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Statistical Convergence via q-Calculus and a Korovkin's Type Approximation Theorem

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Abstract: In this paper, we define and study q-statistical limit point, q-statistical cluster point, q-statistically Cauchy, q-strongly Cesàro and statistically C_1^q-summable sequences. We establish relationships of q-statistical convergence with q-statistically Cauchy, q-strongly Cesàro and statistically C_1^q-summable sequences. Further, we apply q-statistical convergence to prove a Korovkin type approximation theorem.

Keywords: statistical convergence; q-integers; q-Cesàro matrix; q-statistical convergence; q-statistical limit point; q-statistical cluster point; q-statistically Cauchy; Korovkin type approximation theorem

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1. Introduction and Background

Recently, q-calculus appeared as a connection between mathematics and physics. There exist many applications in several areas of mathematics and physics such as orthogonal polynomials, hyper-geometric functions, number theory, complex analysis, combinatorics, matrix summability, approximation theory, quantum physics, particle physics, the theory of relativity, etc. (see [1] for fundamental aspects of quantum calculus). More specifically, q-calculus has a major role in the development of quantum physics. In theories of quantum gravity, q can be thought of as a parameter related to the exponential of the cosmological constant. That is, if $q = 1$, then we recover classical quantum mechanics. For $q \neq 1$, we have a theory of quantum mechanics in a space time with constant curvature. Recently, q-calculus has been used in some matrix and non-matrix summability methods such as q-Cesàro matrix, q-Hausdorff summability and q-statistical convergence (see [2–6]). In approximation theory, it also plays a very important role; e.g., [7–12]. The q-analogs of Bernstein operators and other operators significantly lead to more general results on approximations and show a better rate of convergence than the respective classical operators [13]. Recently, approximation properties for Bernstein operators and their different generalizations have been studied in [14–21].

First, we recall some basic notations for q-calculus ([1,22]). For $q > 0$ and any positive integer τ, a q-integer is defined by ([1,22])

$$[\tau] = [\tau]_q := \begin{cases} \dfrac{1-q^\tau}{1-q}, & q \neq 1, \\ \tau, & q = 1, \end{cases}$$

and the q-factorial by

$$[\tau]! = [\tau]_q! := \begin{cases} [\tau][\tau-1]...[1], & \tau \geq 1, \\ 1, & \tau = 0. \end{cases}$$

For the integers $0 \leq k \leq \tau$, q-binomial coefficients are defined by

$$\left[\begin{array}{c} \tau \\ k \end{array} \right] = \left[\begin{array}{c} \tau \\ k \end{array} \right]_q := \frac{[\tau]_q!}{[k]_q![\tau - k]_q!} \ (\tau \geq k \geq 0).$$

For $q > 0$, we write

$$\mathbb{N}_q := \{[\tau], \text{ with } \tau \in \mathbb{N}\},$$

that is,

$$\mathbb{N}_q = \left\{0, 1, 1 + q, 1 + q + q^2, \ldots.\right\}.$$

For $q = 1$, $\mathbb{N}_q = \mathbb{N}$, the set of nonnegative integers.

In 1951, Fast [23] conceived of the idea of statistical convergence, which was further studied by several authors. Among them, we refer to [24] for the study of several specific operators from the point of view of the q-calculus.

Let $\mathcal{B} \subseteq \mathbb{N}$. Then, $\delta(\mathcal{B}) = \lim_r \frac{1}{r}\#\{k \leq r \ : \ k \in \mathcal{B}\}$ is called the density (also known as asymptotic density or natural density) of $\mathcal{B}$, provided the limit exists, where # denotes the cardinality of the enclosed set. A sequence $\eta = (\eta_k)$ is called statistically convergent to the number s (see [23]) if $\delta(\{k \leq r : |\eta_k - s| > \varepsilon\}) = 0$ for each $\varepsilon > 0$; i.e.,

$$\lim_r \frac{1}{r}\#\{k \leq r : |\eta_k - s| \geq \varepsilon\} = 0$$

and we write $S_t - \lim \eta = s$.

We write S_t for the set of statistically convergent sequences. Note that the ordinary convergence implies statistical convergence, but not conversely. Indeed, such a notion is similar to that of clustering, when studying the eigenvalue or singular value distribution of (preconditioned) matrix-sequences with increasing order (see e.g., [25,26] and references therein): it is worth stressing that the analysis of clustering has important applications when studying the convergence speed of nonstationary iterative solvers for large linear systems [27].

Let $A = (g_{nk})$ be an infinite matrix. $A\eta = (A_n(\eta))_{n=0}^{\infty} = (\sum_{k=0}^{\infty} g_{nk}\eta_k)_{n=0}^{\infty}$ is called the A-transform of a sequence $\eta = (\eta_k)$, provided that $\sum_{k=0}^{\infty} g_{nk}\eta_k$ converges for each $n \in \mathbb{N}$.

A matrix A is said to be regular if the A-transform of all convergent sequences is convergent with the same limit. A is regular [28] if and only if

(i) $\|A\| = \sup_n \sum_{k=0}^{\infty} |g_{nk}| < \infty$,
(ii) $\lim\limits_{n\to\infty} g_{nk} = 0$ for each $k \in \mathbb{N}$,
(iii) $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} g_{nk} = 1$.

Freedman and Sember [29] introduced the notion of A-density for a nonnegative regular matrix $A = (g_{nk})_{n,k=0}^{\infty}$. Let $\mathcal{B} \subseteq \mathbb{N}$ and $\chi_{\mathcal{B}}$ denote the characteristic function of $\mathcal{B}$.

$$\delta_A(\mathcal{B}) = \lim \inf_{n\to\infty} (A\chi_{\mathcal{B}})_n$$

is defined as the A-density of $\mathcal{B}$. If A is replaced by the Cesàro matrix C_1, then A-density is reduced to the natural density. That is,

$$\delta(\mathcal{B}) = \delta_{C_1}(\mathcal{B}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathcal{B}}(k). \tag{1}$$

Gadjiev and Orhan [30] used the idea of statistical convergence in approximation theory and showed that replacing ordinary convergence by statistical convergence leads to a more general approximation, in combination with a higher convergence rate. Since q-statistical convergence is more general than both ordinary convergence and statistical convergence (see Example 1 of Section 2), one can improve several results on approximation theory, where ordinary convergence and statistical convergence both fail to work. This idea motivates us to further study q-analogs of some summability methods and apply them in approximation theory. The Korovkin type approximation is one of the most powerful approaches to approximate any continuous function by a sequence of linear positive operators converging to the identity operator and employing, in general, only a limited information on the given continuous function (e.g., samplings in the case of Bernstein operators). Further, we apply the notion of q-statistical convergence to prove a Korovkin type theorem, which is demonstrated to be more general than the classical as well as statistical versions.

This paper is organized as follows: in Section 2, we study q-statistical convergence, q-statistical limit points and q-statistical cluster points. In Section 3, we define q-statistical Cauchy and find its relation with q-statistical convergence. In Section 4, we introduce two notions, namely q-strongly Cesàro summable and statistically $C_1^{(q)}$-summable sequences, and establish their relationship with q-statistical convergence. In the section, we apply q-statistical convergence in order to study a Korovkin type approximation result, with an example to support our claim that our result is more general than both the cases of ordinary convergence and statistical convergence.

2. q-Statistical Convergence

Defining a q-analog of Cesàro matrix C_1 is not unique (see [2,4,5]). Here, we consider the q-Cesàro matrix, $C_1^{(q)} = (c^1_{nk}(q^k))_{n,k=0}^{\infty}$ defined by

$$c^1_{nk}(q^k) = \begin{cases} \frac{q^k}{[n+1]_q}, & k \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

which is regular for $q \geq 1$ (see Lemma 7 of [4]).

Recently, Aktuğlu and Bekar [4] defined q-density and q-statistical convergence by replacing the matrix A by $C_1^{(q)}$ in (1). That is, for $q \geq 1$

$$\delta_q(\mathcal{B}) = \delta_{C_1^{(q)}}(\mathcal{B}) = \lim \inf_{n\to\infty} (C_1^{(q)} \chi_{\mathcal{B}})_n, \ q \geq 1.$$

For double sequences, see [31,32].

Definition 1 ([4])**.** *A sequence $\eta = (\eta_k)$ is said to be q-statistically convergent to the number L if for every $\varepsilon > 0$, $\delta_q(\mathcal{B}_{\varepsilon,n}) = 0$, where $\mathcal{B}_{\varepsilon,n} = \{k \leq n : |\eta_k - L| \geq \varepsilon\}$. That is, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{[n]} \#\{k \leq n : q^k|\eta_k - L| \geq \varepsilon\} = 0$$

and we write $St_q\text{-}\lim_{n\to\infty} \eta_k = L$.

If $\delta(\mathcal{B}) = 0$ for an infinite set $\mathcal{B}$, then $\delta_q(\mathcal{B}) = 0$. Hence, statistical convergence implies q-statistical convergence, but not conversely (c.f. [4] (Example 15)).

Example 1. *Let $\eta = (\eta_k)$ be defined by (see [4] (Example 15))*

$$(1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, \ldots),$$

where $1's$ (ones) and $0's$ (zeros) occur 2^{2n} and 2^{2n-1} $(n = 0, 1, 2, \ldots)$ times, respectively. Let $\mathcal{B} = \{k \in \mathbb{N} : \eta_k = 1\}$. Then $\lim_{n\to\infty}(C_1^q \chi_{\mathcal{B}})_{2^{2n}-1} = 0$, i.e., $St_q\text{-}\lim_{n\to\infty} \eta_k = 0$ but $\delta(\mathcal{B})$ does not exists, so η is not statistically convergent.

Now, we define q-statistical limit points and q-statistical cluster points of a real number sequence with some examples. For more details, we refer to [33].

Definition 2. *For a subsequence $(\eta_{k(j)})$ of $\eta = (\eta_k)$ and $\mathcal{B} = \{k(j) : j \in \mathbb{N}\}$, we write $\{\eta\}_{\mathcal{B}}$ for $(\eta_{k(j)})$. If $\delta_q(\mathcal{B}) = 0$, $\{\eta\}_{\mathcal{B}}$ is called a subsequence of q-density zero, or a q-thin subsequence. On the other hand, $\{\eta\}_{\mathcal{B}}$ is a q-nonthin subsequence of η if $\mathcal{B}$ fails to have q-density zero.*

Definition 3. *A sequence $\eta = (\eta_k)$ is said to have a q-statistical limit point ς if ς is the limit of a q-nonthin subsequence of η.*

For any sequence η, we denote by L_η, Λ_η and Λ_η^q the set of all ordinary limit points, statistical limit points and q-statistical limit points of η, respectively.

Example 2. *Consider Example 1. Then, $L_\eta = \Lambda_\eta = \{0, 1\}$ and $\Lambda_\eta^q = \{0\}$, since $St_q - \lim \eta = 0$.*

Definition 4. *A sequence $\eta = (\eta_k)$ is said to have a q-statistical cluster point γ if for every $\epsilon > 0$, $\delta_q(\{k \in \mathbb{N} :| \eta_k - \gamma |< \epsilon\}) \neq 0$.*

For a given sequence η, we denote by Γ_η and Γ_η^q the set of all statistical cluster points and q-statistical cluster points of η, respectively. Clearly, $\Gamma_\eta^q \subseteq L_\eta$ for every η. Similar to the result of Fridy [33] (Example 15), we find the following.

Proposition 1. *For any number sequence η, $\Lambda_\eta^q \subseteq \Gamma_\eta^q$.*

We prove the following result, which is the q-analog of the result of Šalát [34].

Theorem 1. *A sequence $\eta = (\eta_k)$ is q-statistically convergent to ℓ if and only if there exists a set $\mathcal{B} = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_q(\mathcal{B}) = 1$ and $\lim_n \eta_{k_n} = \ell$.*

Proof. Suppose $\delta_q(\mathcal{B}) = 1$ for $\mathcal{B} = \{k_1 < k_2 < \cdots < k_n < \cdots\}$ and $\lim_n \eta_{k_n} = \ell$. Then, there is $N \in \mathbb{N}$ for which

$$|\eta_{k_n} - \ell| < \epsilon \text{ for } n > N. \tag{2}$$

Put $\mathcal{B}_\epsilon := \{k \in \mathbb{N} : |\eta_k - \ell| \geq \epsilon\}$ and $\mathcal{B}' = \{k_{N+1}, k_{N+2}, \cdots\}$. Then $\delta_q(\mathcal{B}') = 1$ and $\mathcal{B}_\epsilon \subseteq \mathbb{N} - \mathcal{B}'$, so that $\delta_q(\mathcal{B}_\epsilon) = 0$. Hence, $\eta = (\eta_k)$ is q-statistically convergent to l.

Conversely, let $\eta = (\eta_k)$ be q-statistically convergent to ℓ. Write $\mathcal{B}_r := \{k \in \mathbb{N} : |\eta_k - \ell| \geq 1/r\}$ and $E_r := \{k \in \mathbb{N} : |\eta_k - \ell| < 1/r\}$ $(r = 1, 2, 3, \cdots)$. Then $\delta_q(\mathcal{B}_r) = 0$ and

$$E_1 \supset E_2 \supset \cdots E_i \supset E_{i+1} \supset \cdots \tag{3}$$

and

$$\delta_q(E_r) = 1. \tag{4}$$

To show (η_{k_n}) is convergent to ℓ $(n \in E_r)$, suppose $\lim_{n\to\infty} \eta_{k_n} \neq \ell$. Then, $|\eta_{k_n} - \ell| \geq \epsilon$ for infinitely many terms. Let $E_\epsilon := \{n \in \mathbb{N} : |\eta_{k_n} - \ell| < \epsilon\}$ and $\epsilon > 1/r$ $(r = 1, 2, 3, \cdots)$. Then,

$$\delta_q(E_\epsilon) = 0, \tag{5}$$

and by (3), $E_r \subset E_\epsilon$. Therefore, $\delta_q(E_r) = 0$; i.e., a contradiction to (4). Hence, (η_{k_n}) is convergent to ℓ. □

3. *q*-Statistically Cauchy Sequences

We define a q-analog of statistically Cauchy sequences [35] and we obtain relevant relations with the notion of q-statistical convergence.

Definition 5. *A sequence $\eta = (\eta_k)$ is q-statistically Cauchy if for every $\epsilon > 0$ there exists $C = C(\epsilon)$ such that the set*

$$\{k \leq n :\mid \eta_k - \eta_C \mid \geq \epsilon\}$$

has q-density zero.

Theorem 2. *A sequence $\eta = (\eta_k)$ is q-statistically Cauchy if and only if η is q-statistically convergent.*

Proof. Let η be q-statistically Cauchy but not q-statistically convergent. Then, there exists C such that the set $A_{\epsilon,n} = \{k \leq n :\mid \eta_k - \eta_C \mid \geq \epsilon\}$ has q-density zero. Consequently, $\delta_q(D_{\epsilon,n}) = 1$, where

$$D_{\epsilon,n} = \{k \leq n :\mid \eta_k - \eta_C \mid < \epsilon\}.$$

In particular, we can write

$$\mid \eta_k - \eta_C \mid \leq 2 \mid \eta_k - l \mid < \epsilon \tag{6}$$

if $\mid \eta_k - l \mid < \epsilon/2$. Now, let

$$B_{\epsilon,n} = \{k \leq n :\mid \eta_k - l \mid \geq \epsilon\},$$

$$F_{\epsilon,n} = \{k \leq n :\mid \eta_C - l \mid \geq \epsilon\}.$$

Since η is not q-statistically convergent, $\delta_q(B_{\epsilon,n}) = 1$; i.e., for the set $\delta_q(\{k \leq n :\mid \eta_k - l \mid < \epsilon\}) = 0$. Therefore, by (6), the set

$$\{k \leq n :\mid \eta_k - \eta_C \mid < \epsilon\}$$

has q-density 0; i.e., $\delta_q(A_{\epsilon,n}) = 1$, a contradiction. Hence η is q-statistically convergent.

Conversely, let η be q-statistically convergent to a number l. Then, for every $\epsilon > 0$, the set

$$\{k \leq n :\mid \eta_k - l \mid \geq \epsilon\}$$

has q-density zero. Choose N such that $\mid \eta_N - L \mid \geq \epsilon$. Then $A_{\epsilon,n} \subseteq B_{\epsilon,n} \cup F_{\epsilon,n}$ and therefore $\delta_q(A_{\epsilon,n}) \leq \delta_q(B_{\epsilon,n}) + \delta_q(F_{\epsilon,n}) = 0$. Hence η is q-statistically Cauchy. □

4. q-Strong Cesàro Summability

We define the notion of q-strong Cesàro summability and statistically $C_1^{(q)}$-summable sequences. Then, we describe their relations with the concept of q-statistical convergence.

Definition 6. *A sequence $\eta = (\eta_k)$ is q-strongly Cesàro summable to l, i.e., $[C_1^{(q)}]$-$\lim \eta_k = l$, if*

$$\lim_{n\to\infty} \frac{1}{[n]} \sum_{k=1}^{n} q^k |\eta_k - l| = 0.$$

We write $[C_1^{(q)}]$ for the set of q-strongly Cesàro summable sequences.

A sequence $\eta = (\eta_k)$ is *statistically A-summable* to l [36] if for every $\epsilon > 0$, $\delta(\{j \leq n : |A_j(\eta) - l \mid \geq \epsilon\}) = 0$. Here, we define statistical $C_1^{(q)}$-summability, which is obtained by replacing A by $C_1^{(q)}$, and find its relation with q-statistical convergence.

Definition 7. *A sequence $\eta = (\eta_k)$ is statistically $C_1^{(q)}$-summable to l if for every $\epsilon > 0$, $\delta_q(\{k : |(C_1^{(q)}\eta)_k - l \mid \geq \epsilon\}) = 0$, where*

$$(C_1^{(q)}\eta)_k = \sum_{j=1}^{k} \frac{q^j}{[k]} \eta_j.$$

In the following theorem, we study the relation between q-strong Cesàro summability and q-statistical convergence.

Theorem 3. *q-strongly Cesàro summablity implies q-statistical convergence to the same limit. The converse also holds for a bounded sequence.*

Proof. For any $\eta = (\eta_k)$ and $\epsilon > 0$, we observe that

$$\sum_{k=1}^{n} q^k |\eta_k - l| \geq \big|\{k \leq n : q^k|\eta_k - l| \geq \epsilon\}\big|\epsilon.$$

Hence, $[C_1^{(q)}]\text{-}\lim \eta_k = l$ implies $St_q\text{-}\lim \eta_k = l$.

Conversely, let η be bounded and $St_q\text{-}\lim \eta_k = l$. Let us write $M = \|\eta\|_\infty + \mid l \mid$. For a given $\epsilon > 0$, choose N_ϵ such that for all $n > N_\epsilon$

$$\frac{1}{[n]}\big|\{k : q^k|\eta_k - l| \geq \frac{\epsilon}{2}\}\big| < \epsilon/2M.$$

Write $L_n = \{k : q^k|\eta_k - l| \geq \epsilon/2\}$. For $n > N_\epsilon$, we have

$$\begin{aligned}(1/[n])\sum_{k=1}^{n} q^k \mid \eta_k - l \mid &= (1/[n])\{\sum_{k \in L_n} q^k \mid \eta_k - l \mid + \sum_{k \notin L_n\ ,k \leq n} q^k \mid \eta_k - l \mid\} \\ &< (1/[n])([n]\epsilon/2M)M + (1/[n])([n])(\epsilon/2) \\ &= \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence $[C_1^{(q)}]\text{-}\lim \eta_k = l$. □

The following theorem provides important relations between statistical $C_1^{(q)}$-summability and q-statistical convergence.

Theorem 4. *If a sequence η is bounded, then q-statistical convergence implies statistical $C_1^{(q)}$-summablity, but not conversely.*

Proof. Let $\eta = (\eta_k)$ be bounded and $St_q\text{-}\lim \eta_k = l$. Then

$$\begin{aligned}\mid (C_1^{(q)}\eta)_k - l \mid \leq &\mid \sum_{k \notin K_\epsilon} c_{nk}^q(\eta_k - l) \mid + \mid \sum_{k \in K_\epsilon} c_{nk}^q(\eta_k - l) \mid \\ &\leq \epsilon \sum_{k \notin K_\epsilon} c_{nk}^q + (\sup_k \mid \eta_k - l \mid) \sum_{k \in K_\epsilon} c_{nk}^q.\end{aligned}$$

Then the regularity of the q-Cesàro matrix $C_1^{(q)}$ implies $st\text{-}\lim \mid (C_1^{(q)}\eta)_k - l \mid = 0$.

Conversely, let $\eta = (\eta_k)_{k=0}^{\infty}$ be defined by

$$\eta_k = \begin{cases} \frac{1}{q^k}, k \text{ is even}, \\ \frac{-1}{q^k}, k \text{ is odd}, \end{cases}$$

which is not q-statistically convergent. However, η is $C_1^{(q)}$-summable to 0 and hence statistically $C_1^{(q)}$-summable to 0. □

5. Application of q-Statistical Convergence

We apply the notion of q-statistical convergence to prove a Korovkin type theorem. For further applications of q- and (p, q)-calculus in approximations, we refer to [8,12,18].

Let $C[0,1]$ be the set of all continuous functions on $[0,1]$, which is a Banach space with norm

$$\|\zeta\|_\infty := \sup_{y\in[0,1]} |\zeta(y)|,\ \zeta \in C[0,1].$$

Theorem 5 ([37])**.** *Let $(\mathcal{T}_n)$ be a sequence of linear positive operators (LPOs) from $C[0,1]$ into itself. Then, for all $\zeta \in C[0,1]$, $\lim_n(\mathcal{T}_n\zeta)(x) = \zeta(x)$ uniformly on $[0,1]$ if and only if $\lim_n \mathcal{T}_n(x^i) = x^i$ ($i = 0,1,2$) uniformly on $[0,1]$.*

We prove a Korovkin type approximation theorem for q-statistical convergence analogous to that of given by Gadjiev and Orhan [30]. The Korovkin type approximation theorems have been proved by various authors through different summability methods; e.g., [38–44].

Theorem 6. *Let $(\mathcal{T}_k)$ be a sequence of LPOs from $C[0,1]$ into itself. Then for all $\varrho \in C[0,1]$*

$$St_q\text{-}\lim_{k\to\infty}\left\|\mathcal{T}_k(\varrho) - \varrho\right\|_\infty = 0 \tag{7}$$

if and only if

$$St_q\text{-}\lim_{k\to\infty}\left\|\mathcal{T}_k(1) - 1\right\|_\infty = 0 \tag{8}$$

$$St_q\text{-}\lim_{k\to\infty}\left\|\mathcal{T}_k(s) - v\right\|_\infty = 0 \tag{9}$$

$$St_q\text{-}\lim_{k\to\infty}\left\|\mathcal{T}_k(s^2) - v^2\right\|_\infty = 0. \tag{10}$$

Proof. Since each of $1, v, v^2$ belongs to $C[0,1]$, conditions (8)–(10) follow immediately from (7). Let $\varrho \in C[0,1]$. Since ϱ is bounded on the whole real axis, there exists a constant $Q > 0$ such that

$$|\varrho(s) - \varrho(v)| \le 2Q,\ -\infty < s, v < \infty. \tag{11}$$

In addition, since ϱ is continuous on $[0,1]$, for a given $\varepsilon > 0$, there is a $\delta > 0$ for which

$$|\varrho(s) - \varrho(v)| < \varepsilon, \tag{12}$$

whenever $|s - v| < \delta$ for all v, s.

Using (11) and (12), we obtain

$$|\varrho(s) - \varrho(v)| < \varepsilon + \frac{2Q}{\delta^2}(s-v)^2$$

for all $s \in (-\infty, \infty)$ and $y \in [0,1]$. Then as in [30], we have

$$\| \mathcal{T}_k(\varrho) - \varrho \|_\infty \le K\left(\left\|\mathcal{T}_k(1) - 1\right\|_\infty + \left\|\mathcal{T}_k(s) - s\right\|_\infty + \left\|\mathcal{T}_k(s^2) - s^2\right\|_\infty\right) \tag{13}$$

where $K = \max\{\varepsilon + Q + \frac{2Q}{\delta^2}, \frac{4Qb}{\delta^2}\}$. For any $\lambda > 0$, define the following sets

$$G(\varepsilon, n) = \{k \le n : ||\mathcal{T}_k(\varrho) - \varrho||_\infty \ge \frac{\lambda}{K}\},$$

$$G_1(\varepsilon, n) = \{k \le n : ||\mathcal{T}_k(1) - 1||_\infty \ge \frac{\lambda}{3K}\},$$

$$G_2(\varepsilon, n) = \{k \le n : ||\mathcal{T}_k(s) - v||_\infty \ge \frac{\lambda}{3K}\},$$

$$G_3(\varepsilon, n) = \{k \le n : ||\mathcal{T}_k(s^2) - v^2||_\infty \ge \frac{\lambda}{3K}\}.$$

Then, $G(\varepsilon) \subset G_1(\varepsilon, n) \cup G_2(\varepsilon, n) \cup G_3(\varepsilon, n)$, and so by (13) we obtain

$$\delta_q(G(\varepsilon, n)) \le \delta_q(G_1(\varepsilon, n)) + \delta_q(G_2(\varepsilon, n)) + \delta_q(G_3(\varepsilon, n)).$$

Therefore, using conditions (8)–(10), we finally infer

$$St_q\text{-}\lim_n ||\mathcal{T}_n(\varrho) - \varrho||_\infty = 0.$$

□

Example 3. *Consider the Bernstein operators*

$$B_n(\varrho, y) := \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} \varrho\left(\frac{k}{n}\right), \quad \varrho \in C[0,1].$$

Now, define the operators $P_n : C_B[0,1] \to C[0,1]$ by $P_n = (1+u_n)B_n(\varrho, y)$ where the sequence (u_n) is defined by (Example 15 of [4])

$$(1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, \ldots).$$

Then the sequence (P_n) satisfies conditions (8)–(10). Hence, by Theorem 6, we deduce

$$St_q\text{-}\lim_{n\to\infty} \|P_n(\varrho) - \varrho\|_\infty = 0.$$

Let $P_n(\varrho; 0) = (1+u_n)\varrho(0)$.

Since $B_n(\varrho; 0) = \varrho(0)$, and hence

$$\|P_n(\varrho) - \varrho\|_\infty \ge |P_n(\varrho; 0) - \varrho(0)| = u_n|\varrho(0)|.$$

However, the sequence (u_n) is neither convergent nor statistically convergent, so Theorem 5 as well as Theorem 1 of Gadjiev and Orhan [30] does not hold for (P_n).

Hence, Theorem 6 is stronger than Theorem 5 as well as its statistical version.

6. Concluding Remarks and Suggestions for Further Studies

In this paper, we have defined and studied q-analogs of statistical limit point, statistical cluster point, statistically Cauchy, strongly Cesàro sequences and established the inter-relationships between them as well as with q-statistical convergence. We have also introduced the notion of statistical $C_1^{(q)}$-summablility and obtained its relationship with q-statistical convergence for bounded as well as unbounded sequences. Further, we have applied the notion of q-statistical convergence to prove a Korovkin type theorem to approximate any continuous function, which is demonstrated to be more general than the classical as well as statistical versions. For further studies, we suggest that some results on the statistical convergence of Salat [34] and Schoenberg [45] can be extended for q-analogs. The ideas of q-double Cesàro matrices and the q-statistical convergence of double sequences have been studied by [32], which can be further used in studying approximation results for bivariate operators. Recently, (p, q)-calculus, a more general case than q-calculus, has been used in several studies; e.g., orthogonal polynomials, hyper-geometric functions, inequalities, complex analysis, combinatorics, post-quantum physics, approximation theory, etc. One can think to study the (p, q)-version of Cesàro matrices, strongly Cesàro summability, density and statistical convergence with applications in approximation theory. Finally, since the Korovkin theory has been employed in applied problems in numerical linear algebra for the fast solution of large structured linear systems (see [46,47]), it would be interesting to investigate the use of the new notions also in this context.

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