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The Topology of Quantum Theory and Social Choice

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Abstract: Based on the axioms of quantum theory, we identify a class of topological singularities that encode a fundamental difference between classic and quantum probability, and explain quantum theory's puzzles and phenomena in simple mathematical terms so they are no longer 'quantum paradoxes'. The singularities provide also new experimental insights and predictions that are presented in this article and establish a surprising new connection between the physical and social sciences. The key is the topology of spaces of *quantum events*and of the *frameworks* postulated by these axioms. These are quite different from their counterparts in classic probability and explain mathematically the interference between quantum experiments and the existence of several frameworks or 'violation of unicity' that characterizes quantum physics. They also explain entanglement, the Heisenberg uncertainty principle, order dependence of observations, the conjunction fallacy and geometric phenomena such as Pancharatnam–Berry phases. Somewhat surprisingly, we find that the same topological singularities explain the impossibility of selecting a *social preference* among different individual preferences: which is Arrow's *social choice paradox:* the foundations of social choice and of quantum theory are therefore mathematically equivalent. We identify necessary and sufficient conditions on how to restrict experiments to avoid these singularities and recover unicity, avoiding possible interference between experiments and also quantum paradoxes; the same topological restriction is shown to provide a resolution to the social choice impossibility theorem of Chichilnisky.

Keywords: social choice; topological singularities; unicity assumption; quantum events; framework selection

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1. Introduction

Quantum physics is the most successful scientific theory of all time, having emerged less than a century ago from axioms created by Born [1], Dirac, [2] and von Neumann [3]. Based on the same axioms, we identify here a class of topological singularities that encode the fundamental difference between classic probability and quantum probability, and explain quantum theory's puzzles and phenomena in simple and rigourous mathematical terms so in this sense they cease to be 'quantum paradoxes'. The singularities provide also new experimental insights and predictions that are presented in this article as well as a surprising new connection between the physical and social sciences. The key is the topology of spaces of *quantum events* and of the *frameworks* that are postulated by the quantum axioms, which are quite different from their counterparts in classic physics.

Events are physical phenomena that either occur or do not occur. They are central to any probability theory. In classic probability all experiments are part of one large experiment and events are described within one sample space with a single basis of coordinates or framework (*Events* in classic probability theory are measurable sets such as the Borelian sets in R^n. *Events* in quantum probability are quite different): this is the *unicity assumption* of classic physics (Griffiths, 2003 [4]). The space of all events has no singularity. In quantum probability, instead, *quantum events are* projection maps on a Hilbert space. Quantum theory considers all possible experiments on a physical system and breaks tradition by explicitly accepting that there may be no universal experiment and *no single* framework to describe all observed events (In the famous double-slit experiment, light is observed both as particles and waves. Superposition implies the possibility of having a proposition and its opposite). The multiplicity of frameworks in quantum theory violates unicity: there may be no unique basis of coordinates to describe the results of all possible experiments on a physical system and the violation emerges from the fact that the space of quantum events has singularities. When two frameworks or coordinate systems fail to be orthogonal to each other they give rise to so-called 'interaction' or 'interference' among experiments that is at the heart of quantum theory and distinguishes it from classic physics; a classic example is the two hole experiment discussed below, e.g., Griffiths, 2003 [4]; Gudder[5]. As seen in the examples of the last section, the matter has further ramifications as different frameworks lead to Heisenberg uncertainty and order dependence of experiments.

This article explains, in simple mathematical terms, the genesis of interference, where it comes from, and in particular, when and how it can be avoided. For example, it is well known that when all experiments under consideration are part of a single larger experiment, the unicity of classic physics is recovered within quantum theory, in this case experiments do not interfere and are consistent with each other. Being part of a larger experiment is a sufficient condition to eliminate interference. Is it possible to find necessary as well as sufficient conditions on the range of acceptable quantum experiments to recover unicity? We show that the topological structure of the *spaces of quantum events* —which are also the propositions of quantum logic—explains why experiments interfere, why we typically have no common frameworks, and why quantum logic is more complex and richer than the binary logic of classic physics [5]. We find a necessary and sufficient condition that, when used to restrict the domain of acceptable experiments, ensures that one can select a single framework for all experiments thus eliminating interference. This condition restricts appropriately the domain of experiments so they are consistent and do not interfere with each other. It turns out that this topological restriction by itself, creates a connection between quantum theory and social choice theory, a rather unexpected connection. The necessary and sufficient condition that eliminates interference between experiments turns out to be the same as the restriction required to resolve Arrow's classic impossibility theorem in social choice (Arrow [6]) allowing us to aggregate individual into social preferences. It was shown in 1980 that Arrow's impossibility theorem has a topological structure, see Chichilnisky [7,8]) and here we show that the same topological structure is at the core of the paradoxes of quantum theory. The last section illustrates the theorems and discusses simple and practical examples and new experimental predictions, examples of interference, order dependence of experiments, Heisenberg's uncertainty principle [5,9] and Pancharatnam-Barry geometric phases [10] all of which have a similar topological origin, and their connection with the topology of spheres.

From the outset we note that the case of infinite dimensional Hilbert spaces with complex parameters (which is quite standard) is not included in this paper in order to facilitate the presentation of the topological results. It can be included however at the cost of losing simplicity and a transparent connection to the topology of social choice. Please see also for more of a discussion in Section 3 below.

2. Organization

We start by stating the axioms of quantum theory created by von Neumann. Based on these axioms we define and analyze the spaces of quantum events and of frameworks, showing that their topological structure separates classical events from the events of quantum physics. We then establish the impossibility of selecting a common framework for all experiments, based on the topological singularities within quantum events and frameworks. The same singularities are behind the impossibility of selecting a *common social preference* to different individual preferences: this is the *social choice impossibility theorem*. We establish that a resolution to the social choice problem is the same as a resolution to the violation of unicity, and that both cases require the same topological restriction on the domains of experiments and of preferences, respectively. Finally, we illustrate the results with examples of interference, order dependence of experiments, Heisenberg's uncertainty principle, and Pancharatnam—Barry phases.

3. The Axioms of Quantum Theory

We start with key concepts in quantum theory and show how they differ in topological terms from their classical counterparts. In classic mechanics there are three important components: *states*, *observables* (dynamic variables, such as events) and *dynamics*. The same three components are present in quantum mechanics, but they are described by different objects. In classic mechanics the three components are described by points, functions and trajectories, while in quantum mechanics they are described by entities in a Hilbert space. This fundamental difference arises in great measure because the two theories have different goals, which can be summarized as follows: *"The main goal of classic probability is the construction of a model for a single probabilistic experiment—or subexperiments of a single experiment. Quantum theory is based on Hilbert space probability theory and is much more ambitious. It seeks a mathematical model for the class of all experiments that can be performed on a physical system. Why cannot we construct a classical model for each experiment and then "paste" all the models together? The problem is that we do not know how to do the pasting since we do not know how the various experiments interact of interfere with each other. The pasting is automatically done by the Hilbert space structure."* S. Gudder [5] 1988, (p. 68).

While classic physics attempts to explain the universe, quantum theory shares with general relativity an emphasis on the observer. For this reason "quantum events" are defined as maps rather than as measurable sets of objects as in classic physics. *Quantum events* are a key concept in this article, and they are identified with *projection maps* (see Axiom A below), and with the subspaces of a Hilbert space onto which the projections map. *Frameworks* are orthonormal bases of coordinates of the Hilbert space and can be identified with subspaces of the spaces of events. When two frameworks fail to be orthonormal the corresponding experiments are said to interfere with each other. In classic physics things are different: there is only one experiment—the 'Universe'—and one single framework, so quantum interference is impossible: this is the "unicity hypothesis" that is violated in quantum theory. A key difference is therefore that quantum theory does not assume a single framework nor a single sample space.

In the following we consider Hilbert spaces of finite dimension n, where n is arbitrarily large, namely euclidean spaces R^n, which correspond to physical systems with n degrees of freedom. Under appropriate assumptions the theory presented here can be made applicable to infinite dimensional Hilbert spaces. The finite dimensional case is useful to simplify the presentation and to show that fundamental properties of quantum theory occur even within finite dimensional real Hilbert spaces, even though full generality requires infinite dimensional Hilbert spaces with complex coefficients. Quantum Theory uses infinite dimensional complex Hilbert Spaces which are important to represent certain notions such as wave particle duality. Here, we focus instead on topological issues, and for those, in view of Bohr's and von Neumann's axioms, it suffices to focus on finite dimensional real spaces. The reason is that "observables" in quantum theory are by definition self-adjoint operators, this is von Neumann's first axiom of quantum theory (see Gudder [5]) and it explains the focus on (a) finite dimensions and (b) real spaces. This is because quantum theory's operators are self-adjoint (observables) and an operator is self-adjoint if and only if it is unitarily equivalent to a real valued multiplication operator (see Gudder [5]). Furthermore, the structure of self-adjoint operators in infinite dimensional Hilbert spaces essentially resembles the finite dimensional cases. Therefore, by the above observations it suffices to focus here on (a) finite dimensional and (b) real spaces (see Gudder [5]).

To provide a clear foundation and highlight the differences, we start from basic concepts of *probability theory* and show the difference between classic probability and quantum probability (Observe that the violation of unicity that characterizes quantum theory occurs both in finite as well as in infinite dimensional spaces). In classic theory, the set of individual outcomes of a probabilistic experiment is called a *sample space* and it is a non-empty set denoted X. $\sum$ denotes a $\sigma-$algebra of subsets of X, which is the collection of *outcomes sets* to which probabilities can be assigned, the pair $(X, \sum)$ is called a *measurable space,* and the sets in the $\sigma-$algebra $\sum$ are also called *'events'*. By definition the events are all included within the common sample space X, which is the union of the outcome sets. To facilitate the comparison between classic and quantum theory, the sample space X can be assumed to be a Hilbert space with an attendant orthornomal basis of coordinates; with finite dimensions the sample space is therefore R^n. The basic postulate of unicity that divides classic from quantum physics is as follows:

Definition 1. *The **Unicity** postulate of classic physics is the requirement that all events are included in one single sample space X.*

Assume that the sample space X is a Hilbert space of dimension n, so that $X = R^n$.

Example 2. *In classic probability all events are subsets of a single sample space $X = R^n$; the space of all events is denoted $\sum$ and is the $\sigma-$algebra of Borelian subsets of R^n, so the union of all events is the single sample space R^n. In classic physics, therefore, unicity is satisfied.*

Below we show the difference between classic probability and quantum probability. For a presentation of quantum probability theory, see also Griffiths, 2003 [4].

The following four axioms of quantum theory were introduced by von Neumann [3] and highlight the difference between classic and quantum theories.

Von Neumann's Axioms for Quantum Theory

$(A.1)$ The states of a quantum system are unit vectors in a (complex) Hilbert space H (In what follows we simplify by assuming real Hilbert spaces, because the phenomena we are interested in analysing can be found in these spaces) ,

$(A.2)$ The observables are self-adjoint operators in H (A definition of a self adjoint operator is in Dunford and Schwartz [11]) ,

$(A.3)$ The probability that an observable T has a value in a Borel set $A \subset R$ when the system is in the state Ψ is $< P^T(A)\Psi, \Psi >$ where $P^T\ (.)$ is the resolution of the identity (spectral measure) for T and

$(A.4)$ If the state at time $t = 0$ is Ψ, then at time t it is $\Psi_t = e^{-it\mathcal{H}/h}\Psi$ where $\mathcal{H}$ is the energy observable and h is Planck's constant.

To summarize from Axiom $(A.2)$ above, the observables or events in a quantum theory experiment are not sets but rather self-adjoint operators T defined on the Hilbert space H. In further detail, by Axiom $(A.3)$ above, the results of experiments compute the probability that the observable T has a value in the Borelian set A when the system is in state Ψ and the probability is $< P^T(A)\Psi, \Psi >$ where $P^T\ (.)$ is the resolution of the identity (spectral measure) for T (Definitions and statements of self-adjoint operators, the spectral theorem and the resolution of the identity (spectral measure) are in Dunford and Schwartz [11]) .

The four axioms presented above can be greatly simplified: Gudder [5] (pp. 50–53) shows that these four axioms can be derived from a single axiom if we begin with a probabilistic structure defined on a Hilbert space. As already mentioned, the single basic axiom of quantum theory that separates it from classic physics pertains to the structure of quantum events, which are observations of physical phenomena (as defined above) that either occur or do not occur. The following is the single axiom from which the rest can be derived Gudder [5] (pp. 50–53):

Axiom (A) *The events of a quantum system can be represented by (self-adjoint) projections on a Hilbert space.*

The *events* in quantum theory are observables (as defined above), so quantum theory shares with relativity the emphasis on observations and the observer (Quantum probability theory is presented in Griffiths [4]). The axioms presented above do not specify a particular Hilbert space in which the states are represented, nor which self-adjoint projection operator represents a particular physical observable or event. The next step is to show how Axiom

A determines the space of quantum events. Consider a Hilbert space H of finite dimension n, which could be very large, $H = R^n$, and assume that H is spanned by an orthonormal set of vectors $V = \{V_i\}_{i=1,\dots,n}$ that forms a basis for the space H (This is an arbitrary choice and other bases can be used. The order of the vectors in the basis is not relevant). Then from Axiom A we know that a *quantum event* is a (self adjoint) projection (A self adjoint projection is similar to a product operator, like position or momentum) on a subspace of H and can be identified by a subspace $S \subset H$ spanned by a subset $V_S \subset V$ of basis vectors (The order of the vectors V_i does not matter, and since we are concerned with subspaces, nor does their orientation, as is discussed further below); in words the *event occurs* when an experimental observation lies in the subspace $S \subset H$. The event S corresponds to orthogonal projections P_S whose image covers S. For each event S there is an orthonormal basis of coordinates (Gudder 1988 [5], Griffiths 2003 [4]) that defines the subspace S (Events are identified here with subspaces of H and are therefore given by unordered and unoriented bases of coordinates of the subspaces. Orienting the vectors does not change the main results). If the set V has $n-1$ vectors, it defines an $n-1$ dimensional subspace and together with its orthonormal vector it defines an orthonormal base of coordinates for the entire space R^n, which is also called a *framework* (The basis of coordinates need not be an ordered set and the vectors need not be oriented). As an example if S is a subset of vectors in the basis V, and x is an observable or experiment of our physical system, corresponding to S is the event that occurs when a measurement of x results in a value in S, see, e.g., Gudder [5] (p. 52). Following the above description, a *framework* can be defined as an unordered unoriented orthonormal basis of coordinates of R^n The space of frameworks in R^n is therefore the space of all bases of coordinates (unoriented and unordered, since the order or the orientation of the coordinate vectors of S does not alter the subspace S) and is denoted F^n (No orientation is required, although similar results are obtained if the frameworks are unorderd but oriented bases of coordinates).

The notions of *events* and *frameworks* just defined play a key role in quantum physics. The concept of interference or incompatibility between experiments is a critical new idea that distinguishes quantum probability from classic theory and is identified with the 'violation of unicity' (Griffith 2003 [4], Gudder 1988 [5]): *"A key feature of quantum theory is that while some events may be compatible and share the same framework, or bases of coordinates consisting of vectors that are orthogonal to each other, other events may be incompatible and do not share a common basis of coordinates or framework."* cf. also Busemeyer and Bruza 2012 [9]. When the various bases of coordinates that appear within several quantum experiments include vectors that are not orthogonal to each other this causes experimental 'weirdness' as shown in the illustrations of the last section. It is worth noticing that as we show here the violation of unicity can occur both in finite or infinite dimensional spaces and in real or complex Hilbert spaces. In all cases as seen in the last section it leads to interference between experiments, non-conmuting observations, i.e., the order of the experiments changes the observed results, the probabilistic error known as the "conjunction fallacy" by which two events are deemed to be more likely to occur together than each on its own, Busemeyer and Bruza [9], and to Heisenberg's uncertainty principle. At the heart of quantum theory is the lack of a *common* basis of coordinates for different observations, namely *the lack of a common framework* for all possible experiments on a physical system. The following sections show that this is intrinsically a topological issue.

4. Classic and Quantum Physics with $n \geq 2$ Degrees of Freedom

This section illustrates fundamental differences between classic and quantum physics that emerge from the axioms, starting from the simplest possible examples. Consider initially physical systems with two degrees of freedom, $n = 2$.

Example 3. ***Classic physics.*** *The simplest possible physical system has two degrees of freedom and the Hilbert space for such systems is $H = R^2$. In this case the space of events is the Boolean $\sigma-$algebra of Borelian sets in R^2, and the sample space is their union, namely R^2. Therefore a classic* *event is a Borel set, there is a single sample space* (R^2) *and a common framework for all events, namely a single orthonormal coordinate basis for the space* R^2. *Unicity is satisfied.*

Example 4. ***Quantum physics.*** *From Axiom (A) above, the events of a* ***quantum*** *system with two degrees of freedom* $n = 2$ *are (self-adjoint)(For a definition of self-adjoint operators see Dunford and Schwartz [11]; self adjoint operators are the closest there is to 'multiplication' operators that are used to describe basic observables in physics such as* position *and* momentum *(Gudder [5])) projections of* R^2*, and each can be identified with a one-dimensional subspace (or line through the origin) L in* R^2*. The space of all quantum events* Q^2 *in this case is the space of all one-dimensional subspaces or lines through the origin of* R^2*. Observe that each projection or quantum event can also be identified with an orthonormal unordered and unoriented basis of coordinates of the space* R^2*, namely with a framework in* R^2*, by adding a vector that is orthonormal to the line L in* R^n *(The orthonormal bases of coordinates in* R^2 *has two vectors: one is the vector spanning the line and the second is an orthonormal vector. One can choose the vectors so the space of all lines is included in the space of all orthonormal bases of vectors in* R^2 *and the map is one to one and onto. Observe that the bases are unordered and the vectors are unoriented) .*

In summary:

Lemma 5. *The space* F^2 *of frameworks of a quantum system with two degrees of freedom* $(n = 2)$ *in* R^2 *can be identified with the one dimensional projective space* P^1 *of all lines through the origin in* R^2*, and the space* P^1 *in turn can be identified with the unit circle* S^1 *in* R^2*,* $P^1 \approx S^1$ *(Spanier [12], Milnor and Stasheff [13]).*

Proof. As mentioned in Example 3, each line through the origin in R^2 uniquely defines an unoriented, unordered system of coordinates in R^2 namely a framework. The space of all such lines is by definition the projective space P^1, cf. Spanier [12], Milnor and Stasheff [13] who also show the identification between P^1 and S^1. □

The following result provides a geometric characterization of the spaces of events and frameworks in quantum theory and in classic physics. It is based on the axioms stated above, and uses basic definitions and properties of topological spaces. A basic definition is

Definition 6. *For* $n > 1$*, and* $k < n$*, let* $G(k, n)$ *be the* ***Grassmanian*** manifold of k *planes of* R^n*. (Spanier [12], Milnor and Stasheff [13]).*

Observe that when $k = n - 1$, $G(k, n) = P^{n-1}$: by definition therefore $G(n - 1, n)$ is the *n−1 projective space* in R^n (For definitions and topological properties of Grassmanian manifolds see Milnor and Stasheff, [13]).

The following summarizes and shows the geometrical differences between quantum theory and classic physics:

Lemma 7. *The space of* ***classic events*** $\sum$ ***in*** R^2 *is the Boolean* $\sigma-$*algebra of Borel measurable sets in* R^2*.This is a convex space and is therefore topologically trivial (i.e., all its homotopy groups are zero) (For definition of homotopy groups see Spanier [12]). Unicity is satisfied since there is a* ***unique sample space****, namely* R^2*; the space of* ***classic frameworks*** *has a single element, namely a (single) basis of coordinates for* R^2*. In contrast, the space of* ***quantum events*** Q^2 *in* R^2 *is the space of all unoriented lines through the origin within two dimensional space* R^2 *also called the one-dimensional* projective space P^1*; this space* is *the Grassmanian of* $1-$*spaces in* R^2*, denoted* $G(1, 2)$*, The space* $G(1, 2)$ *can be identified with the unit circle* S^1*,* $P^1 \approx S^1 \approx G(1, 2)$*. When* $n = 2$*, the space of* ***frameworks*** F^2 *in* R^2 *can be identified with the space of quantum events in* R^2*, i.e.,* $F^2 \simeq Q^2$*. Both the space of quantum frameworks* F^2 *and the space of quantum events* Q^2 *can be identified with the projective space* $P^1 \approx G(1, 2) \approx S^1$*. (*$S^1$ *denotes the unit circle in* R^2*.* P^1*,* S^1 *and* $G(1, 2)$ *are not contractible.) Neither the space of quantum events nor the space of frameworks in* R^2 *are contractible.*

Proof. From Axiom (A) above, a quantum event in R^2 is by definition a projection of R^2 and therefore can be identified with a (non zero) subspace of R^2 namely a line through the origin of R^2; in turn each line can be identified with a basis of coordinates in R^2 as seen in Example 4 and in Lemma 5. The rest follows immediately from classic probability theory and the unicity postulate stated above. □

Definition 8. *A singularity is a non- zero element of the homology of the space of events. Since the space of classic events is contractible it has no singularities. In quantum physics the space of events is $Q^2 = S_1$ and therefore has one singularity.*

5. $n \geq 2$ Degrees of Freedom

The next step is to characterize spaces of quantum events and frameworks in systems with $n > 1$ degrees of freedom, and exhibit the difference with the same concepts in classic theory:

Definition 9. *A framework in R^n is an n-dimensional unordered orthonormal basis of coordinates of R^n. The space of frameworks in R^n is a manifold denoted F^n, and it consists of all possible coordinate systems of R^2. (No orientation is required, although similar results are obtained if the frameworks are unordered but oriented bases of coordinates).*

Lemma 10. *For $n > 1$, the manifold F^n of all frameworks in R^n is a connected subset of the manifold Q^n of events in R^n.*

Proof. Consider an event S which by Axiom (A) is a (self-adjoint) projection in R^n. When the image of the projection is an $n-1$ dimensional subspace S of R^n, define an n framework by adding an orthonormal unit vector to the $n-1$ basis of coordinates of the subspace that represents S. This maps events, which are projections into $n-1$ dimensional subspaces, into frameworks of R^n; the map is continuous, one to one and onto the space of all n dimensional bases of coordinates of R^n, namely the space of frameworks F^n, which is a connected space. The manifold F^n of frameworks is therefore contained as a connected subset of the manifold Q^n of quantum events in R^n. □

The following summarizes:

Theorem 11. *In a physical system with $n > 1$ degrees of freedom the space of quantum events Q^n can be identified with the space of all subspaces of R^n and therefore can be identified with the union of the Grassmanian manifolds $G(k, n)$ of k dimensional subspaces of R^n $\forall\, k < n$, $Q^n \approx \cup_{k<n} G(k, n)$. In particular the Grassmanian $G(n-1, n)$ consisting of all the $n-1$ subspaces of R^n can be identified with the space F^n of all frameworks in R^n. In particular, when $n = 2$, the space Q^2 of quantum events of R^2 equals the space of frameworks F^2 of R^2 and can be identified with the projective space $F^2 \approx P^1 \approx S^1$. When $k = n-1$, $G(k, n)$ is $G(n-1, n)$ the $n-1$ projective space and the space F^n of all frameworks of R^n.*

Proof. This follows directly from Lemmas 7 and 9. □

We analyzed the spaces of frameworks and of quantum events in R^n, $n > 1$, and the topological difference between the concept of events in classic physics and in quantum theory. The next sections show the critical role played by the topology of spaces of frameworks and of spaces of events in separating quantum theory from classic physics.

Definition 12. *For each n a singularity is a non-zero element of the $n-th$ homology of the space of events with integer coefficients. Since the space of classic events for every $n > 1$ is contractible, it has no singularities. In quantum physics the space of quantum events is $Q^n \approx \cup_{k<n} G(k, n)$ and therefore it has as many [**singularities**] as generators of the homology of Q^n [12,13].*

Below we explore the practical consequences of these facts with examples illustrating their connection with social choice theory.

6. Unicity and Restricted Domains of Experiments

Unicity plays an important role in separating classic physics from quantum theory, and it is generally not satisfied in quantum experiments in which more than one framework is needed to explain observations (Gudder [5], Griffiths [4], Busemeyer [9] . Indeed we saw that the two theories differ in that quantum theory attempts to explain all possible experiments on a given physical system, which may require different frameworks, while experiments in classic physics are restricted from the outset to be part of one large experiment having a single framework. A natural question is whether it is possible to overcome the lack of unicity in quantum theory by restricting appropriately the domain of experiments that are performed on a physical system. There is a simple answer to this question and it is affirmative. Restricting the domain of quantum theory experiments to all the subexperiments of a single experiment—having a single framework—has the desired effect. In classic physics there is a unique experiment that contains all the rest, and from this unique experiment emerges the classic postulate of unicity. It is known that, under the same conditions, the same is true in quantum theory: if the various experiments within a restricted domain are all part of a single larger experiment, it is always possible to define a common framework for all the quantum experiments, see, e.g., Gudder [5]. In quantum theory these are called *compatible experiments,* Gudder [5]. *Compatible observables* correspond to *noninterfering measurements*, Gudder [5]. In quantum logic there is a parallel mathematical characterization of compatible observables see Gudder [5] (p. 82). Quantum theory under these restricted conditions therefore agrees with classic theory.

The question tackled in this section is whether there are more general domains of experiments where unicity can be recovered without requiring that all the experiments be subsets of a single larger experiment.

In the following we characterize restricted domains of experiments where there is a common framework for any given set of experiments within the domain, without requiring that they are initially subexperiments of one single experiment. The ability to find a common framework for a set of experiments decides whether or not it is possible to reduce quantum theory to classic theory, within a restricted set of experiments. There have been indications that this may be possible: indeed it is known that in some cases it may be possible to 'prepare' appropriately the physical system before carrying out the various experiments, so that all the experiments in the domain can be observed within a common basis of coordinates of frameworks, see, e.g., Cerceda [14] Gudder [5] and Busemeyer and Bruza [9] (p. 158).

What follows provides a formal approach to the same problem: we identify topological conditions on a restricted domain of experiments that ensures the existence of a single or *common framework* for all experiments within the restricted domain. A simple example illustrates the issues:

Example 13. *Consider two different bases of coordinates or frameworks in euclidean space R^2 that are not orthonormal to each other (Two bases of coordinates are called orthonormal to each other, when eachvector in one basis is either the same or orthonormal to all the vectors in the other, for examples and a mathematical discussion see Gudder [5]). As an illustration consider the two orthonormal coordinate systems F_1 and F_2 in R^2 defined as $F_1 = \{(0,1)$ and $(1,0)\}$ and $F_2 = \{(1,1),(1,-1)\}$. F_1 and F_2 are two different orthonormal bases of coordinates or frameworks for R^2 that are not orthonormal to each other since the vector $(0,1)$ is at $45°$ from the vector $(1,1)$. Having two different bases of coordinates (or frameworks) that are not orthonormal to describe the same object can create problems, since it leads to different representations for the same object since, e.g., the vector (x,y) in F_1 is $(x-y,x-y)$ in F_2. The problems can cause violation of unicity, interference and superposition of observations, and can lead to apparent contradictions as is illustrated in the last section of this article, which provides practical examples. Nevertheless, for any two given bases* *of coordinates in R^n such as F_1 and F_2 there is always a change of coordinates that maps one into the other, i.e., there is always a way to translate or to change one basis of coordinates into the other, denoted $F_1 \to F_2$. In this case the map is given by the (self adjoint) matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ that maps the two vectors $(1,0),(0,1)$ into the two vectors $(1,1)$ and $(1,-1)$, and more generally (x,y) into $(x+y, x-y)$.The matrix M can be thought of as a "dictionary" that translates one language or framework into another. For any two given frameworks F_1 and F_2 therefore one can define a common framework by selecting one of the two frameworks: in this case selecting F_2 and changing the coordinates of F_1 correspondingly using the matrix M. This way, we can always define a common framework for any vector $v = (x,y)$ in R^2: we simply consider the new vector $M(v)$. The experiment can now be performed in the same basis of coordinates. At the end one uses the inverse matrix $M^{-1} = \begin{matrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{matrix}$ to translate the results back into their original framework F_1. For those two given frameworks F_1 and F_2, therefore, unicity can be recovered. Allowing changes of coordinates seems a mild and natural way to resolve the problem for having a pair of different frameworks that are not orthogonal and thus it resolves the problem of lack of unicity for the two given frameworks.*

We will show however that this solution, while it works for any two frameworks, does not work in general. For any two given frameworks it is possible to select one, and translate the second into the first as shown above, but the question becomes whether one can always select one framework for any two given frameworks and to do it consistently, therefore resolving the lack of unicity by changes of coordinates.

In trying to do so one runs into a *topological problem* that identifies the nature of lack of unicity. As we saw for any two given bases of coordinates by one can always define a common basis of coordinates, but as we will see the change of coordinates that works for two given bases of coordinates does not work for all others and the selection is not consistent and continuous overall (The obvious example is when averaging the vectors in two bases of coordinates to obtain a common basis: this works in many cases but it does not work when the vectors one is trying to average are 180 degrees apart: if so, when one attempts to average both vectors one gets the zero vector. What results is therefore not a framework and the problem remains unresolved). To recover unicity one needs to be able to select a single basis of coordinates or frameworks for any number of bases of coordinates that may arise from different experiments in a way that (1) does not depend on the order of the two frameworks, (2) when the original frameworks are the same, one keeps the same framework. The map that selects a common framework for any k frameworks must be (3) continuous, so the selection of one framework among two frameworks coming from two different directions yields the same single outcome. In selecting a single framework, continuity is important in order to approximate the outcome by making increasingly accurate measurements. This is also called '*statistical sufficiency*' and is critical for any probabilistic theory. When continuity fails, practically identical experiments will lead to fundamentally different results, causing by itself contradictions and 'weird' observations. We need some definitions:

Definition 14. *A map $\Phi : X^k \to X$ is called **symmetric** if it does not depend on the order of the arguments, namely $\Phi(x_1, ..., x_k) = \Phi \circ \Pi(x_1, ..., x_k)$, where Π is any permutation of $k > 1$ elements.*

Definition 15. *A map $\Phi : X^k \to X$ is called t**he identity on the diagonal** if $\forall x \in X$, $\Phi(x, ..., x) = x$. Equivalently, Φ is the **identity on the diagonal** Δ of the product space X^k, where $\Delta = \{(x_1, ..., x_k) : \forall i, j, x_i = x_j\}$ when $\forall k > 1$, the restriction map $\Phi|\Delta(X^k) : \Delta(X^k) \to X$ is the identify map on $\Delta(X^k)$.*

Let $F \subset R^n$ be the space of frameworks of R^n, $n > 1$.We can now define

Definition 16. *A **framework selection** is a way to select a single framework among any $k > 2$ frameworks, satisfying the conditions (1), (2) and (3) above. Formally, a framework selection is a sequence of continuous maps $\{\Phi_k\}_{k=1,2,...}$ that selects one framework within any set of $k > 1$ frameworks,where $\Phi_k : F^k \to F$, (1) Φ_k is continuous (2) Φ_k is symmetric,and (3) $\Phi_k / \Delta(F^n)^k = id_k$.*

Based on the above example, *unicity* can be defined as the possibility of selecting in a systematic way a single common framework or basis of coordinates for any set of $k > 1$ frameworks.

Remark 17. *Observe that when a framework selection exists, the unicity of frameworks can be recovered by standard changes in coordinates as in Example 11 above.*

Another example already mentioned is as follows

Example 18. *If all experiments within a restricted domain are subsets of a single larger experiment, then a common framework exists and unicity is satisfied, see e.g., Gudder [5] (p. 82) and see also below. Observe that under these conditions the inclusion of each experiment as a subset of a larger experiment provides the framework selection required.*

Is it always possible to select one common framework for any set of frameworks as defined above? In general the answer is negative. We show in the next section that it is impossible to select a common framework for all the bases of coordinates of R^n. As shown below the reason is topological: the ability to select one common framework among several is a property that is only satisfied under certain topological conditions on the domain of frameworks that arises from the various experiments.

7. Why Unicity Fails: Impossibility Theorems for Selecting Frameworks

The next step is to define restricted domains of experiments within which one can recover unicity, and show why the recovery cannot be obtained in general. Starting with simple examples in two dimensional spaces, we extend gradually the results to provide a characterization that is valid for all dimensions (Under certain conditions the results can be extended to Hilbert spaces of infinite dimensions either complex or real, which appear naturally in physical systems with n degress of freedom evolving over time. Dynamics in quantum theory can be formulated both in discrete and in continuous time, Gudder [5]).

First we establish that it is generally impossible to select a single common framework. Then we identify restricted domains of experiments within which a single framework can be selected:

Theorem 19 (Chichilnisky [7])**.** *In experiments with two degrees of freedom where $H = R^2$ there is no way to select a single framework for all experiments on a physical system. Formally, there exists no continuous function Ψ that selects one common framework $\Psi : F^2 \times F^2 \to F^2$ that is independent from the order of the frameworks i.e., $\forall x, y$,$\Psi\ (x, y) = \Psi(y, x)$ and respects unanimity, $\forall x,\ \Psi(x, x) = x$.*

Proof. With two degrees of freedom the space of frameworks F^2 and the space of quantum events Q^2 coincide by Lemma 5; they are both the one-dimensional projective space P^1 and this space can be identified with the circle ($S^1 = \{x = (x_1, x_2) \in R^2 : x_1^2 + x_2^2 = 1\}$). S^1, Spanier [12], and $F^2 \approx S^1 \approx P^1$. Therefore the theorem reduces to the non-existence of a continuous function $\Psi : S^1 \times S^1 \to S^1$ that is symmetric, i.e., $\forall x, y\ \Psi(x, y) = \Psi(y, x)$, and respects unanimity, i.e., $\forall x,\ \Psi(x, x) = x$. By definition, Ψ is the identity map on the diagonal $D = \{(x, y) \in S^1 \times S^1 : x = y\}$ namely $\Psi\ |_D = id_D\ (x, x) = x$. For a given $z \in S^1$ define $A = \{(x, z), \forall x \in S^1\}$ and $B = \{(z, x), \forall x \in S^1\}$. Then $A \cup B$ can be continuously deformed into D within $S^1 \times S^1$ so by definition the degree mod 2 of the map Ψ on D must be the same as the degree mod 2 of the map Ψ on $A \cup B$:

$$deg(\Psi\mid_D) = \deg(\Psi\mid_{A\cup B})\mathrm{mod}2 \tag{1}$$

□

Degree of $\Psi\mid_D$:$S^1 \to S^1$ is 1 since $\Psi\mid_D$ is the identity map, while the degreee $\Psi\mid_{A\cup B}$ is even, since Ψ is symmetric, which is a contradiction with (1). The contradiction arises from assuming that a map Ψ with the stated properties exists and therefore the map Ψ cannot exist. See also [7].

The next result shows why the selection of a single framework is a topological problem, which can only be resolved in spaces that are *contractible* or topologically trivial (The space X can be euclidean, or it can be a manifold in euclidean space or a CW manifold cf [15]), namely in spaces of frameworks that are homotopic to a point or can be continuously deformed through themselves into a point:

Theorem 20 (Chichilnisky and Heal [15])**.** *Let X be a manifold or CW complex (Generally one works on CW manifolds, cf. [7]). There exists a continuous selection map* $\Phi : X^k \to X$ *satisfying axioms (1) (2) and (3) above, if and only if the space X is topologically trivial or contractible, i.e., X is homotopically equivalent to a point.*

Proof. See Chichilnisky and Heal [15]. □

Theorem 21 (Chichilnisky [7])**.** *There is no continuous function* $\Psi : (G(n-1,n))^k \to (G(n-1,n)$ *for any* $n > 1$ *that is symmetric and respects unanimity for all* $k > 1$*.*

Proof. By Theorem 13 the necessary and sufficient condition for the existence of a continuous map $\Psi : X^k \to X$ satisfying the conditions of symmetry and unanimity for all $k > 1$, is that the space X be contractible, see Chichilnisky and Heal [15]. For every $n > 1$,the Grassmanian manifold $G(n-1,n)$ is not a contractible space (Milnor and Stasheff [13]). This completes the proof. □

The above can be summarized as follows:

Theorem 22. *Let H be a finite dimensional Hilbert space,* $H = R^n$*, and* F^n *the space of* its frameworks. *Then* F^n *violates unicity, i.e. there is* no way to select a single framework among *k frameworks* because there exists no continuous map $\Phi : F^k \to F$ selecting a common framework in F^n *for any k frameworks* $\forall k > 1$*.The space of frameworks* F^k *can be identified with the Grassmanian* $G(n-1,n)$ *which is not topologically trivial as required for unicity. Violation of unicity is therefore due to the topology of the space of frameworks* F^k*.*

Proof. See Theorem above, and Chichilnisky [7] and Chichilnisky and Heal (The characterization of the space F defined as the orbits of all orthonormal bases of coordinates of euclidean space R^n, $n \geq 2$ under the action of the symmetry group S_n on n elements is $F = S^1 \times S^2 \times ...S^n$. This is in Chichilnisky [16]). □

The results provided above show the topological origin of the violation of unicity. The next step is to show that by restricting the domain of experiments it is possible to recover unicity: indeed by Theorem 21 the topological condition of *contractibility* is necessary as well as sufficient for unicity. Consider now a physical system with n degrees of freedom and corresponding Hilbert space $H = R^n$.

Theorem 23. *A necessary and sufficient restriction on the experiments of a quantum system with n degrees of freedom to satisfy unicity, is that the corresponding space of frameworks* F^n *is topologically trivial or contractible.*

Proof. This follows from Theorem 20 above. □

8. Quantum Theory and Social Choice

The topological roots of the violation of unicity create an unexpected and fertile connection between quantum theory and social choice theory. Social Choice Theory emerged from the classic theory of elections. The work of Nicolas Marquis de Condorcet (see Arrow [6] and de Condorcet [17]) on the theory of elections focused on explaining how societies take into consideration the preferences of different individuals in arriving at a social decision. Condorcet found an essential paradox that bears his name and appears in majority decisions, that are widely used in voting within democracies. Here is a sketch of the problem: if one individual prefers a to b and b to c, represented (a, b, c), a second individual prefers (c, a, b), and a third prefers (b, c, a) then two out of three individuals prefer a to b, two out of three prefer b to c and two out of three prefer c to a. By transitivity, the choices made by majorities of two individuals are inconsistent, a majority prefers a to c and a majority prefers c to a. Kenneth Arrow uses this 'paradox' as a foundation for a new and general theory that he called Social Choice.

Social choice theory originated with Arrow's impossibility theorem, which defined reasonable axioms for the aggregation of individual into social preferences and proved that they were impossible to achieve [6]. In 1980 social choice theory was redefined as follows: one seeks to define a map Ψ that assigns a social preference to any two or more individual preferences, formally $\Psi : P^k \to P$ where P represents a space of preferences [7,8]. Reasonable conditions are that the map Ψ must be continuous and symmetric, depending on individual's preferences but not on the order of the individuals, and that Ψ respects unanimity so that if both individuals have the same preferences, the social preference is the same. Continuity means that it is possible to approximate the social preference by taking sufficiently accurate measurements of the individual preferences.

In 1980 the social choice problem was rewritten and given a simple geometrical form in [7,8]. Geometrically, linear preferences are vectors in a sphere S^n, where n is the dimension of the space of choices. When $n = 1$, the problem is finding a map that assigns a single *point to* every two points in the circle S^1 in a continuous way that is symmetric, so it does not depend on the order of the preferences, and respects unanimity; Chichilnisky [7,8] established that the problem has no solution: it is not possible to find such maps in the circle S^1, or in higher dimensional spheres S^n, or even in general spaces of preferences that are co-dimension one oriented smooth foliations of R^n. (This result was extended to necessary and sufficient conditions for the existence of such maps on manifolds of any dimension [7,15]. Formally, the problem is the non existence of a continuous function Ψ that assigns to k individual preferences a social preference, $\Psi : P^k \to P$, so that (1) Ψ is symmetric and (2) Ψ respects unanimity, as defined in the previous section. Here $k > 1$ represents the number of individuals and P is the space of preferences. One seeks to define a map Ψ that assigns a common ('social') preference to any two or more individual preferences) [7,8].

There is a deep connection between social choice and the topology of spheres and it comes from the definition of *preferences*. Preferences are *rankings or orders*. A linear function $f : R^n \to R$ defines a ranking $\succ$ as follows $x \succ y \Leftrightarrow f(x) > f(y)$. Linear preferences are defined by linear functions on R^n. A linear function has by definition a constant gradient vector in R^n; and therefore a linear preference is defined by a single unit gradient vector in R^n. The space of linear preferences P can therefore be represented by a set of vectors of length one, the unit circle $S^1 \subset R^2$ or more generally the unit sphere S^n. The space of all smooth preferences P is the space of all smooth co-dimension one oriented foliations of R^n[7,8]. The problem of social choice as introduced in Chichilnisky was formulated as the existence of a continuous function $\Psi : P^k \to P$ satisfying two axioms (1) and (2) above. It was shown in Chichilnisky [7] that this problem has no solution, namely such a map Ψ does not exist. This non-existence result was shown to be a topological property of the space of co-dimension one foliations of R^n. In the special case of linear preferences, the problem reduces to a topological property of spheres of all dimensions (The problem is equivalent under certain conditions to Arrow's Impossibility Theorem Arrow [6]). Chichilnisky [7] established that there is no continuous map $\Psi : (S^n)^k \to S^n$ that is symmetric and is the identity on the diagonal $\Delta \subset S^n$.

To understand the connection between social choice with unicity in physics we need more definitions:

Definition 24. *A vector x in the unit circle S^1 represents a linear preference (or order) on R^2, defined by the linear map $f_x : R^2 \to R$ having the vector x as a gradient, i.e., such that $Df_x = x$. The unit circle $S^1 \subset R^2$ can therefore be identified with the space P of all linear preferences on R^2 (Alternatively the vector is the gradient at $\{0\}$ of a smooth function defined on R^2 that need not be linear) .*

Definition 25. *Let P be the space of smooth preferences or co-dimension one oriented foliations of euclidean space R^n [7] for $n > 1$. When preferences are linear $P = S^n$. For $k > 1$, a continuous function $\Psi : P^k \to P$ satisfying the two axioms (1) and (2) is called a **preference selection** or a **common preference.***

The simplest case is $n = 1$:

Theorem 26 (Chichilnisky [7])**.** *There is no continuous function $\Psi : S^1 \times S^1 \to S^1$ that is symmetric, i.e., $\forall x, y\ \Psi(x, y) = \Psi(y, x)$, and respects unanimity, i.e., $\forall x,\ \Psi(x, x) = x$, i.e., In other words: is not possible to define a common preference for any two individuals with linear preferences.*

Proof. See Theorem above and Chichilnisky [7]. □

The result extends to spheres of all dimensions:

Theorem 27. *For $n > 1$, it is not possible to define a common preference for any $k > 1$ individuals with linear preferences; there is no continuous map $\Psi : (S^n)^k \to S^n$ that is symmetric, and is the identity on the diagonal $\Delta_{(S^n)^k} \subset (S^n)^k$.*

Proof. See Chichilnisky [7]. □

Theorem 25 extends to general spaces P of smooth preferences consisting of oriented codimension-one smooth foliations of R^n:

Theorem 28. *For any $n > 1$, it is not possible to find a common preference for any $k > 1$ smooth preferences on R^n : In particular for $n > 1$, and $\forall k > 1$, there is no continuous map $\Psi : (P)^k \to P$ that is symmetric, and is the identity on the diagonal $\Delta \subset P^k$.*

Proof. See Chichilnisky [7]. □

From the above results we can now formally establish the connection between quantum theory and social choice:

Lemma 29. *The space F^n of frameworks in R^2 can be identified with the space of linear preferences on R^n.*

Proof. When $n = 2$, the result is immediate because the space of frameworks F^2 is in this case the one dimensional projective space P^1 that is the unit circle S^1 Spanier [12], and the space of linear preferences in R^2 is also the unit circle S^1. When $n > 2$, by Theorem 11 the space of frameworks can be identified with the space of $n-1$ subspaces of R^n. $G(n-1, n)$ and each $n-1$ subspace A in R^n defines an orthonormal vector $v(A)$ in R^n as shown in Theorem 13, which in turn can be identified with the gradient of a linear preference in R^n. This completes the proof. □

We have therefore established:

Theorem 30. *For any restricted domain of preferences $M \subset R^n$ $n > 1$, the social choice problem of aggregation of preferences in M is the existence of a map $\Psi : M^k \to M$ satisfying axioms (1) (2) and (3) for all $k > 1$, and this problem formally coincides with the quantum theory problem of existence of unicity for all frameworks within the manifold M.*

Proof. This follows from Lemma 28, from the identity between the axioms (1) and (2) in the two cases, in quantum theory and in social choice, and from the definition of unicity. □

Theorem 31. *The existence of common preferences is equivalent to the existence of common frameworks, or unicity.*

Proof. The equivalence can be seen formally by considering the necessary and sufficient conditions for the existence of a selection of a single framework in restricted domains of experiments. In quantum theory, for any $k \geq 2$ experiments on a given physical system, it may not be possible to define in a continuous way a corresponding common framework f. In social choice theory, instead, it may not be possible to define continuously a common preference in a way that respects unanimity and is anonymous, i.e., is symmetric. In this sense the general mathematical problem underlying quantum theory, which is the 'violation of unicity', can be seen as the non-existence of a continuous map $\Phi : F^k \to F$ assigning a common framework to every $k \geq 2$ frameworks $(f_1...f_k) \in F^k$ in a way that is symmetric and respects unanimity namely $\forall f, \Phi(f, ..., f) = f$. □

9. Examples of 'Weirdness' without Common Frameworks: Conjunction Fallacy, Interference, Heisenberg Uncertainty and Order Dependence

The examples provided below of the conjunction paradox, the order effect in changing observations, and of the Heisenberg Uncertainty Principle, are all obtained by representing observables as self-adjoint operators, which is von Neumann's first axiom of quantum theory (see Gudder [5]). For simplicity, the examples are provided in two or three dimensional real spaces. The examples below are obtained by replacing the standard observables in traditional physics, by observables in quantum theory, which as we saw above, are self-adjoint operators. In fact they are multiplicative real valued operators. We already discussed above that self-adjoint operators (the "observables" in quantum theory) are unitarily equivalent to multiplicative operators that are real valued.

This section illustrates with specific examples the theory developed in this article. It shows how the topology of spaces of events and frameworks that is the main focus of the article, by interfering with the existence of a common framework as shown in previous sections, leads to interference and to the Heisenberg Uncertainty principle, to the observations changing with the order of the experiments, and to the so-called 'conjunction fallacy' where two events together are considered more likely to occur than each on their own. Interference, the Heisenberg Uncertainty principle and order dependence of experiments are instances of 'weirdness' that are commonly associated with quantum theory. We also illustrate the results presented above with a geometric phase (the Pancharatnam—Berry phase) that appears in quantum mechanics, which is a phase difference acquired by a system over the course of a cycle, a phenomenon in which a parameter is slowly changed and then returns to its initial value, executing a closed path or "loop" and where its initial and final states differ in their phases. The examples offered here are taken from the literature to (Busemeyer and Bruza [9], 2012) Ong and Wei-Li Lee [10] Gudder [5]) to help eliminate differences of data interpretation. In addition we offer a new experiment that is a modification of the classic "two slit experiment" that anticipates observations of new phenomena from the theory offered here. Observe that with the topological interpretation of quantum theory provided above, the 'weirdness' phenomena are simply a reflection of the natural topological structure of the problem, namely the general impossibility of finding common frameworks. In this sense there is no weirdness at all. The following examples arise in experiments that have different frameworks. In each case, if the two frameworks were reduced to a common framework, of course, the so called weirdness would dissappear, while in each case one may find a common framework for two specific cases of the given experiments, the above topological results show their strength in that they demonstrate that in general this cannot be achieved: there will always exist two experiments where the common frameworks fail to exist. Quantum theory's violation of unicity has a logical, topological necessity that cannot be avoided. This is an issue that is not contemplated nor considered in the existing literature: we have shown that it is not possible to consistently reinterpret or measure all experiments—and their frameworks—to find always a common framework. The weirdness examples illustrated here will necesssarily emerge for some experiments, no matter how one may change the instruments and redefine the measurements, and therefore the frameworks, in the specific examples presented below.

Example 32. *The conjunction fallacy.*

Tversky & Kahneman, Ref. [18] 1983 defined an important and common probability judgment error, called the 'conjunction fallacy', that is based on the lack of common frameworks. It is the famous 'Linda' problem. Judges are provided a brief story of a woman named Linda who used to be a philosophy student at a liberal university and was active in the anti-nuclear movement. The judges are asked to rank the likelihood of the following events: that Linda is now (a) active in the feminist movement, (b) a bank teller, (c) active in the feminist movement and a bank teller, (d) active in the feminist movement and not a bank teller, and (e) not active in the feminist movement and a bank teller. The conjunction fallacy occurs when option (c) is judged to be more likely that option (b) (even though the latter contains the former). The experimental evidence shows that, surprisingly, people frequently produce conjunction fallacies for the Linda problem and for many other problems as well (Tversky and Kahneman [18] 1983).

In the following we use a geometric approach to quantum theory taken from Busemeyer and Bruza [9] (2012), and explain how this relates to the results of the previous sections of this article. We refer the reader to [9] for further details and for clarifications on the examples.

First we represent two answers to the feminism question by two different frameworks or basis of coordinates for euclidean space R^2. Each framework is given by two orthogonal rays that span a two dimensional space. The answer yes to feminism is represented by a ray labeled $\mathfrak{F}$ and the answer no to the feminism question is represented by an orthogonal ray labeled $-\mathfrak{F}$. This is the first framework. The person's initial belief about the feminism question which is generated from the Linda story, can be represented as a unit length vector labeled S, within the two dimensional space spanned by these two rays. Note that the initial state vector S is close to the ray for yes to feminism, which matches the description of the Linda story. As explained geometrically by Busemeyer and Bruza [9], quantum theory computes probabilities for an event, or for a sequence of events, as follows: first one computes the so called 'amplitude' or inner product of two vectors denoted $< \mathfrak{F} \mid S >$ for transiting from the initial state S to the ray $\mathfrak{F}$—this inner product equals of course the projection of the state S onto the $\mathfrak{F}$ ray, which is the point on the $\mathfrak{F}$ ray that intersects with the line extending up from the S state. The quantum theory axioms postulate that the squared amplitude equals the probability of saying yes to the feminism question starting from the initial state and this is equal to $|< \mathfrak{F} \mid S >|^2 = 0.9755$. Now we introduce the second framework, and rotate the axis to change from one to the other framework. The bank teller question is represented by two orthogonal rays labeled B and $-B$ which are rotated so $-B$ is 20° below $\mathfrak{F}$. This defines the second framework, and it means that being a feminist and not being a bank teller are close in this belief space. The amplitude for transitioning from the initial state S which is close to $\mathfrak{F}$ is also far away from the B ray (S is close to the orthogonal ray $-B$). The amplitude $< B \mid S >$ for transitioning from the initial state S to the ray B equals the projection of the state S onto the B ray which is illustrated by the point along the B ray that intersects with the line segment extending from S up to B. In this second framework, and according to the axioms of quantum theory, the square amplitude equals the probability of saying yes to the bank teller question starting from the initial state and this equals $|< B \mid S >|^2 \ = 0.0245$.

Now consider the sequence of answers in which the person says yes to the feminism question and then says yes to the bank teller question in that order. The order that questions are processed is critical in quantum theory, and here we are assuming that the more likely event is evaluated first. The axioms of quantum theory imply that the amplitude for this sequence of answers equals the amplitude for the path $S \to \mathfrak{F} \to B$ and the latter equals the product of the amplitudes namely $< B \mid \mathfrak{F} > . < \mathfrak{F} \mid S >$. The first transition is from the initial state S to the ray F and the second is from the ray $\mathfrak{F}$ to the state B. The amplitude $< \mathfrak{F} \mid S >$ is the projection from S to $\mathfrak{F}$ which has a square magnitude equal to $|< \mathfrak{F} \mid S >|^2 = 0.9755$, and the amplitude $< B \mid \mathfrak{F} >$ is the projection from the unit length basis vector aligned with $\mathfrak{F}$ to the B ray, which has a square magnitude equal to $|< B \mid \mathfrak{F} >|^2 = 0.0955$. By definition, the probability for the sequence equals the square amplitude for the path is $|< B \mid \mathfrak{F} > . < \mathfrak{F} \mid S >|^2 = (0.9755).(0.0955) = 0.0932$. Note that this probability exceeds the probability of saying yes to the bank teller when starting from the initial state based on the story, $|< B \mid S >|^2 = 0.0245$. In conclusion this simple geometric model reproduces the basic facts of the *conjunction fallacy.*

Example 33. *Order effects in observations.*

The same example can be used to show how quantum theory produces order effects that are observed in attitude research. Note that the probability of the sequence for the order "yes to bank teller and then yes to feminism" is quite different than the probability for the opposite order. The bank teller first sequence has a probability equal to $|< \mathfrak{F} \mid B > . < B \mid S >|^2 = (0.0955)(0.0245) = 0.00234$ which is much smaller than the feminism first sequence $|< B \mid \mathfrak{F} > . < \mathfrak{F} \mid S >|^2 = (0.9755)(0.0955) = 0.0932$. This order effect follows from the fact that a property of *incompatibility* that arises between the feminism question and the bank teller question.

Example 34. *Heisenberg uncertainty principle.*

We have assumed two frameworks, namely that the person is able to answer the feminism question using one basis of coordinates or framework $\{\mathfrak{F}, -\mathfrak{F}\}$ but the person requires a different basis of coordinates or framework $\{B, -B\}$ for answering the bank teller question. Observe that this implies that if the person is definite about the feminism question (in other words the belief state vector S is lined up with the ray $\mathfrak{F}$) then he or she must be indefinite about the bank teller's question, because $\mathfrak{F}$ and B are not orthogonal to each other and can be said to "interact or interfere" with the other. Similarly, if the person is definite with respect to the bank teller question then he or she must be indefinite about the feminism question. This is essentially the *Heisenberg uncertainty principle*.

Example 35. *Violation of unicity.*

Busemeyer and Bruza [9] state that, given that the two questions are treated as incompatible, we must also be violating unicity. Indeed, they say, we are assuming that the person is unable to form a single description (i.e., a single sample space) containing all the possible conjunctions $\{\mathfrak{F} \cap B, \mathfrak{F} \cap -B, -\mathfrak{F} \cap B, -\mathfrak{F} \cap -B\}$. What they do not explain is why this is assumed. This article shows that, for topological reasons that are akin to those of the social choice paradox, this assumption is unavoidable. In other words, it is unavoidable that the person will be unable to form a single description for some basis of coordinates, or frameworks. The results presented here explain the violation of unicity. This implies that necessarily in some cases, the person would have never thought about conjunctions—for example those involving feminism and bank tellers—sufficiently to assign probabilities to all these conjunctions. Instead the person relies in such cases on two separate sample spaces: one based on elementary events $\{\mathfrak{F}, -\mathfrak{F}\}$ for which they are familiar, and a second based on elementary events $\{B, -B\}$ for which they are also familiar. If we did assume unicity in this example, then we could not explain the conjunction fallacy because the joint probabilities can be defined under unicity, and they will always be less than (or equal to) the marginal probabilities. Therefore as stated by Busemeyer and Bruza, to explain the experimental result requires the violation of unicity. The results of this article go further: they explain why the violation of unicity is a necessary logical implication when considering all possible experiments of a given physical system—as is the goal of quantum theory. Furthermore, they illustrate why violation of unicity is, at its core, the same as the paradox of social choice.

10. The Classic Two-Hole Experiment

The two-hole experiment is used as a famous example to show how quantum theory can explain observations that could not be explained with classic probability and physics. In the two-hole experiment below S is a source of electrons all of whom have the same energy but they leave S in all directions and many impinge on a planar screen A. The screen A has two holes, 1 and 2, through which the electrons may pass. Behind the screen we have an electron detector which can be placed at distance x from the center of the screen. The detector records each passage of a single electron traveling from S through a hole in A to the point x, see Gudder [5] (Figure 2.1 p. 58). In a classic analysis of the two hole experiment, e.g., [5] (pp. 58–59), after performing the experiment many times with many different values of x one obtains a probability density $P(x)$ that the electron passes from S to x as a function of x. Since an electron must pass through either hole 1 or hole 2, in classic probability theory $P(x) = P(x_1) + P(x_2)$, where $P(x_i)$ is the chance of arrival coming through $i = 1, 2$. Figure 1 illustrates the observed distribution: the actual experimental result of the two-hole experiment is quite different, and it is shown in this figure (see also Gudder [5] (p. 59)) which forces us to conclude that $P \neq P_1 + P_2$. In this sense the observations contradict classic probability.

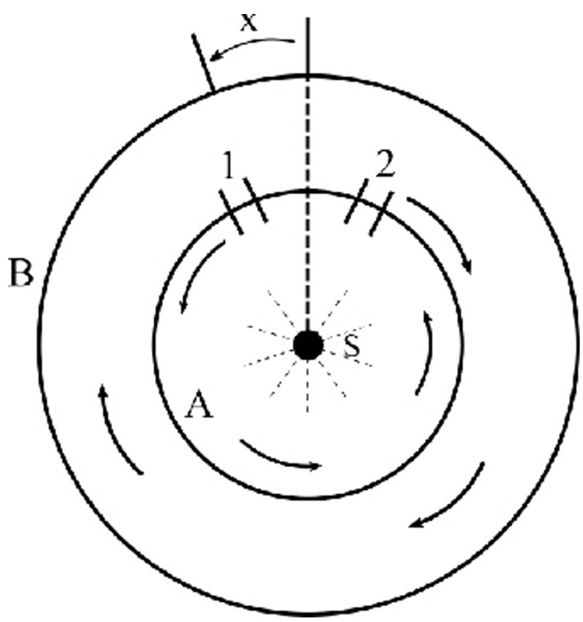

Figure 1. Chichilnisky two rotating hole experiment (top view).

11. The New Two Rotating Hole Experiment

The author proposed a variation of the classic two-hole experiment to predict new experimental observations based on the results of the article. The predicted observations are consistent with but different from the Pancharatnam—Berry Phases results that are analyzed in [10] and are illustrated below. The two rotating hole experiment (side view).is illustrated in Figure 2 below. The two planar screens A and B of the two-hole experiment are replaced by cylinders A and B. The position of each of the two holes 1 and 2 in the cylinder A can be rotated with knobs K_1 and K_2, respectively; each knob can move the respective hole around the entire cylinder A, with K_1 rotating the hole 1 clockwise and K_2 rotating the hole 2 counterclockwise. On the basis of the results presented above, the author's prediction is that as hole 2 is rotated clockwise to the initial position of hole 1, and hole 1 is rotated counterclockwise to the initial position of hole 2, thus reproducing exactly the initial position of the two holes together at the end, the observations of the density distributions on the cylinder B will be different, even though in the final position the positions of the two holes together is indistinguishable from the initial position of the two holes .This prediction remains to be tested experimentally, but it is close to the experimental results that have been obtained in the so called Pancharatnam–Berry phase, which is explained below, and which has been widely accepted, to the extent that with some good will, those can be considered experimental tests of the results of this article.

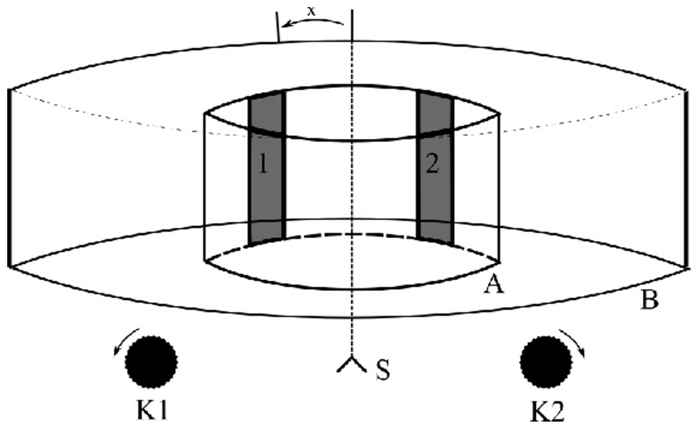

Figure 2. Two rotating hole experiment (side view). The two planar screens A and are here replaced by cylinders. The position of each of the two holes 1 and 2 can be rotated with a knob (K_1 and K_2, respectively) and each knob can move the respective hole around the entire cylinder A.

12. The Pancharatnam—Berry Phase

The Pancharatnam—Berry Phase can be briefly summarized geometrically as follows, for a full presentation see, e.g., [10]. Suppose we travel on a closed path C on a sphere (Earth) while holding a vector V parallel to the surface, i.e., in the local tangent plane (Figure 3 below). At each point, V does not twist around the local vertical axis (the local normal vector n). This is known as *parallel transport* of the vector V around C. When we return to the starting point, we find that in general V makes an angle $\alpha(C)$ with its initial direction: the angle α, which depends only on the particular path C; $\alpha(C)$ is known a the geometric angle, and is the classic analog of the Pancharatnam—Berry phase in quantum physics, see, e.g., Ong and Lee [10].

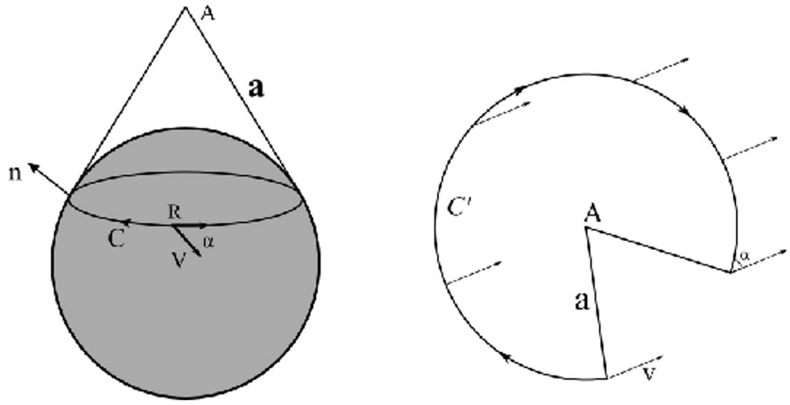

Figure 3. Pancharatnam —Berry Phase: suppose we travel on a closed path C on a sphere (Earth) while holding a vector V paralell to the surface, i.e. in the local tangent plane. At each point, V does not twist around the local vertical axis (the local normal vector n). This is known as *paralell transport* of V around C. When we return to the starting point, we find that in general V makes an angle $\alpha(C)$ with its initial direction: the angle, which depends only on the particular path C is known as the geometric angle, and is the classic analog of the Pancharatnam—Berry phase in quantum physics, see, e.g., Ong and Lee [10].

13. Around the World in 90 Days

In a famous book of the same name, Jules Verne wrote a story around the concept of the "time line" about a gentleman who places a bet on being able to travel around the Earth in 90 days, and thinks he has lost by one day, arriving in 91 days, only to find out that time went slower at the initial location so at their return they had effectively won their bet. This literary piece illustrates the "time line" break in time, so that if one starts traveling around the world along a path such as C in Figure 3 at the end of the journey when one goes all around the world and reaches the initial position, the time measured by a traveling watch will be different than the time at the initial position at the moment of return, measured by a stationary watch. It can be shown that the topological problem posed by Jules Verne is the same as in the Pancharatnam Berry phases. It is well accepted that Barry phases arise from the existence of a singularity, which is the same origin that is postulated here for the basic properties of quantum theory that are described above, a topic that has to be discussed in further writings.

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