

Article

Numerical Processes for Approximating Solutions of Nonlinear Equations

Samundra Regmi ^{1,*} , Ioannis K. Argyros ² , Santhosh George ³  and Christopher I. Argyros ⁴

¹ Department of Mathematics, University of Houston, Houston, TX 77204, USA

² Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA; iargyros@cameron.edu

³ Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Mangaluru 575 025, India; sgeorge@nitk.edu.in

⁴ Department of Computing and Technology, Cameron University, Lawton, OK 73505, USA; christopher.argyros@cameron.edu

* Correspondence: sregmi5@uh.edu

Abstract: In this article, we present generalized conditions of three-step iterative schemes for solving nonlinear equations. The convergence order is shown using Taylor series, but the existence of high-order derivatives is assumed. However, only the first derivative appears on these schemes. Therefore, the hypotheses limit the utilization of the schemes to operators that are at least nine times differentiable, although the schemes may converge. To the best of our knowledge, no semi-local convergence has been given in the setting of a Banach space. Our goal is to extend the applicability of these schemes in both the local and semi-local convergence cases. Moreover, we use our idea of recurrent functions and conditions only on the derivative or divided differences of order one that appear in these schemes. This idea can be applied to extend other high convergence multipoint and multistep schemes. Numerical applications where the convergence criteria are tested complement this article.

Keywords: iterative schemes; Banach space; convergence criterion

MSC: 49M15; 47H17; 65J15; 65G99; 41A25



Citation: Regmi, S.; Argyros, I.K.; George, S.; Argyros, C.I. Numerical Processes for Approximating Solutions of Nonlinear Equations. *Axioms* **2022**, *11*, 307. <https://doi.org/10.3390/axioms11070307>

Academic Editor: Clemente Cesarano

Received: 24 May 2022

Accepted: 23 June 2022

Published: 24 June 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Let M and M_1 denote Banach spaces, D stand for an open set and $F : D \subset M \rightarrow M_1$ be a continuous operator.

We denote by x^* a solution of the nonlinear equation

$$F(x) = 0. \quad (1)$$

Iterative schemes are utilized for solving the nonlinear Equation (1). A plethora of iterative schemes have been employed for approximating x^* [1,2].

In this article, we study the generalized three-step iterative schemes defined for $n = 0, 1, 2, \dots$, by

$$\begin{aligned} y_n &= x_n - M_{1,n}^{-1}F(x_n) \\ z_n &= y_n - M_{2,n}^{-1}F(y_n) \\ x_{n+1} &= z_n - M_{3,n}^{-1}F(z_n), \end{aligned} \quad (2)$$

where $M_{1,n} = M_1(x_n)$, $M_1 : D \rightarrow L(M, M_1)$, $M_{2,n} = M_2(x_n, y_n)$, $M_2 : D \times D \rightarrow L(M, M_1)$, $M_{3,n} = M_3(x_n, y_n, z_n)$, and $M_3 : D \times D \times D \rightarrow L(M, M_1)$.

This scheme generalizes numerous others already in the literature [3–5]. If, e.g.,

$$M_{1,n} = M_{2,n} = F'(x_n), \quad M_{3,n} = O, \quad (3)$$

or

$$M_{1,n} = M_{2,n} = M_{3,n} = F'(x_n) \quad (4)$$

or

$$M_{1,n} = F'(x_n), M_{2,n} = F'(y_n) \text{ and } M_{3,n} = F'(z_n), \quad (5)$$

then Newton–Traub-type methods are obtained.

The convergence order of the specialized schemes was shown to be three, five, and eight, respectively, using Taylor expansions. In the case of order three, the fourth derivative is used. Hence, the assumptions on the ninth derivative reduce the applicability of these schemes [2,4–6]. In particular, even a simple scalar equation cannot be handled with the existing results.

For example: Let $M = M_1 = \mathbb{R}$, $D = [-0.5, 1.5]$. Define scalar function λ on D by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Notice that $t^* = 1$ solves equation $\lambda(t) = 0$ and the third derivative is given by

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously, $\lambda'''(t)$ is not bounded on D . Therefore, the convergence of the scheme (2) is not guaranteed by the previous analyses in [2,4–8]. A plethora of other choices can be found in [4–8]. Therefore, it is important to study the local as well as the semi-local convergence under unifying convergence and weaker than before criteria.

There are two important types of convergence: The semi-local and the local. The semi-local is based on the information about an initial guess to provide criteria guaranteeing the convergence of the scheme; while the local one is based on the information around a solution to find estimates of the radii of the convergence balls.

The local convergence results are important, although the solution is generally unknown since the convergence order of the scheme can be determined. This type of result also demonstrates the degree of difficulty in choosing initial guesses. There are cases when the radius of convergence of the scheme can be found without knowing the solution.

As an example, let $M = M_1 = \mathbb{R}$. Suppose that function F satisfies an autonomous differential [4,6] equation of the form

$$S(F(t)) = F'(t),$$

where S is a continuous function. Notice that $S(F(t_*)) = F'(t_*)$ or $F'(t_*) = S(0)$. In the case of $F(t) = e^t - 1$, we can choose $S(t) = t + 1$ (see also the numerical section).

Moreover, the local results can apply to projection schemes such as Arnoldi's, the generalized minimum residual scheme (GMRES), the generalized conjugate scheme (GCS) for combined Newton/finite projection schemes, and in relation to the mesh independence principle to develop the cheapest and most efficient mesh refinement techniques [5,7,9].

In this article, we introduce a majorant sequence and also use our idea of recurrent functions to extend the applicability of the scheme (2). Our analysis includes error bounds and results on the uniqueness of x^* based on computable Lipschitz constants not given before in [2,4–8] and in other similar studies using the Taylor series. Our idea is very general. Therefore, it applies to other schemes too [9–14].

The rest of the article is set up as follows: In Section 2, we present the results of the local analysis. Section 3 contains the semi-local analysis, whereas in Section 4, special cases are discussed. The numerical experiments are presented in Section 5. Concluding remarks are given in the last Section 6.

2. Local Analysis

Let ℓ_1, ℓ_2 and ℓ_3 be given positive constants. Define function φ_1 on the interval $[0, \frac{1}{\ell_1})$ by

$$\varphi_1(t) = \frac{(\ell_0 + 2\ell_1)t}{2(1 - \ell_1 t)}.$$

Notice that $r_1 = \frac{2}{\ell_0 + 4\ell_1}$ solves equation $\varphi_1(t) - 1 = 0$. Set $\rho_1 = \min\{\frac{1}{\ell_1}, \frac{1}{\ell_2}\}$. Moreover, define function φ_2 on the interval $[0, \rho_1)$ by

$$\varphi_2(t) = \frac{\ell_2 + \frac{\ell_0}{2}\varphi_1(t)t}{1 - \ell_2 t}.$$

Then, $\varphi_2(0) = -1$ and $\varphi_2(t) \rightarrow \infty$ as $t \rightarrow \rho_1^-$. Denote by r_2 the minimal root of function $\varphi_2(t) - 1$ guaranteed to exist by the intermediate value theorem on the interval $(0, \rho_1)$. Furthermore, define function φ_3 on the interval $[0, \rho_2)$ by

$$\varphi_3(t) = \frac{(\ell_3 + \frac{\ell_0}{2}\varphi_2(t)t)}{1 - \ell_3 t},$$

for $\rho_2 = \min\{\frac{1}{\ell_3}, \rho_1\}$. It follows that $\varphi_3(0) = -1$ and $\varphi_3(t) \rightarrow \infty$ as $t \rightarrow \rho_2^-$. Denote by r_3 the minimal root of function $\varphi_3(t) - 1$ in the interval $(0, \rho_2)$.

We then show that r defined by

$$r = \min\{r_1, r_2, r_3\} \tag{6}$$

is a radius of convergence for scheme (2). Set $T = [0, r)$. It then follows that for all $t \in T$

$$\ell_1 t < 1, \ell_2 t < 1, \ell_3 t < 1, \tag{7}$$

$$0 \leq \varphi_1(t) < 1, \tag{8}$$

$$0 \leq \varphi_2(t) < 1, \tag{9}$$

and

$$0 \leq \varphi_3(t) < 1 \tag{10}$$

hold.

Denote by $U(x, \rho)$ the open ball with center $x \in M$ and of radius $\rho > 0$. Moreover, the ball $U[x, \rho]$ denotes the closure of the ball $U(x, \rho)$. Furthermore, by F' , we denote the Fréchet derivative of operator F .

The following conditions are needed to show the local convergence of scheme (2). Suppose:

- (A1) There exists a simple solution $x^* \in D$ of equation $F(x) = 0$.
- (A2) $\|F'(x^*)^{-1}(M_1(x) - F'(x^*))\| \leq \ell_1 \|x - x^*\|$ for all $x \in D$ and some $\ell_1 > 0$. Set $D_1 = U(x^*, \frac{1}{\ell_1}) \cap D$.
- (A3) $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \ell_0 \|x - x^*\|$ for all $x \in D_1$ and some $\ell_0 > 0$.
- (A4) $\|F'(x^*)^{-1}(M_2(x, y) - F'(x^*))\| \leq \ell_2 \|x - x^*\|$ for all $x \in D_1, y = x - F'(x)^{-1}F(x)$, and some constant $\ell_2 > 0$.
- (A5) $\|F'(x^*)^{-1}(M_3(x, y, z) - F'(x^*))\| \leq \ell_3 \|x - x^*\|$ for all $x \in D_1, z = y - M_2(x, y)^{-1}F(y)$, and some constant $\ell_3 > 0$.
- (A6) $U[x^*, r] \subset D$.

The main local convergence result follows for scheme (2).

Theorem 1. Suppose conditions (A1)–(A5) hold. Then, sequence $\{x_n\}$ produced by scheme (2) for $x_0 \in U(x^*, r) - \{x^*\}$ exists in $U(x^*, r)$, remains in $U(x^*, r)$ for all $n = 0, 1, 2, \dots$ and converges to x^* . Moreover, the following estimates hold

$$\|y_n - x^*\| \leq \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{11}$$

$$\|y_n - x^*\| \leq \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{12}$$

and

$$\|y_n - x^*\| \leq \varphi_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{13}$$

where the functions $\varphi_j, j = 1, 2, 3$ are previously defined and radius r is given by (6).

Proof. Mathematical induction is employed to show assertions (11)–(13). Let $v \in U(x^*, r) - \{x^*\}$. Using (A1) and (A2), we obtain

$$\|F'(x^*)^{-1}(M_1(v) - F'(x^*))\| \leq \ell_1\|v - x^*\| \leq \ell_1 r < 1.$$

It follows by (7) and the Banach lemma on invertible operators [2] that $M_1(v)^{-1} \in L(M_1, M)$ and

$$\|M_1(v)^{-1}F'(x^*)\| \leq \frac{1}{1 - \ell_1\|v - x^*\|}. \tag{14}$$

In particular, iterate y_0 is well defined by the first substep of method (2) and (14) for $v = x_0$. Then, we can write by this substep

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - M_{1,0}^{-1}F(x_0) \\ &= M_{1,0}^{-1}F'(x^*)F'(x^*)^{-1} \int_0^1 [(M_{1,0}(x_0) - F'(x^*)) \\ &\quad + (F'(x^*) - \int_0^1 F'(x^* + \theta(x_0 - x^*))d\theta)](x_0 - x^*), \end{aligned} \tag{15}$$

Then, in view of estimate (15) (for $v = x_0$), conditions (A1), (A2), (A3), and identity (15), we get

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{(\ell_1\|x_0 - x^*\| + \frac{\ell_0}{2}\|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \ell_1\|x_0 - x^*\|} \\ &= \frac{(\ell_0 + 2\ell_1)\|x_0 - x^*\|^2}{2(1 - \ell_1\|x_0 - x^*\|)} \\ &= \varphi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < r, \end{aligned} \tag{16}$$

where we also used identity

$$F'(x^*)(x_0 - x^*) - [F(x_0) - F(x^*)] = [F'(x^*) - \int_0^1 F'(x_0 + \theta(x_0 - x^*))d\theta](x_0 - x^*),$$

since $F(x^*) = 0$,

$$\|F'(x^*)^{-1}(M_{1,0}(x_0) - F'(x^*))\| \leq \ell_1\|x_1 - x^*\|,$$

and

$$\|F'(x^*)^{-1}(F'(x^*) - \int_0^1 F'(x_0 + \theta(x_0 - x^*))d\theta)\| \leq \frac{\ell}{2}\|x_0 - x^*\|,$$

and the triangle inequality. It follows from (16), that iterate $y_0 \in U(x^*, r)$, and (11) holds for $n = 0$. Then, using condition (A4),

$$\|F'(x^*)^{-1}(M_{2,0} - F'(x^*))\| \leq \ell_2\|x_0 - x^*\| \leq \ell_2 r < 1.$$

That is $M_{2,0}^{-1} \in L(M_1, M)$,

$$\|M_{2,0}^{-1}F'(x^*)\| \leq \frac{1}{1 - \ell_2\|x_0 - x^*\|} \tag{17}$$

and iterate z_0 exists by the second substep of method (2) for $n = 0$. Then, similarly to the derivation of identity (15), we can also write by this substep

$$\begin{aligned} z_0 - x^* &= y_0 - x^* - M_{2,0}^{-1}F(y_0) \\ &= M_{2,0}^{-1}[(M_{2,0} - F'(x^*)) \\ &\quad + (F'(x^*) - \int_0^1 F'(x^* + \theta(y_0 - x^*))d\theta)](x_0 - x^*). \end{aligned} \tag{18}$$

Then, as in the derivation of estimate (16) but using (17), (A2) and (A4), we obtain

$$\begin{aligned} \|z_0 - x^*\| &\leq \frac{(\ell_2\|x_0 - x^*\| + \frac{\ell_0}{2}\|y_0 - x^*\|)\|x_0 - x^*\|}{1 - \ell_2\|x_0 - x^*\|} \\ &\leq \varphi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|, \end{aligned} \tag{19}$$

Hence, iterate $z_0 \in U(x_0, t^*)$ and (12) holds for $n = 0$. Then, by using (A5), we obtain

$$\|F'(x^*)^{-1}(M_{3,0} - F'(x^*))\| \leq \ell_3\|x_0 - x^*\| \leq \ell_3r < 1$$

Therefore, it follows that $M_{3,0}^{-1} \in L(M_1, M)$,

$$\|M_{3,0}^{-1}F'(x^*)\| \leq \frac{1}{1 - \ell_3\|x_0 - x^*\|} \tag{20}$$

and iterate x_1 is well defined by the third substep of scheme (2) for $n = 0$. Furthermore, by this substep as in (15), we obtain the identity

$$x_1 - x^* = M_{3,0}^{-1}[(M_{3,0} - F'(x^*)) + (F'(x^*) - \int_0^1 F'(x^* + \theta(z_0 - x^*))d\theta)](x_0 - x^*). \tag{21}$$

Then, using (20), (21), (A3) and (A5) as in (16), we have

$$\begin{aligned} \|x_1 - x^*\| &\leq \frac{(\ell_3\|x_0 - x^*\| + \frac{\ell_0}{2}\|z_0 - x^*\|)\|x_0 - x^*\|}{1 - \ell_2\|x_0 - x^*\|} \\ &= \varphi_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{22}$$

It then follows by estimate (22) that iterate $x_1 \in U(x^*, r)$ and (13) holds for $n = 0$. Therefore, the induction for assertions (11)–(13) is completed if x_i, y_i, z_i, x_{i+1} replace x_0, y_0, z_0, x_1 , respectively, in the previous calculations. Finally, from the calculation

$$\|x_{i+1} - x^*\| \leq \lambda\|x_i - x^*\| < r, \tag{23}$$

where $\lambda = \varphi_3(\|x_0 - x^*\|) \in [0, 1)$, we conclude that $\lim_{i \rightarrow \infty} x_i = x^*$ and $x_{i+1} \in U(x^*, r)$. \square

The uniqueness of the solution’s result follows.

Proposition 1. *Suppose that there exists a simple solution $x^* \in D$ of equation $F(x) = 0$, and (A3) holds. Set $D_2 = U(x^*, \frac{2}{\ell_0}) \cap D$. Then, element x^* is the only solution of equation $F(x) = 0$ in region D_2 .*

Proof. Consider $\tilde{x} \in D_2$ with $F(\tilde{x}) = 0$. Define the linear operator $Q = \int_0^1 F'(x^* + \theta(\tilde{x} - x^*))d\theta$. Then, by applying condition (A3)

$$\begin{aligned} \|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \ell_0 \int_0^1 \|x^* + \theta(\tilde{x} - x^*) - x^*\|d\theta \\ \ell_0 \int_0^1 \theta \|\tilde{x} - x^*\|d\theta &< \frac{\ell_0}{2} \frac{2}{\ell_0} = 1. \end{aligned} \tag{24}$$

It follows that the linear operator Q is invertible. Then, the approximation $Q(\tilde{x} - x^*) = F(\tilde{x}) - F(x^*) = 0 - 0 = 0$, gives $\tilde{x} - x^* = Q^{-1}(0) = 0$. Hence, we conclude that $\tilde{x} = x^*$. \square

Remark 1. A similar result was given in ([15], Theorem 1) in the special case when $M = M_1 = \mathbb{R}$ and $M_{1,n} = F'(x_n)$. However, this non-affine invariant form result is not correct, since it corresponds to the (11) estimate which is

$$\|y_n - x^*\| \leq \frac{\ell_1 \|x_n - x^*\|^2}{2(1 - \ell_1 \|x_n - x^*\|)}$$

but which is not implied by (A2). Hence, the proof of Theorem 1 in [15] breaks down at this point. Notice also that in [15] they used $\bar{M}_{1,n} = F'(x_n)^{-1}$, $\bar{M}_{2,n} = M_{2,n}^{-1}$ and $\bar{M}_{3,n} = M_{3,n}^{-1}$.

3. Semi-Local Analysis

The semi-local analysis of iterative scheme (2) is based on some Lipschitz-type conditions relating operators F, F' , and linear operators $M_{j,n}$ to some parameters. Moreover, sequence $\{x_n\}$ is majorized by some scalar sequences depending on some parameters. Suppose:

(H1) There exist $x_0 \in D$, $\eta \geq 0$ such that $F'(x_0)^{-1}, M_{1,0}^{-1} \in L(M_1, M)$ and $\|M_{1,0}^{-1}F(x_0)\| \leq \eta$.

(H2) $\|F'(x_0)^{-1}(M_1(x) - F'(x_0))\| \leq a_1 \|x - x_0\|$ for all $x \in D$ and some $a_1 > 0$. Set $D_3 = U(x_0, \frac{1}{a_1}) \cap D$.

(H3) $\|\int_0^1 F'(x_0)^{-1}(F'(z + \theta(x - z)) - M_3(x, y, z))d\theta\| \leq b_1$

$$\|\int_0^1 F'(x_0)^{-1}(F'(x + \theta(y - x)) - M_1(x))d\theta\| \leq b_2,$$

$$\|\int_0^1 F'(x_0)^{-1}(F'(y + \theta(z - y)) - M_2(x, y))d\theta\| \leq b_3,$$

$$\|F'(x_0)^{-1}(M_2(x, y) - F'(x_0))\| \leq a_2 \|y - x_0\|,$$

$$\|F'(x_0)^{-1}(M_3(x, y, z) - F'(x_0))\| \leq a_3 \|z - x_0\|,$$

where for all $\theta \in [0, 1], x \in D_3$ and y, z are taken from method (26) (or for all $y, z \in D_3$), and b_1, b_2, b_3, a_2 and a_3 are positive constants depending on operators F, F' and $M_{j,n}$.

(H4) $U[x_0, \rho] \subset D$ for some $\rho > 0$ to be given later.

As can be seen by the proof of Theorem 2 that follows the iterates, $\{x_n\}$ lies in the set D_3 which is a more accurate domain than D , since $D_3 \subset D$. This way, at least as tight constants are obtained than if conditions (H3) and (H4) hold only in D (see also the numerical section).

We chose the last two conditions in (H3) this way. However, other choices are also possible [1–4]. Notice that if $a_1 \eta < 1$ and $\tilde{D} = U(y_0, \frac{1}{a_1} - \eta) \cap D$, then $\tilde{D} \subset D_3$, respectively, and even smaller constants “ a ” are obtained, if \tilde{D} replaces D_3 .

Moreover, we define the scalar sequence $\{t_n\}$ by

$$\begin{aligned} t_0 &= 0, s_0 = \eta \\ u_n &= s_n + \frac{b_2(s_n - t_n)}{1 - a_2s_n} \end{aligned} \tag{25}$$

$$\begin{aligned} t_{n+1} &= u_n + \frac{b_3(u_n - s_n)}{1 - a_3u_n}, \\ s_{n+1} &= t_{n+1} + \frac{b_1(t_{n+1} - u_n)}{1 - a_1t_{n+1}}. \end{aligned} \tag{26}$$

This sequence shall be shown to be majorizing for scheme $\{x_n\}$ in Theorem 2. However, first, a convergence result for it is needed.

We then develop results on the convergence of sequence $\{t_n\}$.

Lemma 1. *Suppose*

$$a_2s_n < 1, a_3u_n < 1 \text{ and } a_1t_{n+1} < 1 \tag{27}$$

hold for all $n = 0, 1, 2, \dots$. Then, sequence $\{t_n\}$ is such that $s_n \leq u_n \leq t_{n+1} < \frac{1}{a_1}$ and $\lim_{n \rightarrow \infty} t_n = t^* \leq \frac{1}{a_1}$.

Proof. It follows from (26) and (27) that sequence $\{t_n\}$ is nondecreasing, bounded from above by $\frac{1}{a_1}$ and as such it converges to its unique least upper bound $t^* \in [0, \frac{1}{a_1}]$.

□

The semi-local convergence of method (2) follows next.

Theorem 2. *Under conditions (H1)–(H4), further suppose: conditions of Lemma 1 hold and $\rho = t^*$ in (H4). Then, the sequence $\{x_n\}$ generated by method (2) exists in $U(x_0, t^*)$, stays in $U(x_0, t^*)$ and converges to a solution $x^* \in U[x_0, t^*]$ of equation $F(x) = 0$. Moreover, the following estimates hold*

$$\|y_n - x_n\| \leq s_n - t_n \tag{28}$$

$$\|z_n - y_n\| \leq u_n - s_n, \tag{29}$$

and

$$\|x_{n+1} - z_n\| \leq t_{n+1} - u_n. \tag{30}$$

Proof. Mathematical induction is used to show (29)–(31). Using (H1) and (27)

$$\|y_0 - x_0\| = \|M_{1,0}^{-1}F(x_0)\| \leq \eta = s_0 - t_0,$$

so iterate $y_0 \in U(x_0, t^*)$ and (56) holds for $n = 0$. Let $v \in U(x_0, t^*)$. It then follows from (H3) that

$$\|F'(x_0)^{-1}(M_2(x_0, y_0) - F'(x_0))\| \leq a_2\|y_0 - x_0\| < a_2t^* < 1.$$

That is, $M_2(x_0, y_0)^{-1} \in L(M_1, M)$,

$$\|M_2(x_0, y_0)^{-1}F(x_0)\| \leq \frac{1}{1 - a_2\|y_0 - x_0\|}, \tag{31}$$

and iterate z_0 is well defined by the second substep of method (26) for $n = 0$. By the first substep of method (2)

$$F(y_0) = F(y_0) - F(x_0) - M_{1,0}(y_0 - x_0),$$

$$F'(x_0)^{-1}F(y_0) = \int_0^1 F'(x_0)^{-1}(F'(x_0 + \theta(y_0 - x_0)) - M_1(x_0))d\theta(y_0 - x_0),$$

$$\|F'(x_0)^{-1}F(y_0)\| \leq b_2\|y_0 - x_0\| \leq b_2(s_0 - t_0)$$

and

$$\|z_0 - y_0\| \leq \|M_2(x_0, y_0)^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(y_0)\| \leq u_0 - s_0.$$

Hence, (29) holds for $n = 0$ and

$$\|z_0 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq u_0 - s_0 + s_0 - t_0 = u_0 < t^*.$$

Therefore, iterate, $z_0 \in U(x_0, t^*)$. As in (31), we obtain

$$\|M_3(x_0, y_0, z_0)^{-1}F'(x_0)\| \leq \frac{1}{1 - a_3\|z_0 - x_0\|}.$$

By the second substep of method (2), we can write

$$\begin{aligned} F(z_0) &= F(z_0) - F(y_0) + F(y_0) \\ &= \int_0^1 (F'(y_0 + \theta(z_0 - y_0))d\theta - M_{2,0})(z_0 - y_0). \end{aligned}$$

Consequently

$$\|F'(x_0)^{-1}F(z_0)\| \leq b_3\|z_0 - y_0\| \leq b_3(u_0 - s_0).$$

Then, we obtain

$$\|x_1 - z_0\| \leq \|M_3(x_0, y_0, z_0)^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(z_0)\| \leq t_1 - u_0,$$

and

$$\begin{aligned} \|x_1 - x_0\| &\leq \|x_1 - z_0\| + \|z_0 - y_0\| + \|y_0 - x_0\| \\ &\leq t_1 - u_0 + u_0 - s_0 + s_0 - t_0 = t_1 < t^*. \end{aligned}$$

That is, iterate $x_1 \in U(x_0, t^*)$ and (31) holds for $n = 0$. Moreover, we can write

$$\begin{aligned} F(x_1) &= F(x_1) - F(z_0) - M_{3,0}(x_1 - z_0) \\ &= \int_0^1 [F'(z_0 + \theta(x_1 - z_0))d\theta - M_{3,0}](x_1 - z_0), \end{aligned}$$

$$\|F'(x_0)^{-1}F(x_1)\| \leq b_1\|x_1 - z_0\| \leq b_1(t_1 - u_0),$$

$$\|y_1 - x_1\| \leq \|M_{1,0}^{-1}(x_0)F'(x_0)\| \|F'(x_0)^{-1}F(x_1)\| \leq s_1 - t_1$$

and

$$\|y_1 - x_0\| \leq \|y_1 - x_1\| + \|x_1 - z_0\| \leq s_1 - t_1 + t_1 - u_0 \leq s_1 < t^*,$$

so $y_1 \in U(x_0, t^*)$ and (29) holds for $n = 0$. Simply revisit the preceding estimations with x_k, y_k, z_k, x_{k+1} replacing x_0, y_0, z_0, x_1 , respectively, to terminate the induction for items (29)–(31). Sequence $\{t_k\}$ is complete as convergent. In view of (29)–(31), sequence $\{x_n\}$ is also complete and as such, it converges to some $x^* \in U[x_0, t^*]$. By letting $k \rightarrow \infty$ in the estimate

$$\|F'(x_0)^{-1}F(x_k)\| \leq b_1(t_k - u_{k-1})$$

and using the continuity of F , we conclude that $F(x^*) = 0$. □

A uniqueness result follows.

Proposition 2. Under the conditions of Theorem 2, further suppose that there exists $R \geq t^*$ such that

$$\frac{\ell_0}{2}(R + t^*) < 1.$$

Set $D_4 = U[x_0, t^*] \cap D$. Then, the element x^* is the only solution of equation $F(x) = 0$ in the region D_4 .

Proof. Let $\tilde{x} \in D_4$ be such that $F(\tilde{x}) = 0$. Then, as in Proposition 1, we obtain

$$\begin{aligned} \|F'(x_0)^{-1}(Q - F'(x_0))\| &\leq \ell_0 \int_0^1 ((1 - \theta)\|x_0 - x^*\| + \theta\|x_0 - \tilde{x}\|)d\theta \\ &\leq \frac{\ell_0}{2}(t^* + R) < 1. \end{aligned}$$

Therefore, we deduce that $\tilde{x} = x^*$. \square

4. Special Cases

Let $M_{1,n} = F'(x_n)$, $M_{2,n} = F'(y_n)$ and $M_{3,n} = F'(z_n)$. Then, method (2) reduces to

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(y_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - F'(z_n)^{-1}F(z_n). \end{aligned} \tag{32}$$

This is Newton’s three-step method also called by some Traub’s extended three-step method. It seems to be the most interesting special case of method (2) to consider as an application. Moreover, the semi-local convergence of it uses our new idea of recurrent functions, and the resulting convergence criteria are weaker than those in earlier works for method (32) using the Kantorovich condition $2L_1\eta \leq 1$ [2,4,7,8] (as can also be seen in Example 5.2). Moreover, the error bounds are tighter and the information on the location of the solution is more precise than in the aforementioned works. Finally, in Lemma 2, we gave even weaker convergence criteria for method (32). Hence, this is clearly a most revealing special case to consider, since it can also be connected to earlier works and improve them too.

The following conditions are used.

Suppose:

(H1) There exists $x_0 \in D$, $\eta \geq 0$ such that $F'(x_0)^{-1} \in L(M_1, M)$ and

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta.$$

(H2)

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|$$

for all $x \in D$ and some $L_0 > 0$. Set $D_5 = U(x_0, \frac{1}{L_0}) \cap D$.

(H3) For each $x, y \in D_5$

$$\|F'(x_0)^{-1}(F'(u) - F'(v))\| \leq L\|u - v\|$$

for all $u \in D_5$ and $v = u - F'(u)^{-1}F(u) \in D$ (or all $v \in D_5$) and some $L > 0$.

(H4) $U[x_0, t^*] \subset D$ for some t^* to be given later.

Notice that condition (H3) was used for all $u, v \in D$ and constants L_1 [2,4,7,8] as well as for all $u, v \in D_5$ with constant K [1,3]. That is:

(M1) For each $x, y \in D$

$$\|F'(x_0)^{-1}(x_0)(F'(u) - F'(v))\| \leq L_1\|u - v\|.$$

(M2) For each $x, y \in D_5$

$$\|F'(x_0)^{-1}(F'(u) - F'(v))\| \leq K\|u - v\|.$$

It follows by these definitions that

$$L \leq K \leq L_1 \text{ and } \leq L_1. \tag{33}$$

Hence, any analysis using L improves earlier ones using L_1 or K (see also the numerical section). The sequence $\{t_n\}$ defined by

$$\begin{aligned} t_0 &= 0, s_0 = \eta, \\ u_n &= s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0 s_n)}, \\ t_{n+1} &= u_n + \frac{L(u_n - s_n)^2}{2(1 - L_0 u_n)}, \\ s_{n+1} &= t_{n+1} + \frac{L(t_{n+1} - u_n)^2}{2(1 - L_0 t_{n+1})} \end{aligned} \tag{34}$$

shall be shown to be majorizing for method (32). However, first we need some convergence results for it.

Notice that the corresponding sequences are

$$\begin{aligned} \bar{t}_0 &= 0, \bar{s}_0 = \eta \\ \bar{u}_n &= \bar{s}_n + \frac{K(\bar{s}_n - \bar{t}_n)^2}{2(1 - L_0 \bar{s}_n)}, \\ \bar{t}_{n+1} &= \bar{u}_n + \frac{K(\bar{u}_n - \bar{s}_n)^2}{2(1 - L_0 \bar{u}_n)}, \\ \bar{s}_{n+1} &= \bar{t}_{n+1} + \frac{K(\bar{t}_{n+1} - \bar{u}_n)^2}{2(1 - L_0 \bar{t}_{n+1})}, \\ \bar{\bar{t}}_0 &= 0, \bar{\bar{s}}_0 = \eta \\ \bar{\bar{u}}_n &= \bar{\bar{s}}_n + \frac{L_1(\bar{\bar{s}}_n - \bar{\bar{t}}_n)^2}{2(1 - L_1 \bar{\bar{s}}_n)}, \\ \bar{\bar{t}}_{n+1} &= \bar{\bar{u}}_n + \frac{L_1(\bar{\bar{u}}_n - \bar{\bar{s}}_n)^2}{2(1 - L_1 \bar{\bar{u}}_n)}, \\ \bar{\bar{s}}_{n+1} &= \bar{\bar{t}}_{n+1} + \frac{L_1(\bar{\bar{t}}_{n+1} - \bar{\bar{u}}_n)^2}{2(1 - L_1 \bar{\bar{t}}_{n+1})}. \end{aligned} \tag{35}$$

$$\begin{aligned} \bar{\bar{t}}_0 &= 0, \bar{\bar{s}}_0 = \eta \\ \bar{\bar{u}}_n &= \bar{\bar{s}}_n + \frac{L_1(\bar{\bar{s}}_n - \bar{\bar{t}}_n)^2}{2(1 - L_1 \bar{\bar{s}}_n)}, \\ \bar{\bar{t}}_{n+1} &= \bar{\bar{u}}_n + \frac{L_1(\bar{\bar{u}}_n - \bar{\bar{s}}_n)^2}{2(1 - L_1 \bar{\bar{u}}_n)}, \\ \bar{\bar{s}}_{n+1} &= \bar{\bar{t}}_{n+1} + \frac{L_1(\bar{\bar{t}}_{n+1} - \bar{\bar{u}}_n)^2}{2(1 - L_1 \bar{\bar{t}}_{n+1})}. \end{aligned} \tag{36}$$

We assume that $L_0 \leq K$. Otherwise, replace K by L_0 in sequence (35). It follows from (34) and these definitions that

$$\begin{aligned} t_n &\leq \bar{t}_n \leq \bar{\bar{t}}_n, \\ s_n &\leq \bar{s}_n \leq \bar{\bar{s}}_n, \\ u_n &\leq \bar{u}_n \leq \bar{\bar{u}}_n \end{aligned} \tag{37}$$

and

$$t^* = \lim_{n \rightarrow \infty} t_n \leq s^* = \lim_{n \rightarrow \infty} s_n \leq u^* = \lim_{n \rightarrow \infty} u_n$$

(if these limits exist). Hence, the new majorizing sequence is more precise. The convergence criteria for sequences (35) [1,3] and (36) [2,4,7,8] are:

$$2K\eta \leq 1 \tag{38}$$

and

$$2L_1\eta \leq 1, \tag{39}$$

respectively. However, the convergence criterion for sequence (34) is

$$2L\eta \leq 1. \tag{40}$$

Notice that

$$2L_1\eta \leq 1 \Rightarrow 2K\eta \leq 1 \Rightarrow 2L\eta \leq 1. \tag{41}$$

Condition (40) is weakened further in Lemma 3. It is worth noticing that these benefits are obtained under the same computational cost, since in practice, the computation of the Lipschitz constant L_1 requires that of L_0, K and L as special cases. Notice that criterion (39) is due to Kantorovich [2].

Then, two convergence results for sequence (34) are presented.

Lemma 2. *Suppose*

$$L_0s_n < 1, L_0u_n < 1 \text{ and } L_0t_{n+1} < 1. \tag{42}$$

Then, sequence $\{t_n\}$ is such that $0 \leq t_n \leq s_n \leq t_{n+1}$ and $\lim_{n \rightarrow \infty} t_n = t^ \leq \frac{1}{L_0}$.*

Proof. See Lemma 1. \square

Next, some stronger conditions than (42) are given but are easier to show. However, first, we define polynomials on the interval $[0, 1)$ by

$$\begin{aligned} f_n^{(1)}(t) &= \frac{L}{2}t^{2n-1}\eta + L_0(1 + t + \dots + t^{2n})\eta - 1, \\ f_n^{(2)}(t) &= \frac{L}{2}t^{2n}\eta + L_0(1 + t + \dots + t^{2n+1})\eta - 1, \\ f_n^{(3)}(t) &= \frac{L}{2}t^{2n+1}\eta + L_0(1 + t + \dots + t^{2n+2})\eta - 1, \\ p(t) &= L_0t^3 + (L_0 + \frac{L}{2})t^2 - \frac{L}{2}. \end{aligned}$$

and parameter γ by

$$\gamma = \frac{2L}{L + \sqrt{L^2 + 8L_0L}}.$$

Notice that $\gamma \in (0, 1)$, $p(\gamma) = 0$, whereas the other two roots of p are negative by the Descartes’s rule of signs. Define the parameters

$$a = \frac{L\eta}{2(1 - L_0\eta)}, b = \frac{L(u_0 - s_0)}{2(1 - L_0u_0)}, c = \frac{L(t_1 - u_0)}{2(1 - L_0t_1)} \text{ and } d = \max\{a, b, c\}.$$

Then, we show:

Lemma 3. *Suppose that*

$$0 \leq d \leq \delta < 1 - L_0\eta. \tag{43}$$

*Then, the sequence $\{t_n\}$ generated by (34) is nondecreasing, bounded from above by $t^{**} = \frac{\eta}{1-\delta}$ and converges to its unique least upper bound $t^* \in [0, t^{**}]$. Moreover, the following items hold*

$$0 \leq s_n - t_n \leq \delta(t_n - s_{n-1}) \leq \delta^{2n}(s_0 - t_0), \tag{44}$$

$$0 \leq u_n - s_n \leq \delta(s_n - t_n) \leq \delta^{2n+1}(s_0 - t_0), \tag{45}$$

and

$$0 \leq t_{n+1} - s_n \leq \delta(u_n - s_n) \leq \delta^{2n+2}(s_0 - t_0). \tag{46}$$

Proof. Induction is utilized for items

$$0 \leq \frac{L(s_k - t_k)}{2(1 - L_0s_k)} \leq \gamma, \tag{47}$$

$$0 \leq \frac{L(u_k - s_k)}{2(1 - L_0u_k)} \leq \gamma, \tag{48}$$

$$0 \leq \frac{L(t_{k+1} - u_k)}{2(1 - L_0t_{k+1})} \leq \gamma, \tag{49}$$

and

$$t_k \leq s_k \leq u_k \leq t_{k+1}. \tag{50}$$

These estimates hold for $k = 0$ by (34) and (43). Suppose they hold for all integers smaller or equal to k . Then, we obtain

$$\begin{aligned} t_{k+1} &\leq u_k + \gamma^{2k+2}\eta \leq s_k + \gamma^{2k+1}\eta + \gamma^{2k+2}\eta \\ &\leq t_k + \gamma^{2k}\eta + \gamma^{2k+1}\eta + \gamma^{2k+2}\eta \\ &\vdots \\ &\leq t_0 + \gamma\eta + \dots + \gamma^{2k+2}\eta \\ &= \frac{1 - \gamma^{2k+3}}{1 - \gamma}\eta < \frac{\eta}{1 - \gamma} = t^{**}, \end{aligned}$$

similarly,

$$s_k \leq \frac{1 - \gamma^{2k+1}}{1 - \gamma}\eta \text{ and } u_k = \frac{1 - \gamma^{2k+2}}{1 - \gamma}\eta.$$

Then, evidently, (47) holds if

$$\frac{L}{2}\gamma^{2k}\eta + L_0\gamma\frac{1 - \gamma^{2k+1}}{1 - \gamma}\eta - \gamma \leq 0$$

or

$$f_k^{(1)}(t) \leq 0 \text{ at } t = \gamma. \tag{51}$$

By the definition of $f_k^{(1)}$, we can find a relationship between two consecutive functions:

$$\begin{aligned} f_{k+1}^{(1)}(t) &= f_{k+1}^{(1)}(t) - f_k^{(1)}(t) + f_k^{(1)}(t) \\ &= f_k^{(1)}(t) + \frac{L}{2}t^{2k+1}\eta + L_0(1 + t + \dots + t^{2k+2})\eta - 1 \\ &\quad - \frac{L}{2}t^{2k+1}\eta - L_0(1 + t + \dots + t^{2k})\eta + 1 \\ &= f_k^{(1)}(t) + p(t)t^{2k+1}\eta. \end{aligned} \tag{52}$$

In particular, by the definition of p , we obtain

$$f_{k+1}^{(1)} = f_k^{(1)}(t) \text{ at } t = \gamma. \tag{53}$$

Let function

$$f_\infty^{(1)}(t) = \lim_{k \rightarrow \infty} f_k^{(1)}(t). \tag{54}$$

It follows by the definition of $f_k^{(1)}$ and (54) that

$$f_\infty^{(1)}(t) = \frac{L_0\eta}{1 - t} - 1.$$

Consequently, assertion (51) holds if

$$f_{\infty}^{(1)}(t) \leq 0 \text{ at } t = \gamma,$$

which is true by the right hand side of inequality (43). Similarly, to show (48)

$$\frac{L}{2}\gamma^{2k+1}\eta + L_0\gamma\frac{1-\gamma^{2k+2}}{1-\gamma}\eta - \gamma \leq 0$$

or

$$f_k^{(2)}(t) \leq 0 \text{ at } t = \gamma.$$

This time, we also have

$$f_{k+1}^{(2)}(t) = f_k^{(2)}(t) + p(t)t^{2k}\eta,$$

and for

$$f_{\infty}^{(2)}(t) = \lim_{k \rightarrow \infty} f_k^{(2)}(t) = \frac{L_0\eta}{1-t} - 1 \leq 0$$

at $t = \gamma$. Moreover, (49) holds if

$$\frac{L}{2}\gamma^{2k+2}\eta + \gamma L_0\frac{1-\gamma^{2k+3}}{1-\gamma}\eta - \gamma \leq 0$$

or

$$f_k^{(3)}(t) \leq 0 \text{ at } t = \gamma. \tag{55}$$

However, we have

$$f_{k+1}^{(3)}(t) = f_k^{(3)}(t) + p(t)t^{2k+1}\gamma,$$

so

$$f_{k+1}^{(3)}(t) = f_k^{(3)}(t), \text{ at } t = \gamma.$$

That is, (55) holds if $f_{\infty}^{(3)}(t) = \lim_{k \rightarrow \infty} f_k^{(3)}(t) \leq 0$, at $t = \gamma$. However, again, we obtain

$$f_{\infty}^{(3)}(t) = \frac{L_0\eta}{1-t} - 1.$$

Therefore, assertion (55) holds again by (45). Furthermore, (50) holds by (34) and (47)–(49). The induction for items (47)–(50) is completed. Hence, we deduce $t_k \leq s_k \leq t_{k+1}$ and $\lim_{k \rightarrow \infty} t_k = t^*$. \square

5. Numerical Example

We verify convergence criteria using method (32). Moreover, we compare Lipschitz constants L_0, L, L_1 and K . In particular, the first example is used to show that the ratio $\frac{L_0}{L_1}$ can be arbitrarily small.

Example 1. Let $M = M_1 = \mathbb{R}$. Define function

$$\psi(t) = \delta_0 t + \delta_1 + \delta_2 \sin \delta_3 t, \quad t_0 = 0,$$

where $\delta_j, j = 0, 1, 2, 3$ are fixed parameters. Then, clearly for δ_3 large and δ_2 small, $\frac{L_0}{L_1}$ can be (arbitrarily) small, so that $\frac{L_0}{L_1} \rightarrow 0$.

The parameters L_0, L, K and L_1 are computed in the next example. Moreover, the convergence criteria (46)–(48) and those of Lemma 3 are compared.

Example 2. Let $M = M_1 = \mathbb{R}$. Let us consider a scalar function F defined on the set $D = U[x_0, 1 - q]$ for $q \in (0, \frac{1}{2})$ by

$$F(x) = x^3 - q.$$

Choose $x_0 = 1$. Then, we obtain the estimates $\eta = \frac{1-q}{3}$,

$$\begin{aligned} |F'(x_0)^{-1}(F'(x) - F'(x_0))| &= |x^2 - x_0^2| \\ &\leq |x + x_0||x - x_0| \leq (|x - x_0| + 2|x_0|)|x - x_0| \\ &= (1 - q + 2)|x - x_0| = (3 - q)|x - x_0|, \end{aligned}$$

for all $x \in D$, so $L_0 = 3 - q$, $D_0 = U(x_0, \frac{1}{L_0}) \cap D = U(x_0, \frac{1}{L_0})$,

$$\begin{aligned} |F'(x_0)^{-1}(F'(y) - F'(x))| &= |y^2 - x^2| \\ &\leq |y + x||y - x| \leq (|y - x_0 + x - x_0 + 2x_0|)|y - x| \\ &= (|y - x_0| + |x - x_0| + 2|x_0|)|y - x| \\ &\leq (\frac{1}{L_0} + \frac{1}{L_0} + 2)|y - x| = 2(1 + \frac{1}{L_0})|y - x|, \end{aligned}$$

for all $x, y \in D$ and so $K = 2(1 + \frac{1}{L_0})$.

$$\begin{aligned} |F'(x_0)^{-1}(F'(y) - F'(x))| &= (|y - x_0| + |x - x_0| + 2|x_0|)|y - x| \\ &\leq (1 - q + 1 - q + 2)|y - x| = 2(2 - q)|y - x|, \end{aligned}$$

for all $x, y \in D$ and $L_1 = 2(2 - q)$.

Notice that for all $q \in (0, \frac{1}{2})$

$$L_0 < K < L_1.$$

Next, set $y = x - F'(x)^{-1}F(x)$, $x \in D$. Then, we have

$$y + x = x - F'(x)^{-1}F(x) + x = \frac{5x^3 + q}{3x^2}.$$

Define function \bar{F} on the interval $D = [q, 2 - q]$ by

$$\bar{F}(x) = \frac{5x^3 + q}{3x^2}.$$

Then, we obtain by this definition that

$$\begin{aligned} \bar{F}'(x) &= \frac{15x^4 - 6xq}{9x^4} \\ &= \frac{5(x - q)(x^2 + xq + q^2)}{3x^3}, \end{aligned}$$

where $p = \sqrt[3]{\frac{2q}{5}}$ is the critical point of function \bar{F} . Notice that $q < p < 2 - q$. It follows that this function is decreasing on the interval (q, p) and increasing on the interval $(p, 2 - q)$, since $x^2 + xq + q^2 > 0$ and $x^3 > 0$. So, we can set

$$K_2 = \frac{5(2 - q)^2 + q}{9(2 - q)^2}$$

and

$$K_2 < L_0.$$

However, if $x \in D_0 = [1 - \frac{1}{L_0}, 1 + \frac{1}{L_0}]$, then

$$L = \frac{5\varrho^3 + q}{9\varrho^2},$$

where $\varrho = \frac{4-q}{3-q}$ and $K < K_1$ for all $q \in (0, \frac{1}{2})$. Then, criterion (39) is not satisfied for all $q \in (0, \frac{1}{2})$. Hence, there is no guarantee that scheme (34) converges to $x^* = \sqrt[3]{q}$. Moreover, our earlier criterion (38) holds for $q \in (0.4620, 1]$. Furthermore, the new criterion by solving becomes

$$2\bar{L}\eta \leq 1,$$

where $\bar{L} = \frac{1}{8}(4L_0 + L + \sqrt{L^2 + 8L_0L})$. This condition holds for $q \in (0.4047, 1)$. Clearly, the new results extend the range of values q for which scheme (34) converges.

This range can be extended even further if we apply Lemma 2. Indeed, choose $q = 0.4$, and we have the following Table 1, showing that the conditions of Lemma 2 are satisfied.

Table 1. Sequence (32).

| n | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------|--------|--------|--------|--------|--------|--------|
| u_i | 0.2330 | 0.2945 | 0.3008 | 0.3009 | 0.3009 | 0.3009 |
| s_i | 0.2000 | 0.2896 | 0.3008 | 0.3009 | 0.3009 | 0.3009 |
| t_{n+1} | 0.2341 | 0.2946 | 0.3008 | 0.3009 | 0.3009 | 0.3009 |
| L_0s_i | 0.5200 | 0.7530 | 0.7820 | 0.7824 | 0.7824 | 0.7824 |
| L_0u_i | 0.6058 | 0.7658 | 0.7822 | 0.7824 | 0.7824 | 0.7824 |
| L_0t_{i+1} | 0.6087 | 0.7659 | 0.7822 | 0.7824 | 0.7824 | 0.7824 |

Example 3. Consider $M = M_1 = C[0, 1]$ and $D = U[0, 1]$. Then, the boundary value problem (BVP) [4]

$$\begin{aligned} \zeta(0) &= 0, \zeta(1) = 1, \\ \zeta'' &= -\zeta - \sigma\zeta^2 \end{aligned}$$

can be also given as

$$\zeta(s) = s + \int_0^1 G(s, t)(\zeta^3(t) + \sigma\zeta^2(t))dt$$

where σ is a constant and $G(s, t)$ is the Green's function

$$G(s, t) = \begin{cases} t(1-s), & t \leq s \\ s(1-t), & s < t. \end{cases}$$

Consider $F : D \rightarrow M_1$ as

$$[F(x)](s) = x(s) - s - \int_0^1 G(s, t)(x^3(t) + \sigma x^2(t))dt.$$

Let us set $\zeta_0(s) = s$ and $D = U(\zeta_0, \rho_0)$. Then, clearly $U(\zeta_0, \rho_0) \subset U(0, \rho_0 + 1)$, since $\|\zeta_0\| = 1$. If $2\sigma < 5$. Then, conditions (H1)–(H4) are satisfied for

$$L_0 = \frac{2\sigma + 3\rho_0 + 6}{8}, L = \frac{\sigma + 6\rho_0 + 3}{4}.$$

Hence, $L_0 < L_1$.

The next two examples concern the local convergence of method (34) and the radii r_j, r were computed using Formula (6) and the functions φ_j .

Example 4. If $M = M_1 = C[0, 1]$ is equipped with the max-norm, $D = U[0, 1]$, consider $Q : D \rightarrow M_1$ given as

$$Q(\lambda)(x) = \varphi(x) - 5 \int_0^1 x\tau\lambda(\tau)^3 d\tau. \tag{56}$$

We obtain

$$Q'(\lambda(\xi))(x) = \xi(x) - 15 \int_0^1 x\tau\lambda(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in D.$$

Then, since $x^* = 0$, conditions (A1)–(A5) hold provided that $\ell_0 = \ell_1 = \ell_2 = \ell_3 = 7.5$. Then, the radii are:

$$r_1 = 0.0533 = r, r_2 = 0.1499, \text{ and } r_3 = 0.1660.$$

Example 5. Consider the motion system

$$\mathcal{H}'_1(w_1) = e^{w_1}, \mathcal{H}'_2(w_2) = (e - 1)w_2 + 1, \mathcal{H}'_3(w_3) = 1$$

with $\mathcal{H}_1(0) = \mathcal{H}_2(0) = \mathcal{H}_3(0) = 0$. Let $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$. Let $M = M_1 = \mathbb{R}^3, D = U[0, 1], x^* = (0, 0, 0)^{tr}$. Let function \mathcal{H} on D for $w = (w_1, w_2, w_3)^{tr}$ given as

$$\mathcal{H}(w) = (e^{w_1} - 1, \frac{e - 1}{2}w_2^2 + w_2, w_3)^{tr}.$$

The Fréchet derivative is given by

$$\mathcal{H}'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $\mathcal{H}'(x^*) = I$. Let $w \in \mathbb{R}^3$ with $w = (w_1, w_2, w_3)^{tr}$. Moreover, the nor for $M \in \mathbb{R}^3 \times \mathbb{R}^3$ is

$$\|M\| = \max_{1 \leq k \leq 3} \sum_{i=1}^3 \|m_{k,i}\|.$$

We need to verify conditions (A1)–(A5). To achieve this, we study $\mathcal{G}(t) = e^t - 1$ on $D = [-1, 1]$. We have $t^* = 1$, hence $\mathcal{G}'(t^*) = 1$, and

$$\begin{aligned} |\mathcal{G}'(t) - \mathcal{G}'(t^*)| &= \left| t + \frac{t^2}{2} + \dots + \frac{t^n}{n!} + \dots \right| \\ &= \left| 1 + \frac{t - 0}{2!} + \dots + \frac{(t - 0)^{n-1}}{n!} + \dots \right| |t - 0| \end{aligned}$$

so $\ell_1 = e - 1$. Then, $D_1 = U(x^*, \frac{1}{e-1}) \cap D = U(x^*, \frac{1}{e-1})$. This time we obtain

$$|\mathcal{G}'(t) - \mathcal{G}'(t^*)| \leq \ell_0 |t - 0|,$$

where

$$\ell_0 = 1 + \frac{1}{(e - 1)2!} + \dots + \frac{1}{(e - 1)^{n-1}n!} + \dots \approx 1.43 < \ell_1.$$

Then, we have for $t \in D_1$

$$\begin{aligned} |s| &= |t - \mathcal{G}'(t)^{-1}\mathcal{G}(t)| = |t - 1 + e^{-t}| \\ &= \left| \frac{(-t)^2}{2!} + \dots + \frac{(-t)^n}{n!} + \dots \right| \\ &= |t| \left(\frac{|t|}{2!} + \dots + \frac{|t|^{n-1}}{n!} + \dots \right) \leq \frac{\ell_0 - 1}{e - 1}. \end{aligned}$$

Moreover,

$$\begin{aligned}
 |F'(s) - F'(t^*)| &= |e^s - 1| \\
 &\leq |s|(1 + \frac{|s|}{2!} + \dots + \frac{|s|^{n-1}}{n!} + \dots) \\
 &\leq |t| \frac{\ell_0 - 1}{e - 1} (1 + \frac{\ell_0 - 1}{(e - 1)2!} + \dots + (\frac{\ell_0 - 1}{e - 1})^{n-1} \frac{1}{n!} + \dots) \\
 &= \ell_2(t - 0),
 \end{aligned}$$

where $\ell_2 \approx 0.49 < \ell_1$. We can set $\ell_3 = \ell_2$.

Then, the radii are:

$$r_1 = 0.2409 = r, r_2 = 0.3101, \text{ and } r_3 = 0.3588.$$

In the last example, we revisit the motivational example given in the introduction, where we apply scheme (32).

Example 6. The iterates for the motivational example with $x_0 = 0.85$ are given in Table 2.

Table 2. Sequence (32).

| <i>n</i> | 1 | 2 | 3 | 4 | 5 | 6 |
|----------|--------|--------|--------|--------|--------|--------|
| y_i | 1.1609 | 0.2067 | 0.0846 | 0.0377 | 0.0174 | 0.0081 |
| z_i | 0.3121 | 0.1640 | 0.0695 | 0.0313 | 0.0145 | 0.0068 |
| x_i | 0.8500 | 0.3985 | 0.1399 | 0.0605 | 0.0274 | 0.0127 |

6. Conclusions

Conditions for the convergence of generalized three-step schemes are presented for both the local as well as semi-local case. The sequences generated by these schemes approximate solutions of equation $F(x) = 0$ that are locally unique. The convergence conditions depend on the divided difference of the order of one or the derivative, which appears on the schemes. However, this is not the case with earlier articles utilizing high-order derivatives, which do not appear in the schemes. Moreover, the error analysis is tighter because we show that the iterates remain in a stricter domain than in earlier articles. Hence, the utilization of these schemes is extended with the same or even weaker conditions. Our process does not depend on these schemes. Therefore, it can be employed similarly to extend the usage of the other schemes [9,10,15–18].

Author Contributions: Conceptualization, S.R., I.K.A., S.G. and C.I.A.; methodology, S.R., I.K.A., S.G. and C.I.A.; software, S.R., I.K.A., S.G. and C.I.A.; validation, S.R., I.K.A., S.G. and C.I.A.; formal analysis, S.R., I.K.A., S.G. and C.I.A.; investigation, S.R., I.K.A., S.G. and C.I.A.; resources, S.R., I.K.A., S.G. and C.I.A.; data curation, S.R., I.K.A., S.G. and C.I.A.; writing—original draft preparation, S.R., I.K.A., S.G. and C.I.A.; writing—review and editing, S.R., I.K.A., S.G. and C.I.A.; visualization, S.R., I.K.A., S.G. and C.I.A.; supervision, S.R., I.K.A., S.G. and C.I.A.; project administration, S.R., I.K.A., S.G. and C.I.A.; funding acquisition, S.R., I.K.A., S.G. and C.I.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to express our gratitude to the reviewers for the constructive criticism of this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Argyros, I.K.; Hilout, S. Weaker conditions for the convergence of Newton's method. *J. Complex.* **2012**, *28*, 364–387. [\[CrossRef\]](#)
2. Cordero, A.; Torregrosa, J.R. Variants of Newton's method using fifth-order quadrature formulas. *Appl. Math. Comput.* **2007**, *190*, 686–698. [\[CrossRef\]](#)
3. Argyros, I.K. Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications. *Mathematics* **2021**, *9*, 1942. [\[CrossRef\]](#)
4. Argyros, I.K. *The Theory and Applications of Iteration Methods*, 2nd ed.; Engineering Series; CRC Press, Taylor and Francis Group: Boca Raton, FL, USA, 2022.
5. Kou, J.; Wang, X.; Li, Y. Some eight order root finding three-step methods, *Commun. Nonlinear Sci. Numer. Simulat.* **2010**, *15*, 536–544. [\[CrossRef\]](#)
6. Argyros, I.K.; Magrenan, A.A. *A Contemporary Study of Iterative Methods*; Elsevier (Academic Press): New York, NY, USA, 2018.
7. Grau-Sanchez, M.; Grau, A.; Noguera, M. Ostrowski type methods for solving system of nonlinear equations. *Appl. Math. Comput.* **2011**, *218*, 2377–2385. [\[CrossRef\]](#)
8. Homeier, H.H.H. A modified Newton method with cubic convergence: The multivariate case. *J. Comput. Appl. Math.* **2004**, *169*, 161–169. [\[CrossRef\]](#)
9. Kantorovich, L.V.; Akilov, G.P. *Functional Analysis*; Pergamon Press: Oxford, UK, 1982.
10. Proinov, P.D. New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems. *J. Complex.* **2010**, *26*, 3–42. [\[CrossRef\]](#)
11. Sharma, J.R.; Arora, H. Efficient derivative - free numerical methods for solving systems of nonlinear equations. *Comp. Appl. Math.* **2016**, *35*, 269–284. [\[CrossRef\]](#)
12. Xiao, X.; Yin, H. Achieving higher order of convergence for solving systems of nonlinear equations. *Appl. Math. Comput.* **2017**, *311*, 251–261. [\[CrossRef\]](#)
13. Noor, M.A.; Waseem, M. Some iterative methods for solving a system of nonlinear equations. *Comput. Math. Appl.* **2009**, *57*, 101–106. [\[CrossRef\]](#)
14. Traub, J.F. *Iterative Methods for the Solution of Equations*; Prentice Hall: Englewood Cliffs, NJ, USA, 1964.
15. Ezquerro, J.A.; Hernandez, M.A. *Newton's Method: An Updated Approach of Kantorovich's Theory*; Birkhäuser: Cham, Switzerland, 2018.
16. Nashed, M.Z.; Chen, X. Convergence of Newton-like methods for singular operator equations using outer inverses. *Numer. Math.* **1993**, *66*, 235–257. [\[CrossRef\]](#)
17. Shakhno, S.M.; Gnatyshyn, O.P. On an iterative Method of order 1.839... for solving nonlinear least squares problems. *Appl. Math. Appl.* **2005**, *161*, 253–264.
18. Verma, R. *New Trends in Fractional Programming*; Nova Science Publisher: New York, NY, USA, 2019.