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Positive Solutions for a System of Fractional Boundary Value Problems with r-Laplacian Operators, Uncoupled Nonlocal Conditions and Positive Parameters

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Abstract: In this paper, we investigate the existence and nonexistence of positive solutions for a system of Riemann–Liouville fractional differential equations with r-Laplacian operators, subject to nonlocal uncoupled boundary conditions that contain Riemann–Stieltjes integrals, various fractional derivatives and positive parameters. We first change the unknown functions such that the new boundary conditions have no positive parameters, and then, by using the corresponding Green functions, we equivalently write this new problem as a system of nonlinear integral equations. By constructing an appropriate operator $\mathcal{A}$, the solutions of the integral system are the fixed points of $\mathcal{A}$. Following some assumptions regarding the nonlinearities of the system, we show (by applying the Schauder fixed-point theorem) that operator $\mathcal{A}$ has at least one fixed point, which is a positive solution of our problem, when the positive parameters belong to some intervals. Then, we present intervals for the parameters for which our problem has no positive solution.

Keywords: Riemann–Liouville fractional differential equations; nonlocal boundary conditions; positive parameters; positive solutions; existence; nonexistence

MSC: 34A08; 34B10; 34B18

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1. Introduction

We consider the system of fractional differential equations with r_1-Laplacian and r_2-Laplacian operators

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}u(t))) + \mathfrak{a}(t)\mathfrak{f}(v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}v(t))) + \mathfrak{b}(t)\mathfrak{g}(u(t)) = 0, \ \ t \in (0,1), \end{cases} \tag{1}$$

supplemented with the uncoupled nonlocal boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \ j = 0,\dots,n-2; \ D_{0+}^{\beta_1}u(0) = 0, \ D_{0+}^{\gamma_0}u(1) = \sum_{j=1}^{p}\int_0^1 D_{0+}^{\gamma_j}u(\tau)\,d\mathfrak{H}_j(\tau) + \mathfrak{a}_0, \\ v^{(j)}(0) = 0, \ j = 0,\dots,m-2; \ D_{0+}^{\beta_2}v(0) = 0, \ D_{0+}^{\delta_0}v(1) = \sum_{j=1}^{q}\int_0^1 D_{0+}^{\delta_j}v(\tau)\,d\mathfrak{K}_j(\tau) + \mathfrak{b}_0, \end{cases} \tag{2}$$

where $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1 \in (n-1,n]$, $\beta_2 \in (m-1,m]$, $n, m \in \mathbb{N}$, $n, m \geq 3$, $p, q \in \mathbb{N}$, $\gamma_j \in \mathbb{R}$ for all $j = 0,1,\dots,p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $\delta_j \in \mathbb{R}$ for all $j = 0,1,\dots,q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $r_1, r_2 > 1$, $\varphi_{r_j}(\zeta) = |\zeta|^{r_j-2}\zeta$, $\varphi_{r_j}^{-1} = \varphi_{\varrho_j}$, $\varrho_j = \frac{r_j}{r_j-1}$, $j = 1,2$, $\mathfrak{a}_0$ and $\mathfrak{b}_0$ are positive parameters, the functions $\mathfrak{a}, \mathfrak{b} : [0,1] \to \mathbb{R}_+$ and $\mathfrak{f}, \mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, ($\mathbb{R}_+ = [0,\infty)$), the integrals from (2) are Riemann–Stieltjes integrals with $\mathfrak{H}_i$, $i = 1,\dots,p$ and $\mathfrak{K}_j$, $j = 1,\cdots,q$ functions of bounded variation, and D_{0+}^{κ} denotes the Riemann–Liouville derivative of order κ (for $\kappa = \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_i$ for $i = 0, 1, \ldots, p$, δ_j for $j = 0, 1, \ldots, q$). This paper is motivated by the applications of r-Laplacian operators in various fields such as fluid flow through porous media, nonlinear elasticity, nonlinear electrorheological fluid and glaciology, (for details, see [1] and its references).

In this paper, we provide sufficient conditions for the functions $\mathfrak{f}$ and $\mathfrak{g}$, and intervals for the parameters $\mathfrak{a}_0$ and $\mathfrak{b}_0$ such that problem (1), (2) has at least one positive solution or no positive solution. For the proof of the main existence result, we use the Schauder fixed-point theorem. Using a positive solution of (1), (2) we understand a pair of functions $(u, v) \in (C([0, 1]; \mathbb{R}_+))^2$, satisfying the system (1) and the boundary conditions (2), with $u(t) > 0$ and $v(t) > 0$ for all $t \in (0, 1]$. The method for studying problem (1), (2) consists of the following stages. First, we make a change in the unknown functions such that the new boundary conditions have no positive parameters, and then, by using the corresponding Green functions, we equivalently write this new problem as a system of nonlinear integral equations. By constructing an appropriate operator $\mathcal{A}$, the solutions of the integral system are the fixed points of $\mathcal{A}$. Following some assumptions regarding the nonlinearities of the system, we show that operator $\mathcal{A}$ has at least one fixed point, which is a positive solution of our problem, when the positive parameters belong to certain intervals. Then, we provide intervals for the parameters for which problem (1), (2) has no positive solution. We now present some recent results related to our problem. In [2], by using Guo-Krasnosel'skii fixed-point theorem, the author studied the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}u(t))) + \lambda f(t, u(t), v(t)) = 0, \ \ t \in (0, 1), \\ D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}v(t))) + \mu g(t, u(t), v(t)) = 0, \ \ t \in (0, 1), \end{cases} \tag{3}$$

subject to the boundary conditions (2) with $\mathfrak{a}_0 = \mathfrak{b}_0 = 0$, where $f, g \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and λ, μ are positive parameters. The author presented various intervals for λ and μ, such that problem (3), (2) with $\mathfrak{a}_0 = \mathfrak{b}_0 = 0$ has at least one positive solution ($u(t) > 0$ for all $t \in (0, 1]$, or $v(t) > 0$ for all $t \in (0, 1]$). The author also investigated the nonexistence of positive solutions. In [3], the authors studied the existence and nonexistence of positive solutions for the system (3) with the coupled boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \ j = 0, \ldots, n-2; \ \ D_{0+}^{\beta_1}u(0) = 0, \ \ D_{0+}^{\gamma_0}u(1) = \displaystyle\sum_{j=1}^{p} \int_0^1 D_{0+}^{\gamma_j}v(\tau)\, d\mathfrak{H}_j(\tau), \\ v^{(j)}(0) = 0, \ j = 0, \ldots, m-2; \ \ D_{0+}^{\beta_2}v(0) = 0, \ \ D_{0+}^{\delta_0}v(1) = \displaystyle\sum_{j=1}^{q} \int_0^1 D_{0+}^{\delta_j}u(\tau)\, d\mathfrak{K}_j(\tau), \end{cases}$$

where $\gamma_j \in \mathbb{R}$ for all $j = 0, 1, \ldots, p$, $0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_p \le \delta_0 < \beta_2 - 1$, $\delta_0 \ge 1$, $\delta_j \in \mathbb{R}$ for all $j = 0, 1, \ldots, q$, $0 \le \delta_1 < \delta_2 < \cdots < \delta_q \le \gamma_0 < \beta_1 - 1$, $\gamma_0 \ge 1$, $\mathfrak{H}_i$, $i = 1, \ldots, p$ and $\mathfrak{K}_j$, $j = 1, \ldots, q$ are functions of bounded variation. In [4], the authors investigated the positive solutions for the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + \mathfrak{a}(t)\mathfrak{f}(v(t)) = 0, \ \ t \in (0, 1), \\ D_{0+}^{\beta}v(t) + \mathfrak{b}(t)\mathfrak{g}(u(t)) = 0, \ \ t \in (0, 1), \end{cases}$$

supplemented with the integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(1) = \displaystyle\int_0^1 u(\tau)d\mathfrak{H}(\tau) + \mathfrak{a}_0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(1) = \displaystyle\int_0^1 v(\tau)d\mathfrak{K}(\tau) + \mathfrak{b}_0, \end{cases}$$

where $n - 1 < \alpha \le n$, $m - 1 < \beta \le m$, $n, m \in \mathbb{N}$, $n, m \ge 3$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{f}, \mathfrak{g}$ are nonnegative continuous functions, $\mathfrak{H}$ and $\mathfrak{K}$ are bounded variation functions, and $\mathfrak{a}_0$, $\mathfrak{b}_0$ are positive parameters. Other recent research regarding fractional differential equations and systems of fractional differential equations with or without Laplacian operators and their applications can be found in the papers [5–9], and in the monographs [10–12]. In comparison with other papers, the novelty of our work consists of the combination between the system of fractional differential equations (1), in which sequential fractional derivatives with r-Laplacian operators are considered, and the existence of positive parameters in the general integro-differential boundary conditions (2).

The paper has the following structure. In Section 2, we provide some preliminary results, including the Green functions associated with our problem (1), (2) and their properties. In Section 3, we present the main theorems for the existence and nonexistence of positive solutions for (1), (2). Section 4 contains an example to illustrate our results, and in Section 5, we provide our conclusions.

2. Auxiliary Results

In this section, we present some auxiliary results related to our problem (1), (2) from [2]. We first consider the fractional differential equation

$$D_{0+}^{\alpha_1}(\varphi_{r_1}(D_{0+}^{\beta_1}u(t)))+\mathfrak{h}(t)=0,\ \ t\in(0,1), \tag{4}$$

with the boundary conditions

$$u^{(j)}(0)=0,\ j=0,\ldots,n-2;\ \ D_{0+}^{\beta_1}u(0)=0,\ \ D_{0+}^{\gamma_0}u(1)=\sum_{j=1}^{p}\int_0^1 D_{0+}^{\gamma_j}u(\tau)\,d\mathfrak{H}_j(\tau), \tag{5}$$

where $\alpha_1\in(0,1]$, $\beta_1\in(n-1,n]$, $n\in\mathbb{N}$, $n\geq 3$, $p\in\mathbb{N}$, $\gamma_j\in\mathbb{R}$ for all $j=0,1,\ldots,p$, $0\leq\gamma_1<\gamma_2<\cdots<\gamma_p\leq\gamma_0<\beta_1-1$, $\gamma_0\geq 1$, $\mathfrak{H}_j$, $j=1,\ldots,p$ are bounded variation functions, and $\mathfrak{h}\in C[0,1]$. We denote using

$$\Delta_1=\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma_0)}-\sum_{j=1}^{p}\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\gamma_j)}\int_0^1\zeta^{\beta_1-\gamma_j-1}\,d\mathfrak{H}_j(\zeta).$$

Lemma 1. *If $\Delta_1\neq 0$, then the unique solution $u\in C[0,1]$ of problem (4), (5) is*

$$u(t)=\int_0^1\mathfrak{G}_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}\mathfrak{h}(s))\,ds,\ \ t\in[0,1], \tag{6}$$

where the Green function $\mathfrak{G}_1$ is given by

$$\mathfrak{G}_1(t,s)=\mathfrak{g}_1(t,s)+\frac{t^{\beta_1-1}}{\Delta_1}\sum_{i=1}^{p}\left(\int_0^1\mathfrak{g}_{2i}(\tau,s)\,d\mathfrak{H}_i(\tau)\right),\ \ t,s\in[0,1], \tag{7}$$

with

$$\mathfrak{g}_1(t,s)=\frac{1}{\Gamma(\beta_1)}\begin{cases}t^{\beta_1-1}(1-s)^{\beta_1-\gamma_0-1}-(t-s)^{\beta_1-1}, & 0\leq s\leq t\leq 1,\\ t^{\beta_1-1}(1-s)^{\beta_1-\gamma_0-1}, & 0\leq t\leq s\leq 1,\end{cases}$$

$$\mathfrak{g}_{2i}(\tau,s)=\frac{1}{\Gamma(\beta_1-\gamma_i)}\begin{cases}\tau^{\beta_1-\gamma_i-1}(1-s)^{\beta_1-\gamma_0-1}-(\tau-s)^{\beta_1-\gamma_i-1}, & 0\leq s\leq\tau\leq 1,\\ \tau^{\beta_1-\gamma_i-1}(1-s)^{\beta_1-\gamma_0-1}, & 0\leq\tau\leq s\leq 1,\end{cases}$$

$i=1,\ldots,p.$

Now, we consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha_2}(\varphi_{r_2}(D_{0+}^{\beta_2}v(t)))+\mathfrak{y}(t)=0,\ \ t\in(0,1), \tag{8}$$

with the boundary conditions

$$v^{(j)}(0) = 0,\ j = 0,\dots,m-2;\ D_{0+}^{\beta_2}v(0) = 0,\ D_{0+}^{\delta_0}v(1) = \sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}v(t)\,d\mathfrak{K}_i(t), \quad (9)$$

where $\alpha_2 \in (0,1]$, $\beta_2 \in (m-1,m]$, $m \in \mathbb{N}$, $m \geq 3$, $q \in \mathbb{N}$, $\delta_i \in \mathbb{R}$ for all $i = 0,\dots,q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $\mathfrak{K}_i$, $i = 1,\dots,q$ are bounded variation functions, and $\mathfrak{y} \in C[0,1]$. We denote using

$$\Delta_2 = \frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\delta_0)} - \sum_{j=1}^{q}\frac{\Gamma(\beta_2)}{\Gamma(\beta_2-\delta_j)}\int_0^1 \zeta^{\beta_2-\delta_j-1}\,d\mathfrak{K}_j(\zeta).$$

Lemma 2. *If $\Delta_2 \neq 0$, then the unique solution $v \in C[0,1]$ of problem (8), (9) is*

$$v(t) = \int_0^1 \mathfrak{G}_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}\mathfrak{y}(s))\,ds,\ \ t \in [0,1], \quad (10)$$

where the Green function $\mathfrak{G}_2$ is given by

$$\mathfrak{G}_2(t,s) = \mathfrak{g}_3(t,s) + \frac{t^{\beta_2-1}}{\Delta_2}\sum_{i=1}^{q}\left(\int_0^1 \mathfrak{g}_{4i}(\tau,s)\,d\mathfrak{K}_i(\tau)\right),\ \ t,s \in [0,1], \quad (11)$$

with

$$\begin{aligned}
\mathfrak{g}_3(t,s) &= \frac{1}{\Gamma(\beta_2)}\begin{cases} t^{\beta_2-1}(1-s)^{\beta_2-\delta_0-1} - (t-s)^{\beta_2-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta_2-1}(1-s)^{\beta_2-\delta_0-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\
\mathfrak{g}_{4i}(\tau,s) &= \frac{1}{\Gamma(\beta_2-\delta_i)}\begin{cases} \tau^{\beta_2-\delta_i-1}(1-s)^{\beta_2-\delta_0-1} - (\tau-s)^{\beta_2-\delta_i-1}, & 0 \leq s \leq \tau \leq 1, \\ \tau^{\beta_2-\delta_i-1}(1-s)^{\beta_2-\delta_0-1}, & 0 \leq \tau \leq s \leq 1, \end{cases} \\
& i = 1,\dots,q.
\end{aligned}$$

Lemma 3. *Assume that $\mathfrak{H}_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,p$, and $\mathfrak{K}_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,q$ are nondecreasing functions and $\Delta_1 > 0$, $\Delta_2 > 0$. Then, the Green functions $\mathfrak{G}_1$ and $\mathfrak{G}_2$ given by (7) and (11) have the following properties:*

(a) $\mathfrak{G}_1$, $\mathfrak{G}_2 : [0,1] \times [0,1] \to \mathbb{R}_+$ are continuous functions;

(b) $\mathfrak{G}_1(t,s) \leq \mathfrak{J}_1(s)$ for all t, $s \in [0,1]$, where $\mathfrak{J}_1(s) = \mathfrak{h}_1(s) + \frac{1}{\Delta_1}\sum_{i=1}^{p}\int_0^1 \mathfrak{g}_{2i}(\tau,s)\,d\mathfrak{H}_i(\tau)$, and $\mathfrak{h}_1(s) = \frac{1}{\Gamma(\beta_1)}[(1-s)^{\beta_1-\gamma_0-1} - (1-s)^{\beta_1-1}]$, $s \in [0,1]$;

(c) $\mathfrak{G}_1(t,s) \geq t^{\beta_1-1}\mathfrak{J}_1(s)$ for all t, $s \in [0,1]$;

(d) $\mathfrak{G}_2(t,s) \leq \mathfrak{J}_2(s)$ for all t, $s \in [0,1]$, where $\mathfrak{J}_2(s) = \mathfrak{h}_2(s) + \frac{1}{\Delta_2}\sum_{i=1}^{q}\int_0^1 \mathfrak{g}_{4i}(\tau,s)\,d\mathfrak{K}_i(\tau)$, and $\mathfrak{h}_2(s) = \frac{1}{\Gamma(\beta_2)}[(1-s)^{\beta_2-\delta_0-1} - (1-s)^{\beta_2-1}]$, $s \in [0,1]$;

(e) $\mathfrak{G}_2(t,s) \geq t^{\beta_2-1}\mathfrak{J}_2(s)$ for all t, $s \in [0,1]$.

Lemma 4. *Assume that $\mathfrak{H}_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,p$, and $\mathfrak{K}_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,q$ are nondecreasing functions, $\Delta_1 > 0$, $\Delta_2 > 0$, and $\mathfrak{h}$, $\mathfrak{y} \in C([0,1],\mathbb{R}_+)$. Then, the solutions u and v of problems (4), (5) and (8), (9), respectively, given by (6) and (10) satisfy the inequalities $u(t) \geq 0$, $v(t) \geq 0$, $u(t) \geq t^{\beta_1-1}u(\tau)$, $v(t) \geq t^{\beta_2-1}v(\tau)$ for all $t,\tau \in [0,1]$.*

3. Main Results

In this section, we study the existence and nonexistence of positive solutions for problem (1), (2) by imposing various conditions on the functions $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{f}$ and $\mathfrak{g}$. We present the assumptions that we will use in the sequel.

(*I*1) α_1, $\alpha_2 \in (0,1]$, $\beta_1 \in (n-1,n]$, $\beta_2 \in (m-1,m]$, n, $m \in \mathbb{N}$, n, $m \geq 3$, $p,q \in \mathbb{N}$, $\gamma_j \in \mathbb{R}$ for all $j = 0,1,\dots,p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $\delta_j \in \mathbb{R}$ for all $j = 0,1,\dots,q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, r_1, $r_2 > 1$,

$\varphi_{r_j}(\zeta) = |\zeta|^{r_j-2}\zeta$, $\varphi_{r_j}^{-1} = \varphi_{\varrho_j}$, $\varrho_j = \frac{r_j}{r_j-1}$, $j = 1,2$, $\mathfrak{a}_0 > 0$, $\mathfrak{b}_0 > 0$, $\mathfrak{H}_i$, $i = 1,\dots,p$ and $\mathfrak{K}_j$, $j = 1,\dots,q$ are nondecreasing functions, $\Delta_1 > 0$ and $\Delta_2 > 0$.

$(I2)$ The functions $\mathfrak{a}$, $\mathfrak{b} : [0,1] \to \mathbb{R}_+$ are continuous and there exist $t_1, t_2 \in (0,1)$ such that $\mathfrak{a}(t_1) > 0$, $\mathfrak{b}(t_2) > 0$.

$(I3)$ The functions $\mathfrak{f}$, $\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, and there exists $\mathfrak{c}_0 > 0$ such that $\mathfrak{f}(z) < \frac{\mathfrak{c}_0^{r_1-1}}{L}$, $\mathfrak{g}(z) < \frac{\mathfrak{c}_0^{r_2-1}}{L}$ for all $z \in [0,\mathfrak{c}_0]$, where

$$L = \max\left\{ \frac{\Lambda_i}{\Gamma(\alpha_i+1)} \left(\int_0^1 s^{\alpha_i(\varrho_i-1)} \mathfrak{J}_i(s)\, ds \right)^{r_i-1},\ i = 1,2 \right\},$$

with $\Lambda_1 = \sup_{t\in[0,1]} \mathfrak{a}(t)$, $\Lambda_2 = \sup_{t\in[0,1]} \mathfrak{b}(t)$.

$(I4)$ The functions $\mathfrak{f}$, $\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy the conditions $\lim_{z\to\infty} \frac{\mathfrak{f}(z)}{z^{r_1-1}} = \infty$, $\lim_{z\to\infty} \frac{\mathfrak{g}(z)}{z^{r_2-1}} = \infty$.

Using $(I1)$, $(I2)$ and Lemma 3, we deduce that the constant L from assumption $(I3)$ is positive.

We consider the problems

$$\begin{cases} D_{0+}^{\alpha_1}\left(\varphi_{r_1}(D_{0+}^{\beta_1}x(t))\right) = 0, \quad t \in (0,1), \\ x^{(j)}(0) = 0, \quad j = 0,\dots,n-2, \quad D_{0+}^{\beta_1}x(0) = 0, \quad D_{0+}^{\gamma_0}x(1) = \sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}x(\tau)\, d\mathfrak{H}_i(\tau) + 1, \end{cases} \tag{12}$$

$$\begin{cases} D_{0+}^{\alpha_2}\left(\varphi_{r_2}(D_{0+}^{\beta_2}y(t))\right) = 0, \quad t \in (0,1), \\ y^{(j)}(0) = 0, \quad j = 0,\dots,m-2, \quad D_{0+}^{\beta_2}y(0) = 0, \quad D_{0+}^{\delta_0}y(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\delta_i}y(\tau)\, d\mathfrak{K}_i(\tau) + 1, \end{cases} \tag{13}$$

The aforementioned problems (12) and (13) have the solutions $x(t) = \frac{t^{\beta_1-1}}{\Delta_1}$ and $y(t) = \frac{t^{\beta_2-1}}{\Delta_2}$, $t \in [0,1]$, respectively. Using $(I1)$, we have $x(t) > 0$ and $y(t) > 0$ for all $t \in (0,1]$. For a solution (u,v) of problem (1), (2), we define the functions $h(t) = u(t) - \mathfrak{a}_0 x(t) = u(t) - \frac{\mathfrak{a}_0 t^{\beta_1-1}}{\Delta_1}$, and $k(t) = v(t) - \mathfrak{b}_0 y(t) = v(t) - \frac{\mathfrak{b}_0 t^{\beta_2-1}}{\Delta_2}$, for $t \in [0,1]$. Then (1), (2) can equivalently be written as the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1}\left(\varphi_{r_1}(D_{0+}^{\beta_1}h(t))\right) + \mathfrak{a}(t)\mathfrak{f}(k(t) + \mathfrak{b}_0 y(t)) = 0, \quad t \in (0,1), \\ D_{0+}^{\alpha_2}\left(\varphi_{r_2}(D_{0+}^{\beta_2}k(t))\right) + \mathfrak{b}(t)\mathfrak{g}(h(t) + \mathfrak{a}_0 x(t)) = 0, \quad t \in (0,1), \end{cases} \tag{14}$$

with the boundary conditions

$$\begin{cases} h^{(j)}(0) = 0, \ j = 0,\dots,n-2; \ D_{0+}^{\beta_1}h(0) = 0, \ D_{0+}^{\gamma_0}h(1) = \sum_{j=1}^{p} \int_0^1 D_{0+}^{\gamma_j}h(\tau)\, d\mathfrak{H}_j(\tau), \\ k^{(j)}(0) = 0, \ j = 0,\dots,m-2; \ D_{0+}^{\beta_2}k(0) = 0, \ D_{0+}^{\delta_0}k(1) = \sum_{j=1}^{q} \int_0^1 D_{0+}^{\delta_j}k(\tau)\, d\mathfrak{K}_j(\tau). \end{cases} \tag{15}$$

Using the Green functions $\mathfrak{G}_1$ and $\mathfrak{G}_2$, Lemmas 1 and 2, a pair of functions (h,k) is a solution of problem (14), (15) if, and only if, (h,k) is a solution of the system of nonlinear integral equations

$$\begin{cases} h(t) = \int_0^1 \mathfrak{G}_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\mathfrak{a}(s)\mathfrak{f}(k(s) + \mathfrak{b}_0 y(s))))\, ds, \quad t \in [0,1], \\ k(t) = \int_0^1 \mathfrak{G}_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}(\mathfrak{b}(s)\mathfrak{g}(h(s) + \mathfrak{a}_0 x(s))))\, ds, \quad t \in [0,1]. \end{cases} \tag{16}$$

We consider the Banach space $\mathcal{X} = C[0,1]$ with the supremum norm $\|h\| = \sup_{\tau\in[0,1]} |h(\tau)|$ for $h \in \mathcal{X}$, and the Banach space $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ with the norm $\|(h,k)\|_{\mathcal{Y}} = \max\{\|h\|, \|k\|\}$ for $(h,k) \in \mathcal{Y}$. We define the set $\mathcal{E} = \{(h,k) \in \mathcal{Y},\ 0 \le h(t) \le c_0,\ 0 \le k(t) \le c_0\}$. We also define the operator $\mathcal{A} : \mathcal{E} \to \mathcal{Y}$, $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$,

$$\mathcal{A}_1(h,k)(t) = \int_0^1 \mathfrak{G}_1(t,s)\varphi_{\varrho_1}(I_{0+}^{\alpha_1}(\mathfrak{a}(s)\mathfrak{f}(k(s)+\mathfrak{b}_0 y(s))))\,ds,\ \ t \in [0,1],$$
$$\mathcal{A}_2(h,k)(t) = \int_0^1 \mathfrak{G}_2(t,s)\varphi_{\varrho_2}(I_{0+}^{\alpha_2}(\mathfrak{b}(s)\mathfrak{g}(h(s)+\mathfrak{a}_0 x(s))))\,ds,\ \ t \in [0,1],$$

for $(h,k) \in \mathcal{E}$. We remark that (h,k) is a solution of system (16) if and only if (h,k) is a fixed point of operator $\mathcal{A}$.

Our Theorem 1 is the following existence result for problem (1), (2).

Theorem 1. *We suppose that assumptions $(I1)-(I3)$ are satisfied. Then, there exist $\mathfrak{a}_1 > 0$ and $\mathfrak{b}_1 > 0$ such that for any $\mathfrak{a}_0 \in (0, \mathfrak{a}_1]$ and $\mathfrak{b}_0 \in (0, \mathfrak{b}_1]$, the problem (1), (2) has at least one positive solution.*

Proof. By assumption $(I3)$, we find that there exist $p_0 > 0$ and $q_0 > 0$ such that $\mathfrak{f}(z) \le \frac{c_0^{r_1-1}}{L}$ for all $z \in [0, c_0 + p_0]$, and $\mathfrak{g}(z) \le \frac{c_0^{r_2-1}}{L}$ for all $z \in [0, c_0 + q_0]$. We define $\mathfrak{a}_1 = q_0\Delta_1$ and $\mathfrak{b}_1 = p_0\Delta_2$. Let $\mathfrak{a}_0 \in (0, \mathfrak{a}_1]$ and $\mathfrak{b}_0 \in (0, \mathfrak{b}_1]$. Then, we obtain

$$\mathfrak{f}(k(s)+\mathfrak{b}_0 y(s)) \le \frac{\mathfrak{c}_0^{r_1-1}}{L},\ \ \mathfrak{g}(h(s)+\mathfrak{a}_0 x(s)) \le \frac{\mathfrak{c}_0^{r_2-1}}{L},$$

for all $s \in [0,1]$ and $(h,k) \in \mathcal{E}$. Hence, by using Lemma 4, we deduce that $\mathcal{A}_i(h,k)(t) \ge 0$, $i = 1,2$, for all $t \in [0,1]$ and $(h,k) \in \mathcal{E}$. By Lemma 3, for all $(h,k) \in \mathcal{E}$, we obtain

$$I_{0+}^{\alpha_1}(\mathfrak{a}(s)\mathfrak{f}(k(s)+\mathfrak{b}_0 y(s))) = \frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\mathfrak{f}(k(\tau)+\mathfrak{b}_0 y(\tau))\,d\tau$$
$$\le \frac{\mathfrak{c}_0^{r_1-1}}{L\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau \le \frac{\Lambda_1\mathfrak{c}_0^{r_1-1}}{L\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\,d\tau = \frac{\Lambda_1\mathfrak{c}_0^{r_1-1}s^{\alpha_1}}{L\Gamma(\alpha_1+1)},\ \ \forall s \in [0,1],$$

and then,

$$\mathcal{A}_1(h,k)(t) \le \int_0^1 \mathfrak{J}_1(s)\varphi_{\varrho_1}\left(\frac{\Lambda_1\mathfrak{c}_0^{r_1-1}s^{\alpha_1}}{L\Gamma(\alpha_1+1)}\right)ds$$
$$= \left(\frac{\Lambda_1\mathfrak{c}_0^{r_1-1}}{L\Gamma(\alpha_1+1)}\right)^{\varrho_1-1}\int_0^1 \mathfrak{J}_1(s)s^{\alpha_1(\varrho_1-1)}\,ds \le \mathfrak{c}_0,\ \ \forall t \in [0,1].$$

In a similar manner, for all $(h,k) \in \mathcal{E}$ we have

$$I_{0+}^{\alpha_2}(\mathfrak{b}(s)\mathfrak{g}(h(s)+\mathfrak{a}_0 x(s))) \le \frac{\Lambda_2\mathfrak{c}_0^{r_2-1}s^{\alpha_2}}{L\Gamma(\alpha_2+1)},\ \ \forall s \in [0,1],$$

and

$$\mathcal{A}_2(h,k)(t) \le \int_0^1 \mathfrak{J}_2(s)\varphi_{\varrho_2}\left(\frac{\Lambda_2\mathfrak{c}_0^{r_2-1}s^{\alpha_2}}{L\Gamma(\alpha_2+1)}\right)ds \le \mathfrak{c}_0,\ \ \forall t \in [0,1].$$

Therefore, we find that $\mathcal{A}(\mathcal{E}) \subset \mathcal{E}$. By using standard arguments, we deduce that $\mathcal{A}$ is a completely continuous operator. Therefore, using the Schauder fixed-point theorem, we conclude that $\mathcal{A}$ has a fixed point $(h,k) \in \mathcal{E}$, which is a nonnegative solution for problem (16) or, equivalently, for problem (14), (15). Therefore, (u,v), where $u(t) = h(t) + \mathfrak{a}_0 x(t) = h(t) + \mathfrak{a}_0\frac{t^{\beta_1-1}}{\Delta_1}$, $v(t) = k(t) + \mathfrak{b}_0 y(t) = k(t) + \mathfrak{b}_0\frac{t^{\beta_2-1}}{\Delta_2}$ for $t \in [0,1]$, is a positive solution of problem (1), (2). This solution (u,v) satisfies the conditions $\frac{\mathfrak{a}_0 t^{\beta_1-1}}{\Delta_1} \leq u(t) \leq \frac{\mathfrak{a}_0 t^{\beta_1-1}}{\Delta_1} + \mathfrak{c}_0$ and $\frac{\mathfrak{b}_0 t^{\beta_2-1}}{\Delta_2} \leq v(t) \leq \frac{\mathfrak{b}_0 t^{\beta_2-1}}{\Delta_2} + \mathfrak{c}_0$ for all $t \in [0,1]$. □

Theorem 2 is the following nonexistence result for problem (1), (2).

Theorem 2. *We suppose that assumptions $(I1)$, $(I2)$ and $(I4)$ are satisfied. Then, there exist $\mathfrak{a}_2 > 0$ and $\mathfrak{b}_2 > 0$ such that, for any $\mathfrak{a}_0 \geq \mathfrak{a}_2$ and $\mathfrak{b}_0 \geq \mathfrak{b}_2$, the problem (1), (2) has no positive solution.*

Proof. By $(I2)$, there exist $[\theta_1, \theta_2] \subset (0,1)$, $\theta_1 < \theta_2$ such that $t_1, t_2 \in (\theta_1, \theta_2)$, and then

$$\Xi_1 := \int_{\theta_1}^{\theta_2} \mathfrak{J}_1(s) \left(\int_{\theta_1}^{s} \mathfrak{a}(\tau)(s-\tau)^{\alpha_1-1}\, d\tau \right)^{\varrho_1-1} ds > 0,$$
$$\Xi_2 := \int_{\theta_1}^{\theta_2} \mathfrak{J}_2(s) \left(\int_{\theta_1}^{s} \mathfrak{b}(\tau)(s-\tau)^{\alpha_2-1}\, d\tau \right)^{\varrho_2-1} ds > 0.$$

We consider

$$R = \max\left\{ 2^{r_i-1}\Gamma(\alpha_i)\left(\Xi_i \theta_1^{\beta_1+\beta_2-1}\right)^{1-r_i},\ i = 1,2 \right\}.$$

By using $(I4)$, for teh R defined above, we deduce that there exists $M_0 > 0$ such that $\mathfrak{f}(z) \geq R z^{r_1-1}$, $\mathfrak{g}(z) \geq R z^{r_2-1}$ for all $z \geq M_0$. We define $\mathfrak{a}_2 = \frac{M_0 \Delta_1}{\theta_1^{\beta_1}}$ and $\mathfrak{b}_2 = \frac{M_0 \Delta_2}{\theta_1^{\beta_2-1}}$. Let $\mathfrak{a}_0 \geq \mathfrak{a}_2$ and $\mathfrak{b}_0 \geq \mathfrak{b}_2$. We assume that (u,v) is a positive solution of (1), (2). Then, (h,k) where $h(t) = u(t) - \mathfrak{a}_0 x(t) = u(t) - \mathfrak{a}_0 \frac{t^{\beta_1-1}}{\Delta_1}$, $k(t) = v(t) - \mathfrak{b}_0 y(t) = v(t) - \mathfrak{b}_0 \frac{t^{\beta_2-1}}{\Delta_2}$ for $t \in [0,1]$, is a solution for (14), (15) or equivalently for (16). By using Lemma 4, we have $h(t) \geq t^{\beta_1-1}\|h\|$, $k(t) \geq t^{\beta_2-1}\|k\|$ for all $t \in [0,1]$. Then, $\inf_{t\in[\theta_1,\theta_2]} h(t) \geq \theta_1^{\beta_1-1}\|h\|$, $\inf_{t\in[\theta_1,\theta_2]} k(t) \geq \theta_1^{\beta_2-1}\|k\|$. Using the definition of x and y, we obtain $\inf_{t\in[\theta_1,\theta_2]} x(t) = \frac{\theta_1^{\beta_1-1}}{\Delta_1} = \theta_1^{\beta_1-1}\|x\|$ and $\inf_{t\in[\theta_1,\theta_2]} y(t) = \frac{\theta_1^{\beta_2-1}}{\Delta_2} = \theta_1^{\beta_2-1}\|y\|$. Therefore, we find

$$\begin{aligned}
&\inf_{t\in[\theta_1,\theta_2]} (h(t) + \mathfrak{a}_0 x(t)) \geq \inf_{t\in[\theta_1,\theta_2]} h(t) + \mathfrak{a}_0 \inf_{t\in[\theta_1,\theta_2]} x(t) \\
&\geq \theta_1^{\beta_1-1}\|h\| + \mathfrak{a}_0 \theta_1^{\beta_1-1}\|x\| \geq \theta_1^{\beta_1-1}\|h + \mathfrak{a}_0 x\|, \\
&\inf_{t\in[\theta_1,\theta_2]} (k(t) + \mathfrak{b}_0 y(t)) \geq \inf_{t\in[\theta_1,\theta_2]} k(t) + \mathfrak{b}_0 \inf_{t\in[\theta_1,\theta_2]} y(t) \\
&\geq \theta_1^{\beta_2-1}\|k\| + \mathfrak{b}_0 \theta_1^{\beta_2-1|}\|y\| \geq \theta_1^{\beta_2-1}\|k + \mathfrak{b}_0 y\|.
\end{aligned}$$

In addition, we have

$$\inf_{t\in[\theta_1,\theta_2]} (h(t) + \mathfrak{a}_0 x(t)) \geq \theta_1^{\beta_1-1}\|h\| + \frac{a_0 \theta_1^{\beta_1-1}}{\Delta_1} \geq \frac{\mathfrak{a}_0 \theta_1^{\beta_1-1}}{\Delta_1} \geq \frac{\mathfrak{a}_2 \theta_1^{\beta_1-1}}{\Delta_1} = M_0,$$
$$\inf_{t\in[\theta_1,\theta_2]} (k(t) + \mathfrak{a}_0 y(t)) \geq \theta_1^{\beta_2-1}\|k\| + \frac{b_0 \theta_1^{\beta_2-1}}{\Delta_2} \geq \frac{\mathfrak{b}_0 \theta_1^{\beta_2-1}}{\Delta_2} \geq \frac{\mathfrak{b}_2 \theta_1^{\beta_2-1}}{\Delta_2} = M_0.$$

Now, by using Lemma 4 and the above inequalities, we obtain

$$
\begin{aligned}
&I_{0+}^{\alpha_1}(\mathfrak{a}(s)\mathfrak{f}(k(s)+\mathfrak{b}_0 y(s))) \\
&\geq \frac{1}{\Gamma(\alpha_1)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\mathfrak{f}(k(\tau)+\mathfrak{b}_0 y(\tau))\,d\tau \\
&\geq \frac{R}{\Gamma(\alpha_1)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)(k(\tau)+\mathfrak{b}_0 y(\tau))^{r_1-1}\,d\tau \\
&\geq \frac{R}{\Gamma(\alpha_1)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\left(\inf_{\tau\in[\theta_1,\theta_2]}(k(\tau)+\mathfrak{b}_0 y(\tau))\right)^{r_1-1}d\tau \\
&\geq \frac{RM_0^{r_1-1}}{\Gamma(\alpha_1)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau,\ \ \forall\, s\in[\theta_1,\theta_2],
\end{aligned}
$$

and then

$$
\begin{aligned}
h(\theta_1) &\geq \int_0^1 \theta_1^{\beta_1-1}\mathfrak{J}_1(s)\varphi_{\varrho_1}\left(I_{0+}^{\alpha_1}(\mathfrak{a}(s)\mathfrak{f}(k(s)+\mathfrak{b}_0 y(s)))\right)ds \\
&\geq \int_{\theta_1}^{\theta_2}\theta_1^{\beta_1-1}\mathfrak{J}_1(s)\varphi_{\varrho_1}\left(\frac{RM_0^{r_1-1}}{\Gamma(\alpha_1)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau\right)ds \\
&= \frac{R^{\varrho_1-1}M_0\theta_1^{\beta_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}}\int_{\theta_1}^{\theta_2}\mathfrak{J}_1(s)\left(\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau\right)^{\varrho_1-1}ds \\
&= \frac{R^{\varrho_1-1}M_0\theta_1^{\beta_1-1}\Xi_1}{(\Gamma(\alpha_1))^{\varrho_1-1}} > 0.
\end{aligned}
$$

We deduce that $\|h\| \geq h(\theta_1) > 0$. In a similar manner, we find

$$
\begin{aligned}
&I_{0+}^{\alpha_2}(\mathfrak{b}(s)\mathfrak{g}(h(s)+\mathfrak{a}_0 x(s))) \\
&\geq \frac{R}{\Gamma(\alpha_2)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_2-1}\mathfrak{b}(\tau)\left(\inf_{\tau\in[\theta_1,\theta_2]}(h(\tau)+\mathfrak{a}_0 x(\tau))\right)^{r_2-1}d\tau \\
&\geq \frac{RM_0^{r_2-1}}{\Gamma(\alpha_2)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_2-1}\mathfrak{b}(\tau)\,d\tau,\ \ \forall\, s\in[\theta_1,\theta_2],
\end{aligned}
$$

and so

$$
k(\theta_1) \geq \frac{R^{\varrho_2-1}M_0\theta_1^{\beta_2-1}\Xi_2}{(\Gamma(\alpha_2))^{\varrho_2-1}} > 0.
$$

Therefore, $\|k\| \geq k(\theta_1) > 0$.

Besides, from the above inequalities, we obtain

$$
\begin{aligned}
&I_{0+}^{\alpha_1}(\mathfrak{a}(s)\mathfrak{f}(k(s)+\mathfrak{b}_0 y(s))) \\
&\geq \frac{R}{\Gamma(\alpha_1)}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\left(\inf_{\tau\in[\theta_1,\theta_2]}(k(\tau)+\mathfrak{b}_0 y(\tau))\right)^{r_1-1}d\tau \\
&\geq \frac{R\theta_1^{(\beta_2-1)(r_1-1)}}{\Gamma(\alpha_1)}\|k+\mathfrak{b}_0 y\|^{r_1-1}\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau,\ \ \forall\, s\in[\theta_1,\theta_2],
\end{aligned}
$$

and then

$$
\begin{aligned}
h(\theta_1) &\geq \int_{\theta_1}^{\theta_2}\theta_1^{\beta_1-1}\mathfrak{J}_1(s)\left(\frac{R\theta_1^{(\beta_2-1)(r_1-1)}}{\Gamma(\alpha_1)}\right)^{\varrho_1-1}\|k+\mathfrak{b}_0 y\|\left(\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau\right)^{\varrho_1-1}ds \\
&= \frac{\theta_1^{\beta_1+\beta_2-2}R^{\varrho_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}}\|k+\mathfrak{b}_0 y\|\int_{\theta_1}^{\theta_2}\mathfrak{J}_1(s)\left(\int_{\theta_1}^{s}(s-\tau)^{\alpha_1-1}\mathfrak{a}(\tau)\,d\tau\right)^{\varrho_1-1}ds \\
&= \frac{\theta_1^{\beta_1+\beta_2-2}R^{\varrho_1-1}}{(\Gamma(\alpha_1))^{\varrho_1-1}}\Xi_1\|k+\mathfrak{b}_0 y\| \geq 2\|k+\mathfrak{b}_0 y\| \geq 2\|k\|.
\end{aligned}
$$

Hence,

$$
\|k\| \leq \frac{h(\theta_1)}{2} \leq \frac{\|h\|}{2}. \tag{17}
$$

In a similar manner, we deduce

$$
\begin{aligned}
& I_{0+}^{\alpha_2}(\mathfrak{b}(s)\mathfrak{g}(h(s)+\mathfrak{a}_0 x(s))) \\
& \geq \frac{R}{\Gamma(\alpha_2)} \int_{\theta_1}^{s} (s-\tau)^{\alpha_2-1}\mathfrak{b}(\tau)\left(\inf_{\tau\in[\theta_1,\theta_2]}(h(\tau)+\mathfrak{a}_0 x(\tau))\right)^{r_2-1} d\tau \\
& \geq \frac{R\theta_1^{(\beta_1-1)(r_2-1)}}{\Gamma(\alpha_2)} \|h+\mathfrak{a}_0 x\|^{r_2-1} \int_{\theta_1}^{s} (s-\tau)^{\alpha_2-1}\mathfrak{b}(\tau)\,d\tau, \ \ \forall s\in[\theta_1,\theta_2],
\end{aligned}
$$

and so

$$
\begin{aligned}
k(\theta_1) & \geq \int_{\theta_1}^{\theta_2} \theta_1^{\beta_2-1}\mathfrak{J}_2(s)\left(\frac{R\theta_1^{(\beta_1-1)(r_2-1)}}{\Gamma(\alpha_2)}\right)^{\varrho_2-1} \|h+\mathfrak{a}_0 x\| \left(\int_{\theta_1}^{s}(s-\tau)^{\alpha_2-1}\mathfrak{b}(\tau)\,d\tau\right)^{\varrho_2-1} ds \\
& = \frac{\theta_1^{\beta_1+\beta_2-2}R^{\varrho_2-1}}{(\Gamma(\alpha_2))^{\varrho_2-1}} \|h+\mathfrak{a}_0 x\| \int_{\theta_1}^{\theta_2}\mathfrak{J}_2(s)\left(\int_{\theta_1}^{s}(s-\tau)^{\alpha_2-1}\mathfrak{b}(\tau)\,d\tau\right)^{\varrho_2-1} ds \\
& = \frac{\theta_1^{\beta_1+\beta_2-2}R^{\varrho_2-1}}{(\Gamma(\alpha_2))^{\varrho_2-1}} \Xi_2 \|h+\mathfrak{a}_0 x\| \geq 2\|h+\mathfrak{a}_0 x\| \geq 2\|h\|.
\end{aligned}
$$

Hence,

$$
\|h\| \leq \frac{k(\theta_1)}{2} \leq \frac{\|k\|}{2}. \tag{18}
$$

Therefore, using (17) and (18), we conclude that $\|h\| \leq \frac{\|k\|}{2} \leq \frac{\|h\|}{4}$, which contradicts the inequality $\|h\| > 0$. Then, problem (1), (2) has no positive solution. $\square$

4. An Example

We consider $\alpha_1 = 1/2$, $\alpha_2 = 1/3$, $\beta_1 = 9/4$, $n = 3$, $\beta_2 = 17/5$, $m = 4$, $p = 2$, $q = 1$, $\gamma_0 = 8/7$, $\gamma_1 = 1/5$, $\gamma_2 = 2/3$, $\delta_0 = 11/6$, $\delta_1 = 3/4$, $r_1 = 21/5$, $\varrho_1 = 21/16$, $r_2 = 11/2$, $\varrho_2 = 11/9$, $\mathfrak{a}(t) = 1$, $\mathfrak{b}(t) = 1$ for all $t \in [0,1]$, $\mathfrak{H}_1(t) = 7t/6$ for all $t \in [0,1]$, $\mathfrak{H}_2(t) = \{1/2,\ t\in[0,11/23);\ 17/18,\ t\in[11/23,1]\}$, $\mathfrak{K}_1(t) = \{2,\ t\in[0,2/5);\ 61/21,\ t\in[2/5,1]\}$. We also consider the functions $\mathfrak{f}(z) = \frac{\sigma_1 z^{\omega_1}}{z^{\omega_2}+\sigma_2}$, $\mathfrak{g}(z) = \frac{\sigma_3 z^{\omega_3}}{z^{\omega_4}+\sigma_4}$ for all $z \in \mathbb{R}_+$, with $\sigma_i > 0$, $\omega_i > 0$, $i = 1,\dots,4$, $\omega_1 > \omega_2 + 16/5$, $\omega_3 > \omega_4 + 9/2$. We have $\lim_{z\to\infty}\frac{\mathfrak{f}(z)}{z^{r_1-1}} = \infty$ and $\lim_{z\to\infty}\frac{\mathfrak{g}(z)}{z^{r_2-1}} = \infty$.

Hence, we consider the system of Riemann–Liouville fractional differential equations

$$
\begin{cases}
D_{0+}^{1/2}\left(\varphi_{21/5}(D_{0+}^{9/4}u(t))\right) + \dfrac{\sigma_1(v(t))^{\omega_1}}{(v(t))^{\omega_2}+\sigma_2} = 0, \ \ t\in(0,1), \\
D_{0+}^{1/3}\left(\varphi_{11/2}(D_{0+}^{17/5}v(t))\right) + \dfrac{\sigma_3(u(t))^{\omega_3}}{(u(t))^{\omega_4}+\sigma_4} = 0, \ \ t\in(0,1),
\end{cases} \tag{19}
$$

with the boundary conditions

$$
\begin{cases}
u(0) = u'(0) = 0, \ D_{0+}^{9/4}u(0) = 0, \ D_{0+}^{8/7}u(1) = \dfrac{7}{6}\displaystyle\int_0^1 D_{0+}^{1/5}u(\tau)\,d\tau + \dfrac{4}{9}D_{0+}^{2/3}u\left(\dfrac{11}{23}\right) + \mathfrak{a}_0, \\
v(0) = v'(0) = v''(0) = 0, \ D_{0+}^{17/5}v(0) = 0, \ D_{0+}^{11/6}v(1) = \dfrac{19}{21}D_{0+}^{3/4}v\left(\dfrac{2}{5}\right) + \mathfrak{b}_0.
\end{cases} \tag{20}
$$

We obtain $\Delta_1 \approx 0.19646507 > 0$, $\Delta_2 \approx 2.94848267 > 0$. Therefore, assumptions $(I1)$, $(I2)$ and $(I4)$ are satisfied. In addition, we find

$$
\begin{aligned}
\mathfrak{g}_1(t,s) &= \frac{1}{\Gamma(9/4)}\begin{cases} t^{5/4}(1-s)^{3/28}-(t-s)^{5/4}, & 0\le s\le t\le 1,\\ t^{5/4}(1-s)^{3/28}, & 0\le t\le s\le 1,\end{cases}\\
\mathfrak{g}_{21}(t,s) &= \frac{1}{\Gamma(41/20)}\begin{cases} t^{21/20}(1-s)^{3/28}-(t-s)^{21/20}, & 0\le s\le t\le 1,\\ t^{21/20}(1-s)^{3/28}, & 0\le t\le s\le 1,\end{cases}\\
\mathfrak{g}_{22}(t,s) &= \frac{1}{\Gamma(19/12)}\begin{cases} t^{7/12}(1-s)^{3/28}-(t-s)^{7/12}, & 0\le s\le t\le 1,\\ t^{7/12}(1-s)^{3/28}, & 0\le t\le s\le 1,\end{cases}\\
\mathfrak{g}_{3}(t,s) &= \frac{1}{\Gamma(17/5)}\begin{cases} t^{12/5}(1-s)^{17/30}-(t-s)^{12/5}, & 0\le s\le t\le 1,\\ t^{12/5}(1-s)^{17/30}, & 0\le t\le s\le 1,\end{cases}\\
\mathfrak{g}_{41}(t,s) &= \frac{1}{\Gamma(53/20)}\begin{cases} t^{33/20}(1-s)^{17/30}-(t-s)^{33/20}, & 0\le s\le t\le 1,\\ t^{33/20}(1-s)^{17/30}, & 0\le t\le s\le 1,\end{cases}\\
\mathfrak{G}_1(t,s) &= \mathfrak{g}_1(t,s)+\frac{t^{5/4}}{\Delta_1}\left[\frac{7}{6}\int_0^1 \mathfrak{g}_{21}(\tau,s)\,d\tau+\frac{4}{9}\mathfrak{g}_{22}\left(\frac{11}{23},s\right)\right],\\
\mathfrak{G}_2(t,s) &= \mathfrak{g}_3(t,s)+\frac{19t^{12/5}}{21\Delta_2}\mathfrak{g}_{41}\left(\frac{2}{5},s\right),\\
\mathfrak{h}_1(s) &= \frac{1}{\Gamma(9/4)}\left[(1-s)^{3/28}-(1-s)^{5/4}\right],\\
\mathfrak{h}_2(s) &= \frac{1}{\Gamma(17/5)}\left[(1-s)^{17/30}-(1-s)^{12/5}\right],
\end{aligned}
$$

for all $t,s\in[0,1]$. In addition, we deduce

$$
\mathfrak{J}_1(s)=\begin{cases}
\mathfrak{h}_1(s)+\dfrac{1}{\Delta_1}\left\{\dfrac{7}{6\Gamma(61/20)}(1-s)^{3/28}-\dfrac{7}{6\Gamma(61/20)}(1-s)^{41/20}\right.\\
\quad\left.+\dfrac{4}{9\Gamma(19/12)}\left[\left(\dfrac{11}{23}\right)^{7/12}(1-s)^{3/28}-\left(\dfrac{11}{23}-s\right)^{7/12}\right]\right\}, & 0\le s<\dfrac{11}{23},\\
\mathfrak{h}_1(s)+\dfrac{1}{\Delta_1}\left\{\dfrac{7}{6\Gamma(61/20)}(1-s)^{3/28}-\dfrac{7}{6\Gamma(61/20)}(1-s)^{41/20}\right.\\
\quad\left.+\dfrac{4}{9\Gamma(19/12)}\left(\dfrac{11}{23}\right)^{7/12}(1-s)^{3/28}\right], & \dfrac{11}{23}\le s\le 1,
\end{cases}
$$

$$
\mathfrak{J}_2(s)=\begin{cases}
\mathfrak{h}_2(s)+\dfrac{19}{21\Delta_2\Gamma(53/20)}\left[\left(\dfrac{2}{5}\right)^{33/20}(1-s)^{17/30}-\left(\dfrac{2}{5}-s\right)^{33/20}\right], & 0\le s<\dfrac{2}{5},\\
\mathfrak{h}_2(s)+\dfrac{19}{21\Delta_2\Gamma(53/20)}\left(\dfrac{2}{5}\right)^{33/20}(1-s)^{17/30}, & \dfrac{2}{5}\le s\le 1,
\end{cases}
$$

After some computations, we obtain $\int_0^1 s^{5/32}\mathfrak{J}_1(s)\,ds\approx 2.7671383$, $\int_0^1 s^{2/27}\mathfrak{J}_2(s)\,ds\approx 0.12990129$, $\Lambda_1=1$, $\Lambda_2=1$ and $L\approx 29.30581677$. We take $\mathfrak{c}_0=1$ and, if we choose σ_i, $i=1,\ldots,4$ which satisfy the conditions $\sigma_1<\frac{1+\sigma_2}{L}$ and $\sigma_3<\frac{1+\sigma_4}{L}$, then we deduce $\mathfrak{f}(z)\le\frac{\sigma_1}{1+\sigma_2}<\frac{1}{L}$ and $\mathfrak{g}(z)\le\frac{\sigma_3}{1+\sigma_4}<\frac{1}{L}$ for all $z\in[0,1]$. For example, if $\sigma_2=1$ and $\sigma_4=2$, then for $\sigma_1\le 0.068$ and $\sigma_3\le 0.102$, the above conditions for $\mathfrak{f}$ and $\mathfrak{g}$ are satisfied. Hence, assumption $(I3)$ is also satisfied. Using Theorems 1 and 2, we conclude that there exist $\mathfrak{a}_1$, $\mathfrak{b}_1$, $\mathfrak{a}_2$, $\mathfrak{b}_2$ such that, for any $\mathfrak{a}_0\in(0,\mathfrak{a}_1]$ and $\mathfrak{b}_0\in(0,\mathfrak{b}_1]$ there exists at least one positive solution of problem (19), (20), and, for any $\mathfrak{a}_0\ge\mathfrak{a}_2$ and $\mathfrak{b}_0\ge\mathfrak{b}_2$, there exists no positive solution of (19), (20).

5. Conclusions

In this paper, we studied the system of Riemann–Liouville fractional differential Equation (1) with r_1-Laplacian and r_2-Laplacian operators, supplemented with the nonlocal uncoupled boundary conditions (2), which contain fractional derivatives of various orders, Riemann–Stieltjes integrals, and two positive parameters. The functions $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{f}$ and $\mathfrak{g}$ from the system are continuous ones and satisfy some additional assumptions. We presented some auxiliary results, including the associated Green functions with their properties. Then, we investigated problem (1), (2) in some stages. First, we made a change in the unknown functions, such that the new boundary conditions have no positive parameters, and then, by using the Green functions, we equivalently wrote this new problem as the system of nonlinear integral equations (16). By constructing an appropriate operator $\mathcal{A}$, the solutions of the integral system are the fixed points of $\mathcal{A}$. By applying the Schauder fixed-point theorem, we showed that the operator $\mathcal{A}$ has at least one fixed point, which is a positive solution of our problem, when the positive parameters belong to some intervals. Then, we provided intervals for the parameters for which problem (1), (2) has no positive solution. We also presented an example to illustrate our obtained results.

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