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Article

New Classes of Degenerate Unified Polynomials

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Abstract: In this paper, we introduce a class of new classes of degenerate unified polynomials and we show some algebraic and differential properties. This class includes the Appell-type classical polynomials and their most relevant generalizations. Most of the results are proved by using generating function methods and we illustrate our results with some examples.

Keywords: Bernoulli polynomials; Euler polynomials; Genocchi polynomials; Apostol-type polynomials; degenerate polynomials

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1. Introduction

The classical three polynomials, Bernoulli polynomials (BP), $B_n(x)$, Euler polynomials (EP), $E_n(x)$, and Genocchi polynomials (GP), $G_n(x)$, were introduced some centuries ago, and they have been used in different mathematical problems. Mainly in the calculus of finite differences and number theory, e.g., [1–3]. We recall that have the following exponential-generating functions

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\ |t|<2\pi,\qquad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\ |t|<\pi,$$

and

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\ |t|<\pi.$$

As a consequence of its importance many extensions for these polynomials and others with similar structures have been studied, achieving certain enthralling results [4–7]. For example, generalized Bernoulli, $B_n^{(\alpha)}(x)$, Euler, $E_n^{(\alpha)}(x)$, and Genocchi, $G_n^{(\alpha)}(x)$, polynomials of order α are given by

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!},\quad |t|<2\pi,\qquad \left(\frac{2}{e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}E_n^{\alpha}(x)\frac{t^n}{n!},\ |t|<\pi,$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}G_n^{\alpha}(x)\frac{t^n}{n!},\ |t|<\pi,$$

respectively, see [8,9]. On other hand, Apostol [10] defined and infrastructures the extended form of achieving Bernoulli polynomials and numbers, which are known as the Apostol–Bernoulli polynomials (ABP), $\mathcal{B}_n(x;\lambda)$, defined using the following generating function:

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{t^n}{n!},$$

where $|t| < 2\pi$ when $\lambda = 1$ and $|t| < |\log \lambda|$ when $\lambda \neq 1$. Motivated by this result, Srivastava and Luo in [11] (p. 292, Equation (9)), [12] (p. 917, Equation (1)) and [13] (p. 395, Equation (1.18)) introduced the Apostol–Bernoulli polynomials, $\mathcal{B}_n^{(\alpha)}(x;\lambda)$, the Apostol–Euler polynomials (AEP), $\mathcal{E}_n^{(\alpha)}(x;\lambda)$, and the Apostol–Genocchi (APG), $\mathcal{G}_n^{\alpha}(x;\lambda)$, polynomials of order α. We recall that

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \quad \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$$

and

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!},$$

holds for given particular values of the variable t. Recently, in [14] introduced and studied properties of a class of polynomials, $\mathcal{U}_n(x;\lambda;\mu)$, called unified Bernoulli–Euler polynomials of Apostol type (UBEPA) and defined by the following power series.

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t + (1-\mu)} e^{xt} = \sum_{n=0}^{\infty} \mathcal{U}_n(x;\lambda;\mu)\frac{t^n}{n!}, \tag{1}$$

where

$$\left|\ln\left(\frac{\lambda}{1-\mu}\right) + t\right| < \pi, \quad 0 \leq \mu < 1$$

and

$$\left|\ln\left(\frac{\lambda}{\mu-1}\right) + t\right| < 2\pi, \quad \text{otherwise.}$$

Note that for particular values in the parameters μ and λ, we can obtain in (1), the polynomials of Bernoulli, Euler, Apostol–Bernoulli, and Apostol–Euler. However, they do not unify the polynomials of order α, nor consider the polynomials called Frobenius–Euler (FEP), $H_n(x;u)$, that it is are defined through the generating function:

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x;u)\frac{t^n}{n!}, \ |t| < \left|\log \frac{1}{u}\right|.$$

For detail about Frobenius–Euler polynomials, see [15] and [16] (p. 2, Def. 1).

In the last decade, so-called degenerate polynomials have received great attention from several researchers due to their multiple properties and applications in science and engineering, as well as in mathematics (see [17–20]). This type of polynomials was initiated by L. Carlitz when introduced (see [21]) the degenerate Bernoulli polynomials (DBP), $\mathfrak{B}_n(x;a)$, using the following generating function

$$\frac{t}{(1+at)^{\frac{1}{a}} - 1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x;a)\frac{t^n}{n!}. \tag{2}$$

In a similar way, the degenerate Euler polynomials (DEP), $\mathfrak{E}_n(x;a)$, the degenerate Genocchi polynomials (DGP), $\mathfrak{G}_n(x;a)$, and the degenerate Frobenius–Euler polynomials (DFEP), $\mathfrak{H}_n(x;a)$, are given by means of the corresponding generating functions;

$$\frac{2}{(1+at)^{\frac{1}{a}} + 1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x;a)\frac{t^n}{n!}, \tag{3}$$

$$\frac{2t}{(1+at)^{\frac{1}{a}}+1}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{G}_n(x;a)\frac{t^n}{n!} \tag{4}$$

and

$$\frac{1-u}{(1+at)^{\frac{1}{a}}-u}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{H}_n(x;a;u)\frac{t^n}{n!}. \tag{5}$$

See [22,23]. The authors of [20] (p. 3, Equation (2.1)) introduces a unified class of the degenerate Apostol-type polynomials

$$\left(\frac{2^{\nu}t^{v}}{\gamma(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathcal{P}_n^{(\alpha)}(x;a;\gamma,\nu,v)\frac{t^n}{n!}. \tag{6}$$

Observe that, for particular parameters, v, γ, α, and ν, we obtain the polynomials (2), (3), and (4). However, it is not possible to obtain (5), immediately.

Motivated by (1) and (6), we introduce a new version of unified degenerate polynomials and numbers, that relates to all of the above polynomials mentioned. Several important recurrence relations and explicit representations for these polynomials are derived.

2. Preliminaries

Let $\mathbb{R}^*$ be the set of the non-zero real numbers and $\mathbb{R}_+$ positive real numbers. For complex sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, we recall the following identity

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}b_k. \tag{7}$$

See [24] (p. 18, Equation 0.36) and [25] (p. 463, Def. 9.4.6). Further, recursive formula for binomial coefficient (see [26] (p. 13, Equation (5))) is given by

$$\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k},\ r \in \mathbb{C},\ k \in \mathbb{Z}. \tag{8}$$

For any natural number n, the forward difference Δ is given by

$$\Delta u(n) = u(n+1) - u(n).$$

On other hand, the Taylor series for the natural logarithm (see [24] (p. 53, Equation 1.511)) is given by

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n, \quad -1 < x \leq 1. \tag{9}$$

For $a \in \mathbb{R}^*$, we recall that

$$(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty}(x|a)_n\frac{t^n}{n!}, \tag{10}$$

where

$$(x|a)_0 = 1,\ (x|a)_n = x(x-a)\ldots(x-(n-1)a),\ (n\geq 1)$$

and

$$\lim_{a\to 0}(1+at)^{\frac{x}{a}} = e^{xt}, \quad n \geq 0. \tag{11}$$

For more detail see [27].

3. New Classes of Degenerate Unified Polynomials

Given the results mentioned in Section 1, we focus our attention on new unified presentations of generalized polynomials of type Generalized Apostle type. More specifically, we define degenerate unified polynomials and study their properties using power series.

Definition 1. *Let $a \in \mathbb{R}^*$, $\theta \in \mathbb{C}$, $\alpha \in \mathbb{Z}$, $\lambda \in \mathbb{R}_+ \cup \{0\}$ and $\mu, \rho \in \mathbb{R}_+ - \{1\}$. We define the degenerate unifies given polynomials $\mathcal{U}_n^{(\alpha)}(x; a; \lambda; \mu; \theta; \rho)$ by the following power series:*

$$\left[\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\rho)}\right]^{\alpha}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha)}(x; a; \lambda; \mu; \theta; \rho)\frac{t^n}{n!}. \tag{12}$$

Furthermore, the degenerate unified numbers, denoted

$$\mathcal{U}_n^{(\alpha)}(a; \lambda; \mu; \theta; \rho),$$

are given by

$$\mathcal{U}_n^{(\alpha)}(0, a; \lambda; \mu; \theta; \rho) := \mathcal{U}_n^{(\alpha)}(a; \lambda; \mu; \theta; \rho).$$

In case $\mu = \rho$ and $\theta = 1$, we denote simply by $\mathcal{U}_n^{(\alpha)}(x; a; \lambda; \mu)$.

Remark 1. *The radius of convergence of (12) is given as:*

$$\left|\ln\left(\frac{\lambda}{1-\rho}\right)+t\right| < \pi, \quad 0 \le \rho < 1$$

and

$$\left|\ln\left(\frac{\lambda}{\rho-1}\right)+t\right| < 2\pi, \quad \textit{otherwise.}$$

Remark 2. *By (11), we have that the degenerate unified polynomials (12) converge to unified Bernoulli–Euler polynomials of Apostol type (1) when $\alpha = \theta = 1$, $\mu = \rho$ and $a \to 0$.*

$$\begin{aligned}
\lim_{a\to 0}\sum_{n=0}^{\infty}\mathcal{U}_n^{(\alpha)}(x; a; \lambda; \mu; \theta; \rho)\frac{t^n}{n!} &= \lim_{a\to 0}\left[\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\rho)}\right]^{\alpha}(1+at)^{\frac{x}{a}} \\
&= \left[\frac{2-\mu+\frac{\mu}{2}t}{\lambda e^t+(1-\mu)}\right]e^{xt} \\
&= \sum_{n=0}^{\infty}\mathcal{U}_n(x; \lambda; \mu)\frac{t^n}{n!}.
\end{aligned}$$

Remark 3. *From the Definition 1, the degenerate unified numbers $\mathcal{U}_n^{(\alpha)}(a; \lambda; \mu; \theta; \rho)$ is given by the following series:*

$$\left[\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\rho)}\right]^{\alpha} = \sum_{n=0}^{\infty}\mathcal{U}_n^{(\alpha)}(a; \lambda; \mu; \theta; \rho)\frac{t^n}{n!}.$$

The tables below (Tables 1–4) summarize the standard notation for several sub-classes degenerate unified polynomials $\mathcal{U}_n^{(\alpha)}(x; a; \lambda; \mu; \theta; \rho)$.

Table 1. Parameters for the degenerate polynomials.

Parameters	Generating Functions	Polynomials
$\mu=\rho=2,\quad \alpha=1, \lambda=1, \theta=1$	$\dfrac{t(1+at)^{\frac{x}{a}}}{(1+at)^{\frac{1}{a}}-1}=\sum_{n=0}^{\infty}\mathfrak{B}_n(x;a)\dfrac{t^n}{n!}$	The DBP
$\mu=\rho=0,\quad \alpha=\lambda=1$	$\dfrac{2(1+at)^{\frac{x}{a}}}{(1+at)^{\frac{1}{a}}+1}=\sum_{n=0}^{\infty}\mathfrak{E}_n(x;a)\dfrac{t^n}{n!}$	The DEP
$\mu=2, \alpha=\theta=1, \lambda=\rho=\frac{1}{2}$	$\dfrac{2t(1+at)^{\frac{x}{a}}}{(1+at)^{\frac{1}{a}}+1}=\sum_{n=0}^{\infty}\mathfrak{G}(x;a)\dfrac{t^n}{n!}$	the DGP
$\mu=4h, \alpha=1, \lambda=2, \theta=0, \rho=1+2h$	$\dfrac{(1-h)(1+at)^{\frac{x}{a}}}{(1+at)^{\frac{1}{a}}-h}=\sum_{n=0}^{\infty}\mathfrak{H}_n(x;a;h)\dfrac{t^n}{n!}$	The DFEP

Table 2. Parameters for the Apostol type polynomials of order α.

Parameters	Generating Functions	Polynomials
$\mu=\rho=2, \theta=1$ and $a\to 0$	$\left(\dfrac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\dfrac{t^n}{n!}$	The ABP of order α
$\mu=\rho=0$ and $a\to 0$	$\left(\dfrac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\dfrac{t^n}{n!}$	The AEP of order α
$\mu=2, \rho=\frac{1}{2}, \lambda=\frac{h}{2}, \theta=1$ and $a\to 0$	$\left(\dfrac{2t}{he^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x;h)\dfrac{t^n}{n!}$	The AGP of order α

Table 3. Parameters for the classical polynomials.

Parameters	Generating Functions	Polynomials
$\alpha=\theta=1, \mu=\rho=2, \lambda=1$ and $a\to 0$	$\dfrac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\dfrac{t^n}{n!}$	The BP
$\alpha=1, \mu=\rho=0, \lambda=1$ and $a\to 0$	$\dfrac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\dfrac{t^n}{n!}$	The EP
$\alpha=\theta=1, \lambda=\frac{1}{2}\ \mu=2, \rho=\frac{1}{2}$ and $a\to 0$	$\dfrac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\dfrac{t^n}{n!}$	The GP

Table 4. Parameters for the Frobenius–Euler polynomials.

Parameters	Generating Functions	Polynomials
$\mu=4h, \alpha=1, \theta=0, \rho=1+2h, \lambda=2$ and $a\to 0$	$\dfrac{1-h}{e^t-h}e^{xt}=\sum_{n=0}^{\infty}H_n(x;h)\dfrac{t^n}{n!}$	The FEP

For specific parameters, we calculate the firsts degenerate unified polynomials in the followings two examples (Figures 1 and 2).

Example 1. *For $\lambda=\theta=a=1$, $\alpha=2$, and $\mu=\rho=3$ the first few degenerate unified polynomials are given as:*

$$\begin{aligned}
\mathcal{U}_0^{(2)}(x;1;1;3;1;3) &= 1,\\
\mathcal{U}_1^{(2)}(x;1;1;3;1;3) &= x-1,\\
\mathcal{U}_2^{(2)}(x;1;1;3;1;3) &= x^2-3x-\frac{3}{2},\\
\mathcal{U}_3^{(2)}(x;1;1;3;1;3) &= x^3-6x^2+\frac{x}{2}-3,\\
\mathcal{U}_4^{(2)}(x;1;1;3;1;3) &= x^4-10x^3+14x^2-17x-6,\\
\mathcal{U}_5^{(2)}(x;1;1;3;1;3) &= x^5-15x^4+50x^3-90x^2+24x.
\end{aligned}$$

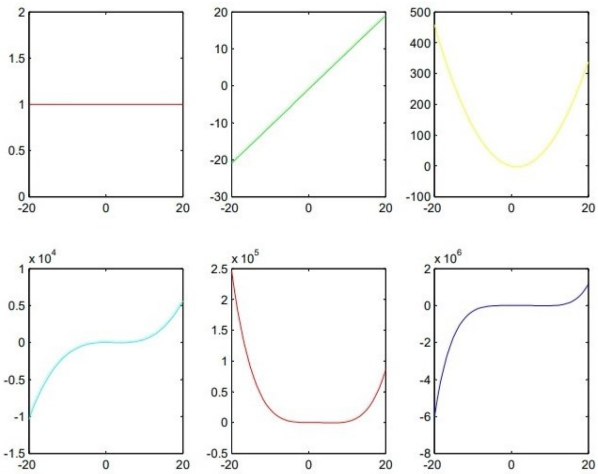

Figure 1. Polynomials of the Example 1.

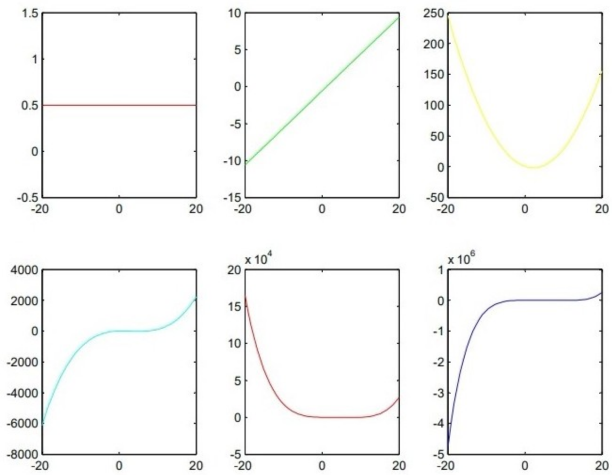

Figure 2. Polynomials of the Example 2.

Example 2. *For $\lambda = -1, \theta = 1$, $a = 2$, $\alpha = 1$, and $\mu = \rho = 4$ the first few degenerate unified polynomials are given as:*

$$\begin{aligned}
\mathcal{U}_0^{(1)}(x;2;-1;4;1;4) &= \frac{1}{2},\\
\mathcal{U}_1^{(1)}(x;2;-1;4;1;4) &= \frac{x}{2} - \frac{5}{8},\\
\mathcal{U}_2^{(1)}(x;2;-1;4;1;4) &= \frac{x^2}{2} - \frac{9x}{4} + \frac{7}{16},
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_3^{(1)}(x;2;-1;4;1;4) &= \frac{x^3}{2} - \frac{39x^2}{8} + \frac{145x}{16} - \frac{75}{64},\\
\mathcal{U}_4^{(1)}(x;2;-1;4;1;4) &= \frac{x^4}{2} - \frac{17x^3}{2} + \frac{317x^2}{8} - \frac{863x}{16} + \frac{357}{64},\\
\mathcal{U}_5^{(1)}(x;2;-1;4;1;4) &= \frac{x^5}{2} - \frac{105x^4}{8} + \frac{895x^3}{8} - \frac{12,015x^2}{32} + \frac{27,413x}{64} - \frac{9735}{256}.
\end{aligned}$$

4. Properties

In this section, we state some properties for the new classes of degenerate unified polynomials using generating function approach. Initially, we can use the generating function to develop a recurrence relation for our polynomials.

Theorem 1. *Let n be non-negative integer. For $\alpha, \beta \in \mathbb{Z}$, we have*

$$\mathcal{U}_n^{(\alpha+\beta)}(x+y;a;\lambda;\mu;\theta;\rho) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\mathcal{U}_k^{(\beta)}(y;a;\lambda;\mu;\theta;\rho).$$

Proof. Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha+\beta)}(x+y;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!} \sum_{k=0}^{\infty} \mathcal{U}_k^{(\beta)}(y;a;\lambda;\mu;\theta;\rho)\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\mathcal{U}_k^{(\beta)}(y;a;\lambda;\mu;\theta;\rho)\frac{t^n}{(n-k)!k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\mathcal{U}_k^{(\beta)}(y;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!},
\end{aligned}$$

where we used (7). Then,

$$\sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha+\beta)}(x+y;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\mathcal{U}_k^{(\beta)}(y;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!}.$$

Hence, comparing the coefficients, we obtain the result. □

Corollary 1. *Let n be non-negative integer, we have*

$$\mathcal{U}_n^{(\alpha)}(x+y;a;\lambda;\mu;\theta;\rho) = \sum_{k=0}^{n} \binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)(y|a)_k. \tag{13}$$

In particular, for $x := 0$ and $y := x$, the above relation becomes

$$\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho) = \sum_{k=0}^{n} \binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)(x|a)_k. \tag{14}$$

Proof. By (10) and (7), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha)}(x+y;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!} &= \left[\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\rho)}\right]^{\alpha}(1+at)^{\frac{x+y}{a}} \\
&= \sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!}(1+at)^{\frac{y}{a}} \\
&= \sum_{n=0}^{\infty} \mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!}\sum_{k=0}^{\infty}(y|a)_k\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^n}{(n-k)!}(y|a)_k\frac{1}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!}(y|a)_k.
\end{aligned}$$

Comparing the coefficients, we obtain (13). □

Corollary 2. *The following statements hold:*

$$\Delta\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho) = \mathcal{U}_{n+1}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)\sum_{k=1}^{n}\binom{n}{k}\mathcal{U}_{n+1-k}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)(x|a)_k.$$

Proof. By (14) and (8),

$$\begin{aligned}\Delta\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho) &= \sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{U}_{n+1-k}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)(x|a)_k \\ &\quad - \sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)(x|a)_k \\ &= \sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n+1-k}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)(x|a)_k \\ &= \mathcal{U}_{n+1}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)\sum_{k=1}^{n}\binom{n}{k}\mathcal{U}_{n+1-k}^{(\alpha)}(a;\lambda;\mu;\theta;\rho)(x|a)_k.\end{aligned}$$

□

For the following proposition, we recall that $\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\mu) := \mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta)$.

Proposition 1. *The following identities hold:*

(a) $\mathcal{U}_n^{(1)}(x;a;\lambda;\mu;1) = \left(\frac{1}{2-2\mu}\right)\left[(2-\mu)\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu};1;0\right) + \frac{n\mu}{2}\mathcal{P}_{n-1}^{(1)}\left(x;a;\frac{\lambda}{1-\mu};1;0\right)\right].$

(b) $\mathcal{U}_n^{(1)}(x;a;\lambda;\mu;\theta) = \left(\frac{1}{1-\mu}\right)\left[(1-\frac{\mu}{2})\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu};1;0\right) + \frac{\mu}{2}\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu};0;\theta\right)\right].$

Proof. We have

$$\frac{2-\mu+\frac{\mu}{2}t}{\lambda(1+at)^{\frac{1}{a}}+(1-\mu)} = \frac{2-\mu+\frac{\mu}{2}t}{2(1-\mu)}\left(\frac{2}{1+\dfrac{\lambda(1+at)^{1/a}}{1-\mu}}\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{U}_n^{(1)}(x;a;\lambda;\mu;1)\frac{t^n}{n!} &= \left(\frac{2-\mu+\frac{\mu}{2}t}{2(1-\mu)}\right)\sum_{n=0}^{\infty}\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu}\right)\frac{t^n}{n!} \\ &= \left(\frac{1}{2(1-\mu)}\right)\left[(2-\mu)\sum_{n=0}^{\infty}\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu}\right)\frac{t^n}{n!} + \frac{\mu}{2}\sum_{n=0}^{\infty}\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu}\right)\frac{t^{n+1}}{n!}\right] \\ &= \left(\frac{1}{2(1-\mu)}\right)\sum_{n=0}^{\infty}\left[(2-\mu)\mathcal{P}_n^{(1)}\left(x;a;\frac{\lambda}{1-\mu}\right) + \frac{n\mu}{2}\mathcal{P}_{n-1}^{(1)}\left(x;a;\frac{\lambda}{1-\mu}\right)\right]\frac{t^n}{n!},\end{aligned}$$

where we used (6). Thus, we obtain item (*a*). On other hand, observe that

$$\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\mu)} = \frac{1}{1-\mu}\left[\left(1-\frac{\mu}{2}\right)\frac{2}{\frac{\lambda}{1-\mu}(1+at)^{\frac{1}{a}}+1} - \frac{\mu}{2}\frac{t^{\theta}}{\frac{\lambda}{\mu-1}(1+at)^{\frac{1}{a}}-1}\right].$$

From the above, it follows (*b*). □

Proposition 2. *The following statements hold:*

$$\frac{\partial}{\partial x}\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho) = \sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\mathcal{U}_{n-1-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{nk!a^k}{k+1}.$$

Proof. By (7) and (9), we have

$$\begin{aligned}
\frac{\partial}{\partial x}\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho) &= \left[\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\rho)}\right]^{\alpha}\frac{\partial}{\partial x}(1+at)^{\frac{x}{a}} \\
&= \frac{1}{a}\left[\frac{2-\mu+\frac{\mu}{2}t^{\theta}}{\lambda(1+at)^{\frac{1}{a}}+(1-\rho)}\right]^{\alpha}(1+at)^{\frac{x}{a}}\ln(1+at) \\
&= \frac{1}{a}\left(\sum_{n=0}^{\infty}\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(at)^{n+1}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^{n+1}}{(n-k)!}\frac{(-1)^k}{k+1}a^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_{n-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{t^{n+1}}{n!}\frac{(-1)^k}{k+1}a^k k! \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}\mathcal{U}_{n-1-k}^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\frac{nk!a^k(-1)^k}{k+1}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients, we obtain the expected result. □

Proposition 3. *The degenerate unified polynomials* $\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)$*, satisfy the following relation:*

$$\left[\lambda\mathcal{U}_n^{(\alpha)}(x+1;a;\lambda;\mu;\theta;\rho)+\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\right] = 2\sum_{k=0}^{n}\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\mathcal{U}_n^{(-1)}(a;\lambda;0;\theta;0). \tag{15}$$

Proof. Using (12), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[\lambda\mathcal{U}_n^{(\alpha)}(x+1;a;\lambda;\mu)+\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu)\right]\frac{t^n}{n!} \\
&= \lambda\left[\frac{2-\mu+\frac{\mu}{2}t}{\lambda(1+at)^{\frac{1}{a}}+(1-\mu)}\right]^{\alpha}(1+at)^{\frac{x+1}{a}} + \left[\frac{2-\mu+\frac{\mu}{2}t}{\lambda(1+at)^{\frac{1}{a}}+(1-\mu)}\right]^{\alpha}(1+at)^{\frac{x}{a}} \\
&= \left[\frac{2-\mu+\frac{\mu}{2}t}{\lambda(1+at)^{\frac{1}{a}}+(1-\mu)}\right]^{\alpha}(1+at)^{\frac{x}{a}}\left(1+\lambda(1+at)^{\frac{1}{a}}\right) \\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathcal{U}_n^{(\alpha)}(x;a;\lambda;\mu;\theta;\rho)\mathcal{U}_n^{(-1)}(a;\lambda;0;\theta;0)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we obtain the identity (15) at once. □

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