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Article

Third-Order Differential Subordinations Using Fractional Integral of Gaussian Hypergeometric Function

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Abstract: Sanford S. Miller and Petru T. Mocanu's theory of second-order differential subordinations was extended for the case of third-order differential subordinations by José A. Antonino and Sanford S. Miller in 2011. In this paper, new results are proved regarding third-order differential subordinations that extend the ones involving the classical second-order differential subordination theory. A method for finding a dominant of a third-order differential subordination is provided when the behavior of the function is not known on the boundary of the unit disc. Additionally, a new method for obtaining the best dominant of a third-order differential subordination is presented. This newly proposed method essentially consists of finding the univalent solution for the differential equation that corresponds to the differential subordination considered in the investigation; previous results involving third-order differential subordinations have been obtained mainly by investigating specific classes of admissible functions. The fractional integral of the Gaussian hypergeometric function, previously associated with the theory of fuzzy differential subordination, is used in this paper to obtain an interesting third-order differential subordination by involving a specific convex function. The best dominant is also provided, and the example presented proves the importance of the theoretical results involving the fractional integral of the Gaussian hypergeometric function.

Keywords: analytic function; convex function; third-order differential subordination; best dominant; fractional integral; Gaussian hypergeometric function

MSC: 30C45; 30C80; 33C05

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1. Introduction

The methods associated with the notion of differential subordination, introduced by S.S. Miller and P.T. Mocanu in [1,2], paved the way for more easily proving the already established results and inspired a tremendous amount of new studies involving the methods specific to this theory. The main features of the theory of differential subordination are given in the book of S.S. Miller and P.T. Mocanu, published in 2000 [3]. In 2011, J.A. Antonino and S.S. Miller [4] extended some of the results established for second-order differential subordinations and laid the foundation for the investigations into third-order differential subordinations. Using the results proved in [4], many studies have referred to third-order differential subordinations obtained using different operators. The Liu–Srivastava operator and meromorphic multivalent functions are considered for obtaining interesting new results regarding third-order differential subordinations in [5], generalized Mittag–Leffler functions were considered for obtaining third-order differential subordinations in studies such as [6,7], and linear operators defined using the class of meromorphic multivalent functions were applied to generate a new outcome regarding third-order differential subordinations in [8,9]. The applications of third-order differential subordinations involving the Srivastava–Attiya operator are presented in [10] and those involving a generalized Struve function are presented in [11]. Starlike functions are associated with third-order differential subordinations in [12], a linear operator defined by including ζ-generalized Hurwitz–Lerch zeta functions is used for applications regarding third-order differential subordinations in [13], and a differential operator is involved in obtaining third-order differential subordinations in [14].

The research presented in this paper continues the idea initiated in [4] of extending the basic knowledge related to the second-order differential subordinations presented in [3]. Hence, new results are established for third-order differential subordinations. The main contribution of this paper resides in providing a new means of investigation regarding third-order differential subordinations. Using the ideas found in [4], additional new results involving higher-order differential subordinations have been obtained mainly by investigating particular classes of admissible functions. The research presented here gives a new method for obtaining the best dominant of a third-order differential subordination which essentially consists of finding the univalent solutions to the differential equations that correspond to the differential subordinations considered in the investigation. Additionally, a method for finding a dominant of a third-order differential subordination is provided when the behavior on the boundary of the unit disc of the dominant is not known. Fractional calculus aspects are added to the investigation, motivated by the numerous interesting results generated by the association of fractional calculus and the third-order differential subordinations that can be seen in published papers, such as [15–17].

The general context of the research is given by the following notations and definitions.

Let $H(U)$ stand for the class of holomorphic functions in $U = \{z \in \mathbb{C} : |z| < 1\}$, the unit disk of the complex plane for which the associated notations

$$\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } \partial U = \{z \in \mathbb{C} : |z| = 1\}$$

are used. Considering a complex number a and a positive integer n, define the class of functions

$$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}$$

and use the notations $H_0 = H[0, 1]$ and $H_1 = H[1, 1]$.

Let $A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\}$ and write $A_1 = A$.

Consider $S = \{f \in A : f(z) = z + a_2 z^2 + \dots,\ z \in U\}$ to be the class of univalent functions on the unit disk U. Denote this by

$$S^* = \left\{f \in A : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0\right\}$$

which is the class of starlike functions in the unit disk U, and

$$K = \left\{f \in A : \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ f(0) = 0,\ f'(0) \neq 0,\ z \in U\right\}$$

which is the class of convex functions in the unit disk U. The following notion of subordination is considered:

Definition 1. *Let f and F be members of $H(U)$. The function f is said to be subordinate to F, written as $f \prec F$, $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ and such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$, if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$, [3,18].*

Consider $\Omega, \Delta \subset \mathbb{C}$, p to be an analytic function in U with $p(0) = a$, $a \in \mathbb{C}$, and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. In [1–3], the properties of the function have been considered in order to satisfy the following implication:

$$\{\psi(p(z), zp'(z), z^2 p''(z) : z \in U\} \subset \Omega \Rightarrow p(U) \subset \Delta. \quad (1)$$

In [4], the theory of differential subordinations has been adapted for the third-order case considering $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$ and determining the properties of the function p such that:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z) : z \in U\} \subset \Omega \Rightarrow p(U) \subset \Delta. \tag{2}$$

Definition 2. *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, and let h be univalent in U. If p is analytic in U and satisfies the third-order differential subordination*

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z) : z \in U\} \prec h(z) \tag{3}$$

then p is called a solution to the differential subordination. A univalent function q is called a dominant of the solutions of the differential subordination, or, more simply, a dominant, if $p \prec q$ for all p that satisfy (3). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all the dominants q of (3) is called the best dominant of (3). Note that the best dominant is unique up to a rotation of U [4].

Definition 3. *Let Q denote the set of functions q that is analytic and univalent on $\overline{U} \setminus E(q)$ where*

$$E(q) = \left\{\zeta \in \partial U;\ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and are such that $\mathrm{Min}|q'(\zeta)| = \rho > 0$, for $q \in \partial U \setminus E(q)$. The subclass of Q for which q(0) = a is denoted by Q(a) [4].

Definition 4. *Let Ω be a set in $\mathbb{C}$, $q \in Q$, and $n \geq 2$. The class of admissible operators $\Psi_n[\Omega, q]$ consists of those $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the admissibility condition [4]*

$$\psi(r, s, t, u; z) \notin \Omega,\ z \in U,\ r = q(w),\ s = nwq'(w) \tag{4}$$

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \geq n\left[\mathrm{Re}\frac{wq''(w)}{q'(w)} + 1\right],\ \mathrm{Re}\frac{u}{s} \geq n^2\mathrm{Re}\frac{w^2q''(w)}{q'(w)},\ w \in \partial U \setminus E(q).$$

Some other notions related to the differential subordination theory are also necessary for the investigation.

Definition 5. *A function $f \in H(U)$ is said to be close to convex if there exists a convex function $\varphi \in K$ such that [19]*

$$\mathrm{Re}\frac{f'(z)}{\varphi'(z)} > 0,\ z \in U.$$

Remark 1. *Based on Alexander's duality theorem [19], we know that $\varphi \in K$ if and only if $g(z) = z\varphi'(z) \in S^*$. We then have that a function $f \in A$ is said to be close to convex if and only if there exists a starlike function $g \in S^*$ such that [19]*

$$\mathrm{Re}\frac{f'(z)}{g(z)} > 0,\ z \in U.$$

The notion related to fractional calculus that is used for illustrating certain applications of the theoretical results proved in the next section is the fractional integral of the Gaussian hypergeometric function [20].

The Gaussian hypergeometric function is given in [3] as follows:

Definition 6. *Let $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$. The function*

$$F(a,b,c;z) = 1 + \frac{ab}{c}\cdot\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \ldots$$
$$= \sum_{k=0}^{\infty} \frac{(a)_k\cdot(b)_k}{(c)_k}\cdot\frac{z^k}{k!}$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\cdot\frac{z^k}{k!},\ z\in U, \tag{5}$$

is called a Gaussian hypergeometric function [3], where

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\ldots(d+k-1) \text{ with } (d)_0 = 1,$$

and

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt$$

with

$$\Gamma(z+1) = z\cdot\Gamma(z),\ \Gamma(1) = 1,\ \Gamma(n+1) = n!$$

The first univalence results were obtained for the Gaussian hypergeometric function in [21] and began to be associated with different operators and with the fractional integral as shown in a study published in 1997 [22]. Furthermore, the univalence properties of the Gaussian hypergeometric function were obtained by also using the convexity aspects [23–25]. In recent investigations, certain univalence conditions have been extended [26], and it has also been proved that the Gaussian hypergeometric function belongs to the class of Caratheodory functions in [27].

Definition 7. *The fractional integral of order λ $(\lambda > 0)$ is defined for a function f by the following expression:*

$$D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}dt, \tag{6}$$

where f is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{1-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$ [28,29].

Definition 8. *Let a, b, and c be complex numbers with $c \neq 0, -1, -2, \ldots$, and $\lambda > 0$. We define the fractional integral of the Gaussian hypergeometric function [20]:*

$$D_z^{-\lambda}(a,b,c;z) = \frac{1}{\Gamma(\lambda)}\int_0^{\infty}\frac{F(a,b,c;t)}{(z-t)^{1-\lambda}}dt$$
$$= \frac{1}{\Gamma(\lambda)}\int_0^z \frac{\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\cdot\frac{z^k}{k!}}{(z-t)^{1-\lambda}}dt$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(\lambda+k+1)}\cdot z^{k+\lambda},\ z\in U. \tag{7}$$

The following lemma is a necessary tool for establishing the proofs of the theorems in the next section.

Lemma 1. *Let $q \in Q(a)$ and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in U, with $p(z) \not\equiv a$*, and* $n \geq 2$*. If p is not subordinate to q, then there exist points* $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$*, and* $\zeta_0 \in \partial U \setminus E(q)$ *for which* $p(U_{r_0}) \subset q(U)$ *and* $p(z_0) = q(\zeta_0)$ *such that the following conditions are satisfied* (Antonino–Miller–Mocanu, [4,5])*:*

(i) $\operatorname{Re} \dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} \geq 0$ *and* $\left| \dfrac{zp'(z)}{q'(\zeta)} \right| \leq n$;

(ii) $z_0 p'(z_0) = n\zeta_0 q'(\zeta_0)$;

(iii) $\operatorname{Re}\left(\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq n \operatorname{Re}\left(\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right)$;

(iv) $\operatorname{Re} \dfrac{z_0^2 p'''(z_0)}{p'(z_0)} \geq n^2 \operatorname{Re} \dfrac{\zeta_0 q'''(\zeta_0)}{q'(\zeta_0)}$.

The theorems obtained in this research and presented in the next section of the paper provide extensions to the third-order differential subordinations of certain classical results established by Miller and Mocanu in [3] for the second-order differential subordination theory that are not investigated by Antonino and Miller in [4], nor by other authors. A method for finding a dominant of a third-order differential subordination is established in Theorem 1 considering the idea that the behavior of the function is not known on the boundary of U. Theorems 2 and 3 provide the means for obtaining the best dominant of a third-order differential subordination involving the function $p \in H[a, n]$ when $n = 2$ and $n > 2$, respectively. Theorem 4 provides the means for obtaining the best dominant of a third-order differential subordination involving a certain convex function. In Corollary 1, the fractional integral of the Gaussian hypergeometric function is used for studying a particular third-order differential subordination as an application for the results established in Theorem 4. A numerical example is also constructed based on this particular outcome.

2. Main Results

In the following theorems, the results that add knowledge to the development of the third-order differential subordination theory are proved by extending the well-known results of Miller and Mocanu regarding the second-order differential subordinations contained in [3].

In the following theorem, the means for finding a dominant for a third-order differential subordination are provided for the case in which the dominant q has an unknown behavior on ∂U.

Theorem 1. *Let* $h, q \in S$, $q(0) = a$*. Denote* $h_\varrho(z) = h(\varrho z)$, $q_\varrho(z) = q(\varrho z)$*. Let* $p \in H[a, n]$ *and suppose that*

$$\operatorname{Re} \frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \text{ and} \left| \frac{zp'(z)}{q'(\zeta)} \right| \leq n.$$

Let $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$*, which verifies one of the following conditions:*

(p) $\psi \in \psi_n[h, q_\rho]$ *for a certain* $\rho \in (0, 1)$*; or*

(pp) *there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \psi_n[h_\rho, q_\rho]$*, for all* $\rho \in (\rho_0, 1)$.

Let $\phi : D \to \mathbb{C}$ *be analytic in D, where* $D \subset \mathbb{C}$ *is the domain. If*

$$p(z) + [zp'(z) + z^2 p''(z) + z^3 p'''(z)]\phi(p(z))$$

is an analytic function in U, then

$$p(z) + [zp'(z) + z^2 p''(z) + z^3 p'''(z)]\phi(p(z)) \prec h(z) \tag{8}$$

implies

$$p(z) \prec q(z),\ z \in U.$$

Proof. Suppose that the conditions of Lemma 1 are satisfied by functions h and q on $\overline{U}$.

Case (p): Define $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ by the relation:

$$\psi(r, s, v, u; z) = r + [s + v + u]\phi(r). \tag{9}$$

Taking $r = p(z)$, $s = zp'(z)$, $v = z^2p''(z)$, and $u = z^3p'''(z)$, (9) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = p(z) + [zp'(z) + z^2p''(z) + z^3p'''(z)]\phi(p(z)). \tag{10}$$

By applying (10), Subordination (8) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z). \tag{11}$$

Subordination (11) can be seen as the following sets' inclusion relation:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)\} \subset h(U). \tag{12}$$

For $z = z_0$, Relation (12) becomes:

$$\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0), z_0^3p'''(z_0); z_0) \in h(U). \tag{13}$$

Suppose that $p \not\prec q_\rho(z)$, $z \in U$. In this case, Lemma 1 yields points $z_0 = r_0e^{i\theta_0}$ and $\zeta_0 \in \partial U \setminus E(q_\rho(z))$, such that

$$p(z_0) = q_\rho(z_0),\ z_0p'(z_0) = n\zeta_0q_\rho'(\zeta_0),\ z_0^2p''(z_0) = t, \text{ and } z_0^3p'''(z_0) = u.$$

Because $\psi \in \psi_n[h, q_\rho]$, ψ is an admissible function that verifies the admissibility Condition (4); hence,

$$\psi(q_\rho(\zeta_0), n\zeta_0q_\rho'(\zeta_0), t, u) \notin h(U). \tag{14}$$

For $r = p(z_0)$, $s = z_0p'(z_0)$, $t = z_0^2p''(z_0)$, and $u = z_0^3p'''(z_0)$, using Relation (14), we conclude that:

$$\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0), z_0^3p'''(z_0); z_0) \notin h(U). \tag{15}$$

Since Condition (15) contradicts Relation (13), it results that the supposition is not true, and we must have

$$p(z) \prec q_\rho(z),\ z \in U. \tag{16}$$

Since $q_\rho(z) = q(\rho z)$, $\rho \in (0, 1)$, we denote

$$w(z) = \rho q,\ w(U) = U. \tag{17}$$

Using (17), we obtain

$$q_\rho(z) \prec q(w(z)) = q(\rho z). \tag{18}$$

Using (16) and (18), we have

$$p(z) \prec q_\rho(z) \prec q(z) \Rightarrow p(z) \prec q(z),\ z \in U.$$

Case (pp): Consider $p_\rho(z) = p(\rho z)$. Then, we have:

$$\begin{gathered} zp_\rho'(z) = z[p(\rho z)]' = \rho zp'(\rho z),\ z^2p_\rho''(z) = \rho z^2p'(\rho z) + z^3\rho^2p''(\rho z), \\ z^3p_\rho'''(z) = 2z^3\rho^2p''(\rho z) + z^4\rho^3p'''(\rho z). \end{gathered} \tag{19}$$

Using $p(z) = p_\rho(z)$ in (12), we obtain:

$$\{\psi(p_\rho(z), zp'_\rho(z), z^2 p''_\rho(z), z^3 p'''_\rho(z); z)\} \subset h_\rho(U). \tag{20}$$

Applying (19) in (20), we obtain:

$$\{\psi(p(\rho z), z\rho p'(\rho z), \rho z^2 p'(\rho z) + z^3\rho^2 p''(\rho z), 2z^3\rho^2 p''(\rho z) + z^4\rho^3 p'''(\rho z); \rho z)\} \subset h_\rho(U).$$

For $w(z) = \rho z$, $w(0) = 0$, $|w(z)| < 1$, we obtain:

$$p_\rho(z) \prec q_\rho(\rho z), \text{ for all } \rho \in (0,1).$$

Using the limiting procedure when $\rho \to 1$, we obtain:

$$\lim_{\substack{\rho\to 1\\ \rho<1}} p_\rho(z) \prec \lim_{\substack{\rho\to 1\\ \rho<1}} q_\rho(\rho z),$$

hence,

$$p(z) \prec q(z).$$

□

The following theorem gives us the sufficient conditions to obtain the best dominant for a third-order differential subordination involving a function $p \in H[a, n]$, when $n = 2$.

Theorem 2. *Let $h \in S$, the function $\phi : D \to \mathbb{C}$, analytic in D, and an analytic function $p \in H[a, 2]$. Assume that the differential equation*

$$q(z) + [zq'(z) + z^2 q''(z) + z^3 q'''(z)]\phi(q(z)) = h(z), \tag{21}$$

has an analytical solution q in U, with $q(0) = a$, which verifies the conditions:

$$\operatorname{Re}\frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq 2.$$

Consider $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$, which verifies one of the following conditions:

(r) $q \in Q$ and $\psi \in \rho_2[h, q]$;
(rr) $q \in S$ and $\psi \in \psi_2[h, q_\rho]$ for a certain $\rho \in (0, 1)$;
(rr) $q \in S$ and there exists $\rho_0 \in (0, 1)$ such that $\psi \in \psi_2[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $p \in H[a, 2]$, and the function

$$p(z) + [zp'(z) + z^2 p''(z) + z^3 p'''(z)]\phi(p(z))$$

is an analytic function in U, then

$$p(z) + [zp'(z) + z^2 p''(z) + z^3 p'''(z)]\phi(p(z)) \prec h(z) \tag{22}$$

implies that

$$p(z) \prec q(z),$$

and q is the best dominant.

Proof. As assumed in Theorem 1, suppose that the conditions of Lemma 1 are satisfied by functions q and h on the closed disk $\overline{U}$. Applying Theorem 1, from Relation (22), we have that $p(z) \prec q(z)$. Since q verifies Equation (21), it results that q is a solution of Subordination (22); then, q will have as a dominant all of the dominants of this subordination. Since q is a solution of Equation (21), we conclude that q is the best dominant of the subordination given by (22). □

The following theorem gives us the possibility to obtain the best dominant for a third-order differential subordination involving a function $p \in H[a,n]$, when $n > 2$.

Theorem 3. *Let $h \in S$, let $\phi : D \to \mathbb{C}$ be an analytic function in D, and consider the analytic function $p \in H[a,n]$. Assume that the differential equation*

$$q(z) + \{nzq'(z)[n + (n-1)(n-2)] + n^2z^2q''(z)[1 + 3n^2(n-1)] + n^3z^3q'''(z)\}\phi(q(z)) = h(z) \tag{23}$$

has a solution q with $q(0) = a$ and verifies the conditions:

$$\mathrm{Re}\frac{\zeta q''(\zeta)}{q'(\zeta)} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq n.$$

Let $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$, which verifies one of the following conditions:

(r) $q \in Q$ and $\psi \in \psi_n[h,q]$;
(rr) $q \in S$ and $\psi \in \psi_n[h,q_\rho]$ for a certain $\rho \in (0,1)$;
(rrr) $q \in S$ and there exists $\rho_0 \in (0,1)$ such that $\psi \in \psi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $p \in H[a,2]$ and the function

$$p(z) + [zp'(z) + z^2p''(z) + z^3p'''(z)]\phi(p(z))$$

is an analytic function in U, then

$$p(z) + [zp'(z) + z^2p''(z) + z^3p'''(z)]\phi(p(z)) \prec h(z) \tag{24}$$

implies that

$$p(z) \prec q(z),$$

and q is the best dominant.

Proof. As assumed before, suppose that the conditions of Lemma 1 are satisfied by the functions q and h on the closed disk $\overline{U}$. From Theorems 1 and 2, using Relation (24), we have that

$$p(z) \prec q(z),\ z \in U.$$

If we take $p(z) = q(z^n)$, we can write:

$$zp'(z) = nz^nq'(z^n),\ z^2p''(z) = n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n),$$
$$z^3p'''(z) = n(n-1)(n-2)z^nq'(z^n) + 3n^2(n-1)z^{2n}q''(z^n) + n^3z^{3n}q'''(z^n).$$

If we replace $z = z^n$ in (23), we have

$$q(z^n) + [nz^nq'(z^n)(n^2 - 2n + 2) + n^2z^{2n}q''(z^n)(3n^3 - 3n^2 + 1) + n^3z^{3n}q'''(z^n)]\phi(q(z^n)) = h(z^n).$$

If we take $p(z) = q(z^n)$, then

$$\begin{aligned} &p(z) + [z'(z) + z^2p''(z) + z^3p'''(z)]\phi(p(z)) \\ &= q(z^n) + [nz^nq'(z^n)(n^2 - 2n + 2) + n^2z^{2n}q''(z^n)(3n^3 - 3n^2 + 1) \\ &+ n^3z^{3n}q'''(z^n)]\phi(q(z^n)) = h(z^n). \end{aligned} \tag{25}$$

Substituting (25) into (24), we have

$$h(z^n) \prec h(z). \tag{26}$$

From Subordination (26), we obtain that $q(z^n)$ is a dominant of Subordination (24). Since $p(U) = q(U)$, it results that q is the best dominant. $\square$

In the next theorem, the best dominant of a third-order differential subordination involving a certain convex function is established.

Theorem 4. *Let $q \in K$ and consider the functions $\theta, \phi \in H(D)$, where $D \subset \mathbb{C}$ is a domain such that $\phi(U) \subset D$ and $\phi(w) \neq 0$, $w \in q(U)$. We denote this by*

$$Q(z) = zq'(z)\phi(q(z)),\ h(z) = \theta(q(z)) + Q(z)$$

and consider that:

(i) $Q \in S^*$;

(ii) $\mathrm{Re}\dfrac{zh'(z)}{Q(z)} > 0$, $z \in U$. *If function $p(z) \in H(U)$ is given by (7) with $p(0) = q(0) = 0$;*

(iii) and it satisfies the conditions $\mathrm{Re}\dfrac{\zeta q''(\zeta)}{q'(\zeta)} > 0$ *and* $\left|\dfrac{zp'(z)}{q'(\zeta)}\right| \leq n$, *where* $z \in U$, $\zeta \in \partial U \setminus E(q)$, *then*

$$\begin{aligned} &\theta(p(z)) + zp'(z)\phi(p(z)) + z^2p''(z) + z^3p'''(z) \\ &\prec \theta(q(z)) + zq'(z)\phi(q(z)) + z^2q''(z) + z^3q'''(z) = h(z) \end{aligned} \tag{27}$$

implies

$$p(z) \prec q(z),$$

and the function q is the best dominant of the subordination given in (27).

Proof. Suppose that the conditions of Lemma 1 are satisfied by the functions h and q on $\overline{U}$. This assumption does not restrict the generality of the problem because we will otherwise be able to replace the functions $p(z)$, $q(z)$, and $h(z)$ with

$$p_r(z) = p(rz),\ q_r(z) = q(rz),\ \text{and } h_r(z) = h(rz),$$

where $0 < r < 1$, using functions that satisfy the conditions of Lemma 1 in the closed disk $\overline{U}$. From Subordination (27), using Definition 1, we have

$$\{\theta(p(z)) + zp'(z)\phi(p(z)) + z^2p''(z) + z^3p'''(z)\} \subset h(U). \tag{28}$$

For $z = z_0$, Relation (28) becomes

$$\theta(p(z_0) + z_0p'(z_0)\phi(p(z_0)) + z_0^2p''(z_0) + z_0^3p'''(z_0) \in h(U). \tag{29}$$

Since the function q is a starlike function, from Relation (ii) we have that h is a close-to-convex function, hence, a univalent function in U. We define the function $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ given by the relation

$$\psi(r, s, v, u; z) = \theta(r) + s\phi(r) + v + u,\ r, s, v, u \in \mathbb{C}. \tag{30}$$

For $r = p(z)$, $s = zp'(z)$, $t = z^2p''(z)$, $u = z^3p'''(z)$, Relation (30) becomes:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z)) = \theta(p(z)) + zp'(z)\phi(p(z)) + z^2p''(z) + z^3p'''(z). \tag{31}$$

For $r = q(z)$, $s = zq'(z)$, $v = z^2q''(z)$, $u = z^3q'''(z)$, Relation (30) becomes

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z)) = \theta(q(z)) + zq'(z)\phi(q(z)) + z^2q''(z) + z^3q'''(z). \tag{32}$$

Using (31) and (32), Subordination (27) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec \psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z). \quad (33)$$

In order to prove that $p \prec q$, Lemma 1 (Antonino–Miller–Mocanu) will be applied. For this, assume that $p \not\prec q$. Then, from Lemma 1, we have that there are points $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, and $\zeta_0 \in \partial U \setminus E(q)$ such that:

$$p(z_0) = q(\zeta_0),\ z_0p'(z_0) = n\zeta_0 q'(\zeta_0),\ t = z_0^2 p''(z_0), \text{ and } u = z_0^3 p'''(z_0), \quad (34)$$

which satisfy the inequalities:

$$\mathrm{Re}\left(\frac{z_0p''(z_0)}{p'(z_0)} + 1\right) \geq n\mathrm{Re}\left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right)$$

and

$$\mathrm{Re}\frac{z_0p''(z_0)}{p'(z_0)} \geq n^2\mathrm{Re}\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}.$$

For $r = q(\zeta_0)$, $s = n\zeta_0 q'(\zeta_0)$, t, and u, which satisfy Condition (34), from Definition (34), we obtain:

$$\psi(q(\zeta_0), n\zeta_0 q'(\zeta_0), z_0^2 q''(z_0), z_0^3 q'''(z_0); z_0) \notin h(U). \quad (35)$$

Using the relations given in (34) and (35), we have

$$\psi(q(z_0), nz_0q'(z_0), z_0^2q''(z_0), z_0^3q'''(z_0)) = \psi(p(z_0), z_0q'(z_0), z_0^2q''(z_0), z_0^3q'''(z_0)) \notin h(U). \quad (36)$$

We also have

$$\begin{aligned} &\psi(p(z_0), z_0q'(z_0), z_0^2q''(z_0), z_0^3q'''(z_0); z_0) \\ &= \theta(p(z_0)) + z_0p'(z_0)\phi(p(z_0)) + z_0^2p''(z_0) + z_0^3p'''(z_0). \end{aligned} \quad (37)$$

Using (37) in (36), we obtain

$$\theta(p(z_0)) + z_0p'(z_0)\phi(p(z_0)) + z_0^2p''(z_0) + z_0^3p'''(z_0) \notin h(U). \quad (38)$$

Relation (38) contradicts Relation (29), hence we conclude that the assumption we have made is not true, and we must have:

$$p(z) \prec q(z),\ z \in U. \quad (39)$$

On the other hand, because the function q is convex, it satisfies the equation:

$$h(z) = \theta(q(z)) + zp'(z)\phi(q(z)) + z^2q''(z) + z^3q'''(z);$$

hence, it is the best dominant of the differential Subordination (27). □

Remark 2. *Using the functions $p(z) = D_z^{-\lambda}F(a,b,c;z)$ given by (7) in Theorem 4, in Definition 8 and $q(z) = z + z^2$, which is convex in U, we have the next application given in the form of a corollary.*

Corollary 1. *Let $q(z) = z + z^2$, $q \in K$ and the functions $\theta, \phi \in H(D)$, where $D \subset \mathbb{C}$ is a domain such that $q(U) \subset D$, and $\phi(w) \neq 0$, $w \in q(U)$. We denote this by*

$$\begin{aligned} Q(z) &= z(1+2z)\phi(z+z^2), \\ h(z) &= \theta(z+z^2) + z(1+2z)\phi(z+z^2) + 2z^2 \end{aligned} \quad (40)$$

and assume that:

(i) $Q \in S^*$;

(ii) $\operatorname{Re}\dfrac{zh'(z)}{Q(z)} > 0$, $z \in U$. *If function* $D_z^{-\lambda}F(a,b,c;z) \in H(U)$ *given by* (7) *with* $D_z^{-\lambda}F(a,b,c;0) = q(0) = 0$, $D_z^{-\lambda}F(U) \subset D$, *satisfies:*

(iii) $\operatorname{Re}\dfrac{\zeta q''(\zeta)}{q'(\zeta)} = \operatorname{Re}\dfrac{2\zeta}{2\zeta+1} = 2\cdot\dfrac{2+\cos a}{5+4\cos a} > 0$ *and* $\left|\dfrac{z(D_z^{-\lambda}F(a,b,c;z))'}{2\zeta+1}\right| \le n$, *where* $z \in U$, $\zeta \in \partial U \setminus E(q)$, *then*

$$\begin{aligned}&\theta(D_z^{-\lambda}F(a,b,c;z)) + z(D_z^{-\lambda}F(a,b,c;z))'\phi(D_z^{-\lambda}F(a,b,c;z))\\&+z^2(D_z^{-\lambda}F(a,b,c;z))'' + z^3(D_z^{-\lambda}F(a,b,c;z))'''\\&\prec \theta(z+z^2) + z(1+2z)\phi(z+z^2) + 2z^2 = h(z)\end{aligned} \tag{41}$$

implies

$$D_z^{-\lambda}F(a,b,c;z) \prec q(z) = z + z^2$$

and q is the best dominant of Subordination (41).

Proof. We show that the function $q(z) = z + z^2$ is convex. For this, we calculate $q'(z) = 1 + 2z$, $q''(z) = 2$,

$$\begin{aligned}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}+1\right) &= \operatorname{Re}\left(\frac{2z}{2z+1}+1\right) = \operatorname{Re}\left(\frac{2z+1-1}{2z+1}+1\right)\\&= 2 - \operatorname{Re}\frac{1}{2z+1} = 2 - \operatorname{Re}\frac{1}{2\rho\cos\alpha + i2\rho\sin\alpha}\\&= 2 - \operatorname{Re}\frac{2\rho\cos\alpha+1}{(4\rho^2\cos^2\alpha + 4\rho\cos\alpha + 1) + 4\rho^2\sin^2\alpha}\\&= 2 - \operatorname{Re}\frac{2\rho\cos\alpha+1}{4\rho^2+4\rho\cos\alpha+1} = \frac{8\rho^2+6\rho\cos\alpha+1}{4\rho^2+4\rho\cos\alpha+1}.\end{aligned}$$

Since $\rho \to 1^-$,

$$\lim_{\rho\to 1^-}\frac{8\rho^2+6\rho\cos\alpha+1}{4\rho^2+4\rho\cos\alpha+1} = \frac{8+6\cos\alpha+1}{5+4\cos\alpha} = \frac{6+6(1+\cos\alpha)}{1+4(1+\cos\alpha)} > 0,$$

We conclude that
$$\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}+1\right) > 0,\ z \in U,$$

hence $q \in K$. Using Relation (39) from the proof of Theorem 1 for $q(z) = z + z^2$, we have

$$D_z^{-\lambda}F(a,b,c;z) \prec z + z^2,\ z \in U.$$

Since $q(z) = z + z^2$ is a univalent solution of the equation given in (40), q is the best dominant of Subordination (41). □

Example 1. *Using function* $D_z^{-\lambda}F(a,b,c;z)$ *given by* (7) *with* $\lambda = -1$, $a = -2$, $b = 1+i$, *and* $c = 1-i$, *we consider the functions*

$$p(z) = D_z^{-1}F(-2,1+i,1-i;z) = z - iz^2 + \frac{-8+6i}{30}z^3 \text{ and } q(z) = z - z^2,$$

which are convex in U, and $\theta, \phi \in H(D)$, $D \subset \mathbb{C}$, where D is a domain $D \subset q(U)$. Using Theorem 4, we obtain:

$$Q(z) = z(1-2z)\phi(z-z^2),\ h(z) = \theta(z-z^2) + z(1-2z)\phi(z-z^2) - 2z^2$$

and we assume that:

(i) $Q \in S^*$,

(ii) $\operatorname{Re}\dfrac{zh'(z)}{Q(z)} > 0,\ z \in U$. *If the function $D_z^{-1}F(-2,1+i,1-i;z) \in H(U)$, $D_z^{-1}F(-2,1+i,1-i;0) = q(0) = 0$ satisfies the conditions*

(iii) $\operatorname{Re}\dfrac{\zeta q''(\zeta)}{q'(\zeta)} = \operatorname{Re}\dfrac{-2\zeta}{1-2\zeta} > 0$ *and*

$$\left| \frac{z\left(1 - 2iz + \dfrac{(-8+6i)z^2}{30}\right)}{1-2\zeta} \right| \leq n,$$

when $z \in U$, $\zeta \in \partial U$, then

$$\begin{aligned}
&\theta\left(1 - iz^2 + \frac{(-8+6i)z^3}{30}\right) + z\left[1 - 2iz + \frac{(-8+6i)z^2}{10}\right]\phi\left(z - iz^2 + \frac{(-8+6i)z^3}{30}\right) \\
&+ z^2\left[-2i + \frac{(-8+6i)z}{5}\right] + z^3 \cdot \frac{-8+6i}{5} \\
&\prec \theta(z-z^2) + z(1-2z)\phi(z-z^2) - 2z^2 = h(z)
\end{aligned} \tag{42}$$

implies

$$z - iz^2 + \frac{(-8+6i)z^3}{30} \prec z - z^2$$

and

$q(z) = z - z^2$ *is the best dominant. Indeed,*

$$\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}\right) = \operatorname{Re}\frac{-2\zeta}{1-2\zeta} = \operatorname{Re}\frac{8\rho^2 - 8\rho\cos\alpha + 1}{4\rho^2 - 4\rho\cos\alpha + 1} > 0.$$

Since

$$\lim_{\rho \to 1^-} \frac{8\rho^2 - 8\rho\cos\alpha + 1}{4\rho^2 - 4\rho\cos\alpha + 1} = \frac{1 + 8(1-\cos\alpha)}{1 + 4(1-\cos\alpha)} > 0.$$

We show that q is convex:

$$\begin{aligned}
\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right) &= \operatorname{Re}\left(\frac{-2z}{1-2z} + 1\right) = 2 - \operatorname{Re}\frac{1}{1-2z} \\
&= 2 - \operatorname{Re}\frac{1 - 2\rho\cos\alpha}{4\rho^2 - 4\rho\cos\alpha + 1} \\
&= \frac{8\rho^2 - 6\rho\cos\alpha + 1}{4\rho^2 - 4\rho\cos\alpha + 1} > 0,
\end{aligned}$$

because

$$\lim_{\rho \to 1^-} \frac{8\rho^2 - 6\rho\cos\alpha + 1}{4\rho^2 - 4\rho\cos\alpha + 1} = \frac{8 - 6\cos\alpha + 1}{5 - 4\cos\alpha} = \frac{2 + 6(1-\cos\alpha)}{1 + 4(1-\cos\alpha)} > 0.$$

Since the conditions of Theorem 4 are satisfied, it follows that Subordination (42) implies

$$z - iz^2 + \frac{(-8+6i)z^3}{30} \prec z - z^2,\ z \in U.$$

Since $q(z) = z - z^2$ satisfies the equation

$$h(z) = \theta(z - z^2) + z(1-2z)\phi(z - z^2) - 2z^2$$

it turns out that it is the best dominant of Subordination (42).

3. Conclusions

The study presented in the paper provides a new outcome regarding the extension of the third-order differential subordination theory to several results proved by Miller and Mocanu in [3] related to the classical second-order differential subordination, which were not considered for such an extension before. The four theorems show the results obtained for third-order differential subordination following the classical results known for second-order differential subordination. In Theorem 1, a new means of finding a dominant of a third-order differential subordination is provided when the behavior of the dominant is not known on the boundary of the unit disc where it is defined. The next two theorems highlight the methods for obtaining the best dominants of third-order differential subordinations when the function p involved belongs to the class $H[a,n]$ with $n = 2$ in Theorem 2 and with $n > 2$ in Theorem 3. The proposed methods extend the known results given in [3] and have not been previously obtained. The results presented in Theorems 2 and 3 show that the problem of finding the best dominant of a third-order differential subordination is basically solved when the univalent solution of the corresponding differential equation is found. In Theorem 4, the best dominant is given for a third-order differential subordination involving a certain convex function. Next, as an application for the proved results, a nice corollary emerges when considering the fractional integral of the Gaussian hypergeometric function and a certain convex function for obtaining a particular third-order differential subordination, for which the best dominant is also provided. An example of the way the theoretical results obtained in Corollary 1 can be used is also included as a closure to this study.

The applications of the results presented in the paper may emerge in the field of inequalities involving the fractional integral of the Gaussian hypergeometric function, as suggested in [30]. Certain applications of fractional calculus are nicely presented in the introduction of [31] and can inspire future uses of the results presented in this paper involving the fractional integral of the Gaussian hypergeometric function. Additionally, applications regarding fluid mechanics can be further obtained following ideas from [32].

The results obtained in this study are going to be used further for obtaining new third-order differential subordinations since they are part of the basic knowledge related to the theory of third-order differential subordinations. Additionally, this study may inspire the use of other fractional operators to replace the fractional integral of the Gaussian hypergeometric function used as the application here. Other differential–integral operators may be associated with the study of third-order differential subordinations following this pattern. Moreover, the dual theory of third-order differential superordination can be applied to investigating similar third-order differential superordinations, as completed in [33], which can be connected to the results presented here through sandwich-type results, as seen in [34,35]. The study exposed in this paper can be extended to fourth-order differential subordinations for analytic univalent functions; certain results are already being obtained in works such as [36,37] and for multivalent functions, as seen in [38].

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