} \quad |\eta - 1| \geq N, \end{cases}$$

where

$$N := \left|1 - \frac{2\left[2(1+\alpha)x^2-1\right]}{3\alpha x^2(e^m-1)}\right|.$$

5. Conclusions

In the current investigation, we have established a new comprehensive subclass $\mathfrak{G}_\Sigma^\alpha(x,\mu,\lambda,\delta)$ of normalized analytic bi-univalent functions that involve Gegenbauer polynomials and the zero-truncated Poisson distribution series. First, we have provided the best estimates for the first initial Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, and then we solved the Fekete–Szegö inequality problem. Moreover, by setting the appropriate values of the parameters δ, λ, and μ, we obtain similar findings for the subclasses ${}^1\mathfrak{G}^{\alpha}_{\Sigma}(x, \mu, \lambda)$, ${}^2\mathfrak{G}^{\alpha}_{\Sigma}(x, \mu, \delta)$, ${}^3\mathfrak{G}^{\alpha}_{\Sigma}(x, \mu)$, and ${}^4\mathfrak{G}^{\alpha}_{\Sigma}(x)$.

The results presented in the present work will lead to many different results for the subclasses of Legendre polynomials $\mathfrak{G}^{1/2}_{\Sigma}(x, \mu, \lambda, \delta)$ and Chebyshev polynomials of the second kind $\mathfrak{G}^{1}_{\Sigma}(x, \mu, \lambda, \delta)$.

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