

Article

Abramovsky—Gribov—Kancheli Theorem in the Physics of Black Holes

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Abstract: The proof of the Abramovsky—Gribov—Kancheli (AGK) theorem for black hole physics is given. Based on the AGK relations, a formula for the luminosity of a black hole as a function of the mass of the black hole is derived. The correspondence to experimental data is considered. It is shown that the black holes of the galaxies NGC3842 and NGC4889 do not differ from those of the other galaxies.

Keywords: black hole; black hole mass; luminosity of black hole; galaxy; AGK-rules; photon

1. Introduction

Observations of recent decades have shown that all massive galaxies have supermassive black holes in their centers. In most cases, the calculation of the masses of these black holes is carried out using empirical scaling relations between black hole mass, luminosity, and galaxy velocity dispersion. These phenomenological empirical relations work quite well and allow us to calculate the masses of black holes with quite a good rate of accuracy [1]. However, in [2], it is stated that two galaxies with a mass of 9.7 billion solar masses and with a mass of 21 billion solar masses are not governed by this phenomenological correlation. In [3], it is emphasized that in the future, with an increase in the diameter of telescopes, it will be possible to more reliably determine these correlations.

In this paper, we will not touch on the phenomenological relationship between the black hole mass and the galactic velocity dispersion. We will consider the relationship between the black hole mass and luminosity, based on a quantum mechanical point of view. Taking the Abramovsky—Gribov—Kancheli theorem (AGK) [4,5], we will show that the dependence of the luminosity of a black hole on its mass has the form:

$$L = a M_{BH}, \quad (1)$$

where the parameter a does not depend on the black hole mass but depends only on the density of the particles surrounding the black hole. Here, the mass of a black hole is expressed in units of solar mass, and the luminosity of a black hole is expressed in units of solar luminosity.

The AGK theorem was derived 47 years ago for hadron–hadron interaction in Regge theory. As an example, let us derive the first AGK relation. We consider the simplest amplitude diagram of a two-reggeon branching (Figure 1). Here, we follow the work of [6].

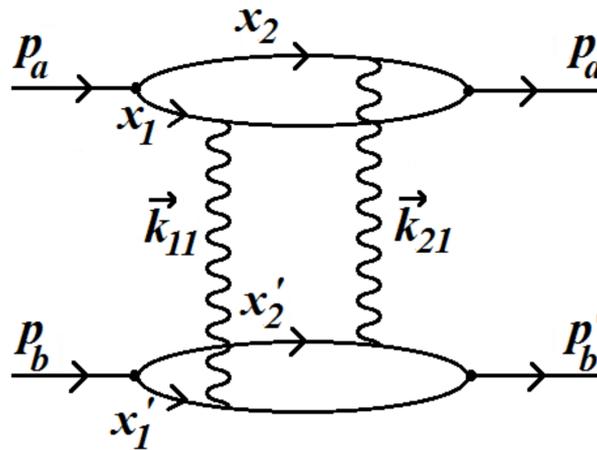


Figure 1. The simplest amplitude diagram of a two-reggeon branching in the interaction of two hadrons with four-momenta p_a and p_b . The wavy lines designate reggeons, and the solid lines denote scalar particles. See text for more details.

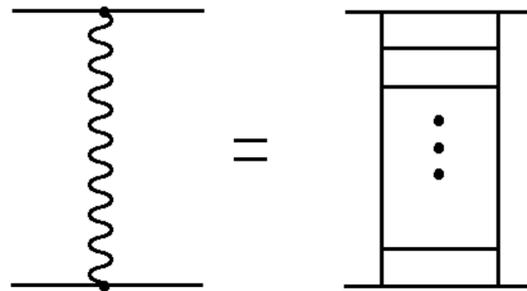


Figure 2. The simplest model of a reggeon. The lines are as in Figure 1.

Contribution of the diagram of Figure 1 to the amplitude of scattering is as follows:

$$iA(s, t) = s \int dx_1 dx_2 \delta(1 - x_1 - x_2) dx'_1 dx'_2 \delta(1 - x'_1 - x'_2) w(x_1, x_2) w(x'_1, x'_2) \times \int d\Omega_2 (iD_1(s_1, k_1^{\perp 2})) (iD_2(s_2, k_2^{\perp 2})). \tag{2}$$

Here, $x_1, x_2, x'_1,$ and x'_2 are Feynman variables of partons that compose hadrons with four-momenta p_a and p_b ; $w(x_1, x_2)$ and $w(x'_1, x'_2)$ are probabilities that each hadron has two partons; $s_1 = s x_1 x'_1,$ $s_2 = s x_2 x'_2,$ and $s = (p_a + p_b)^2$ are the squares of the total energy in the center-of-mass system, and $t = (p_a - p'_a)^2$ is the square of the transferred four-momentum.

The amplitudes of the one-reggeon exchange, $iD_j(s_j, k_j^{\perp 2})$, where $k_j^{\perp 2}$ is the perpendicular part of the square of the transferred reggeon momentum, we will present in the form of the simplest multi-peripheral diagram, as shown in Figure 2. Phase volume of the two-reggeon branching is as follows:

$$d\Omega_2 = 2(2\pi)^2 \delta(\vec{Q} - \vec{k}_1 - \vec{k}_2) \frac{d^2 k_1^\perp}{2(2\pi)^2} \frac{d^2 k_2^\perp}{2(2\pi)^2},$$

where \vec{Q} is the transverse part of the total transferred four-momentum $p_a - p'_a$.

Let us consider the inelastic processes corresponding to the exchange of one or more reggeons. We apply the generalized Cutkosky unitarity bound [7], which defines the imaginary part of any Feynman diagram as the sum of contributions:

$$\text{Im} A(s, t) = \frac{1}{2i} \sum_c \delta_c A(s, t) = \frac{1}{2} \int d\tau_{n_c} A(2 \rightarrow n_c) A(n_c \rightarrow 2), \tag{3}$$

of all sorts of cuttings along some line to the associated amplitudes, the left $A(2 \rightarrow n_c)$ and the right $A(n_c \rightarrow 2) = A^*(2 \rightarrow n_c)$. The discontinuity $\delta_c A(s, t)$ along s in the cutting is associated with the production of the state c , $d\tau_{n_c}$ is the partial phase volume, and n_c is the number of lines in the diagram cut by the contour c and displayed on a mass surface. The summation in Equation (3) runs over all admissible cuttings. Since the contribution of any reggeon diagram is a convergent sum of the inputs of Feynman diagrams, the generalized Cutkosky unitarity bound can be directly applied to any reggeon diagram. It can be shown that the cutting line of a one-reggeon amplitude inside the reggeon diagram should run along the entire reggeon, or not to cut it at all.

For the amplitude of the two-reggeon scattering, the absorption parts of the reggeon diagram can be of the following three types:

- 1) The cutting line is drawn between the reggeons (Figure 3). This corresponds to the process of diffraction dissociation of projectile particles. The absorption part F_2^0 is equal to:

$$F_2^0(s, t) = s \int dx_1 dx_2 \delta(1 - x_1 - x_2) dx'_1 dx'_2 \delta(1 - x'_1 - x'_2) w(x_1, x_2) w(x'_1, x'_2) \times \int d\Omega_2 \{ [i D_1(s_1, k_1^{\perp 2})] [i D_2(s_2, k_2^{\perp 2})]^* + [i D_2(s_2, k_2^{\perp 2})] [i D_1(s_1, k_1^{\perp 2})]^* \}. \tag{4}$$

The two terms in the integrand correspond to two possible positions of reggeons in relation to the cutting line.

- 2) The cutting line is drawn along one of the reggeons (Figure 4). This is a contribution from the interference of the amplitude of the formation of a multi-peripheral particle comb and the same comb with an additional exchange of reggeon. The equation for the absorption part F_2^1 looks like:

$$F_2^1(s, t) = s \int dx_1 dx_2 \delta(1 - x_1 - x_2) dx'_1 dx'_2 \delta(1 - x'_1 - x'_2) w(x_1, x_2) w(x'_1, x'_2) \times \int d\Omega_2 \{ [i D_1(s_1, k_1^{\perp 2})] [2\text{Im} D_2(s_2, k_2^{\perp 2})] + [2\text{Im} D_2(s_2, k_2^{\perp 2})] [i D_1(s_1, k_1^{\perp 2})]^* + [2\text{Im} D_1(s_1, k_1^{\perp 2})] [i D_2(s_2, k_2^{\perp 2})]^* + [i D_2(s_2, k_2^{\perp 2})] [2\text{Im} D_1(s_1, k_1^{\perp 2})] \}. \tag{5}$$

Here, the first two terms correspond to the cut reggeon D_2 and the position of the reggeon D_1 to the left and to the right of the cutting line, and the last two terms correspond to the reggeon cut D_1 and the position of the reggeon D_2 to the left and to the right of the cutting line.

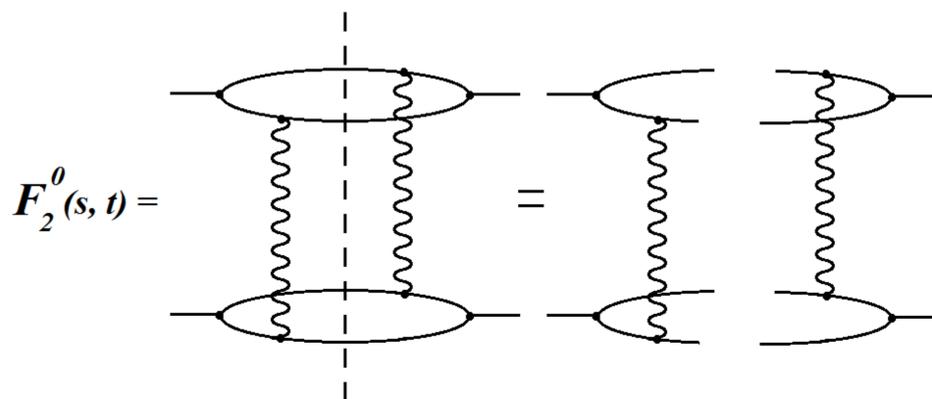


Figure 3. Cut of the double reggeon diagram between the reggeons. The lines are as in Figure 1.

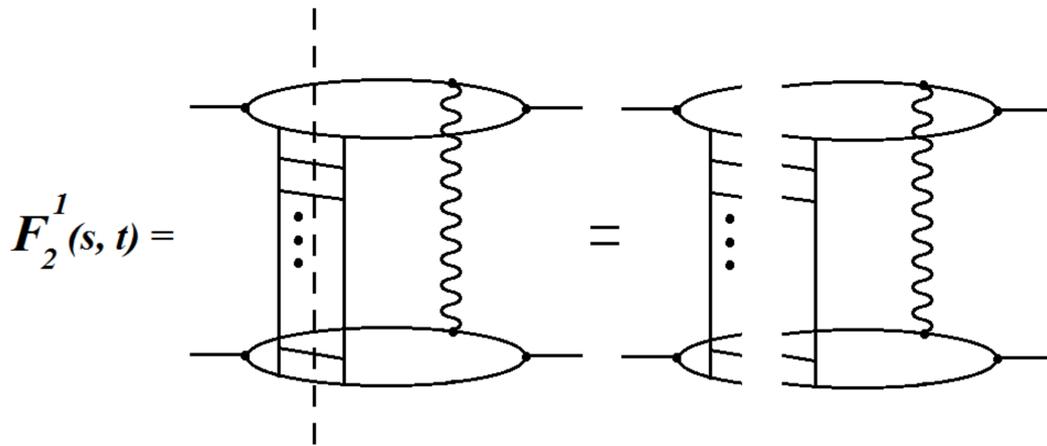


Figure 4. Cut along one of the reggeons. The lines are as in Figure 1.

- 3) The cut line is drawn simultaneously on both reggeons (Figure 5). This cutting corresponds to the formation of two multi-peripheral combs. The corresponding absorption part F_2^2 is represented as:

$$F_2^2(s, t) = s \int dx_1 dx_2 \delta(1 - x_1 - x_2) dx'_1 dx'_2 \delta(1 - x'_1 - x'_2) w(x_1, x_2) w(x'_1, x'_2) \times \int d\Omega_2 [2\text{Im} D_1(s_1, k_1^{\perp 2})] [2\text{Im} D_2(s_2, k_2^{\perp 2})]. \tag{6}$$

Both reggeons are cut here.

Adding these three absorption parts, we obtain the doubled imaginary part of the amplitude of the two-reggeon branching:

$$2\text{Im}A(s, t) = 2s \int dx_1 dx_2 \delta(1 - x_1 - x_2) dx'_1 dx'_2 \delta(1 - x'_1 - x'_2) w(x_1, x_2) w(x'_1, x'_2) \times \int d\Omega_2 [\text{Re}D_1(s_1, k_1^{\perp 2}) \text{Re}D_2(s_2, k_2^{\perp 2}) - \text{Im}D_1(s_1, k_1^{\perp 2}) \text{Im}D_2(s_2, k_2^{\perp 2})]. \tag{7}$$

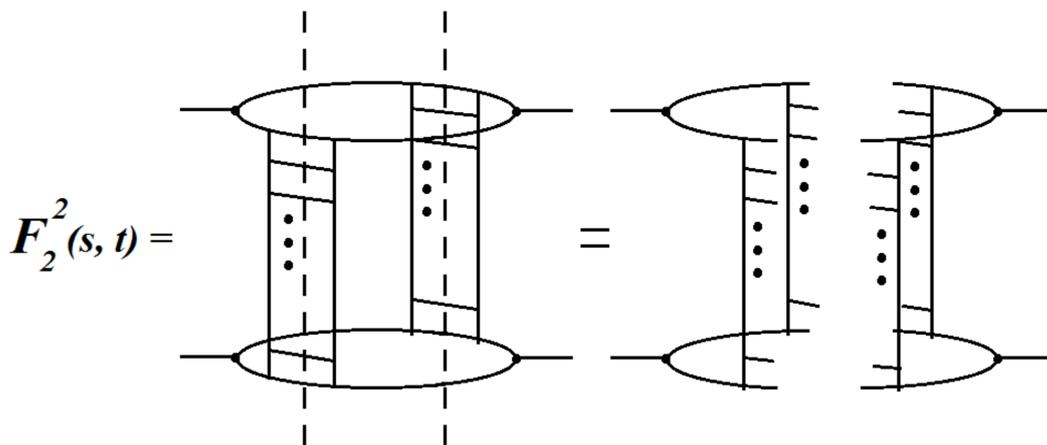


Figure 5. Cut along both reggeons. The lines are as in Figure 1.

Similarly, for the absorption parts of the amplitudes with a cutting of m from n reggeons:

$$F_n^m(s, t) = (-1)^{n-m} C_n^m s \int dx_1 \dots dx_n \delta\left(1 - \sum_1^n x_j\right) dx'_1 \dots dx'_n \delta\left(1 - \sum_1^n x'_j\right) w(x_1, \dots, x_n) w(x'_1, \dots, x'_n) \times \int d\Omega_n \prod_{j=1}^n 2\text{Im} D_j(s_j, k_j^{\perp 2}). \tag{8}$$

From Equation (8), the next relations follow:

$$\begin{aligned} \sum_{m=1}^n m F_n^m(s, t) &= 0, \\ \dots\dots\dots \\ \sum_{m=l}^n m(m-1) \dots (m-l+1) F_n^m(s, t) &= 0. \end{aligned} \tag{9}$$

These AGK relations were derived for hadron physics. The significant distances are determined by the functions $w(x_1, \dots, x_n)$ and, in order of magnitude, are equal to 10^{-15} m. In the Appendix A, we present a more general proof of the AGK theorem, which will not fix the distance. The proof comes from the unitarity of the S -matrix. It proves the theorem not only for the case of elastic scattering but also for any number of initial states. Therefore, it turned out that the theorem is valid for all particles with different energies surrounding a black hole.

The standard scheme for constructing a theory is as follows. The external field is chosen so that it is absent in the distant past and the distant future. In these in- and out-areas, it is possible to unambiguously determine the state of the particle and vacuum. As a result of the action of the external field in the process of evolution of the system in the initial vacuum state, the production of particles starts, so that the result of the evolution of the state corresponding to the in-vacuum does not coincide with the out-vacuum state. Complete information about the processes of particle production, scattering, and annihilation is contained in a unitary operator connecting the in- and out- states, which is called the S -matrix.

The applicability of the S -matrix theory to the black hole radiation problem was shown in [8]. Let us consider the simplest process of absorption of a charged particle (proton) by a black hole. At the initial time, the particle is sufficiently distant from the black hole, so that its interaction is negligible. This allows the establishment of an in-state, including a particle and a black hole. Subsequently, the particle is attracted to a black hole, increasing its momentum and energy. From a certain moment, the particle begins to emit photons in a bremsstrahlung, dumping a part of the energy received from a black hole.

We cannot “turn off” the gravitational field of a black hole. As the particle approaches it, the interaction will increase. The resulting state is a black hole, a particle, and photons will not be an out-state. However, according to this state, we perform the convolution of the T -matrices in the unitarity bound. So, we get diagrams in which there will be free particles in the initial and final states.

It should be noted that the remote observer will be able to register only those photons that fly outside of the Schwarzschild radius. Particles that fall inside the Schwarzschild radius remain “invisible” to an observer. Therefore, we must average on these states. That is, instead of pure quantum-mechanical states, we must research inclusive cross-sections.

2. The Dependence of the Luminosity of a Black Hole on Its Mass

We consider the following picture of the dependence of a black hole on its mass. Let us consider a proton on a long distance from a black hole. As it approaches the black hole, the momentum and energy of this proton increase, and it begins to emit photons in a bremsstrahlung. The corresponding Feynman diagram has the following form (Figure 6). We consider the simplest black hole consisting only of neutrons. A part of the photons and the proton will pass through the Schwarzschild radius, but a certain number of photons will be emitted into the surrounding space. These photons will carry

the energy “sucked off” from the black hole. To determine the probabilities of the processes, we have to multiply by the complex conjugate amplitude, as shown in Figure 7.

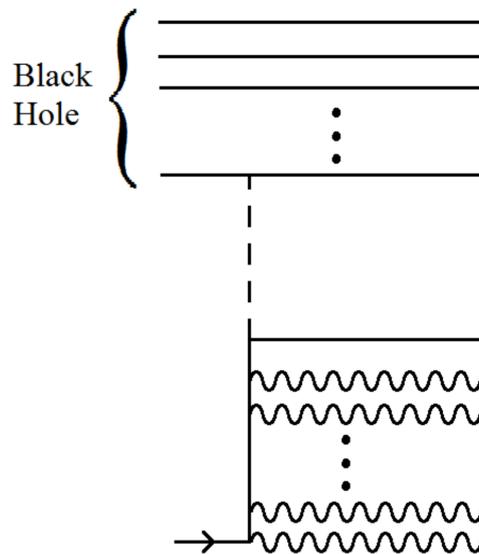


Figure 6. The amplitude of absorption of a proton by a black hole. Solid lines in curly brackets denote the protons inside the black hole. Wavy lines represent photons.

We get the square of the amplitude modulus corresponding to the exclusive photon emission cross-section in the field of a black hole by a proton. Integrating over the phase volume of all particles, including black hole particles, we obtain the imaginary part of the amplitude of elastic scattering of a proton by a black hole.

If a black hole attracts several protons, we get the Feynman diagrams in the form of Figure 8.

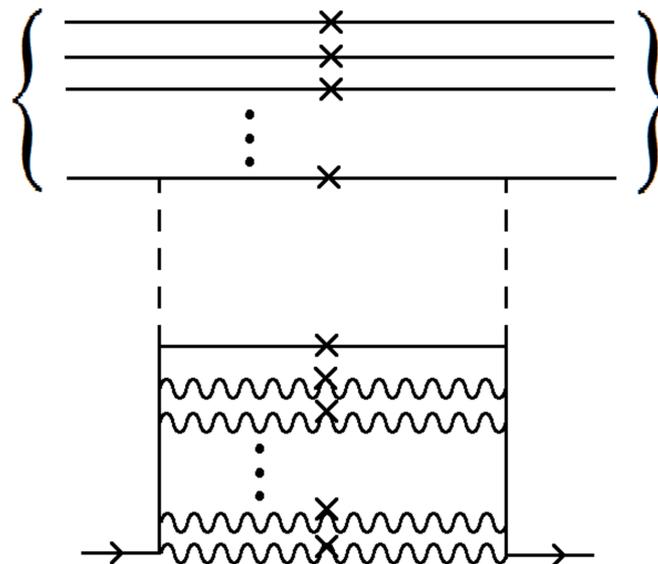


Figure 7. Square of the modulus of the absorption amplitude of a proton by a black hole. The lines are as in Figure 6.

Taking the square of the module of this diagram (Figure 9), we obtain an exclusive cross-section for the production of photons from protons in the field of a black hole. Integrating over the phase volume, we obtain the contribution into the total cross-section of this process.

From the presence of the imaginary parts shown in Figures 7 and 9, it follows that there are full amplitudes corresponding to the elastic scattering of one or several protons on the black hole, as shown in Figures 10 and 11.

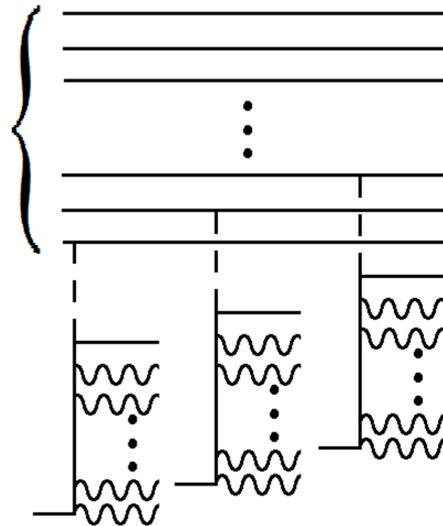


Figure 8. Absorption amplitude of several protons by a black hole. The lines are as in Figure 6

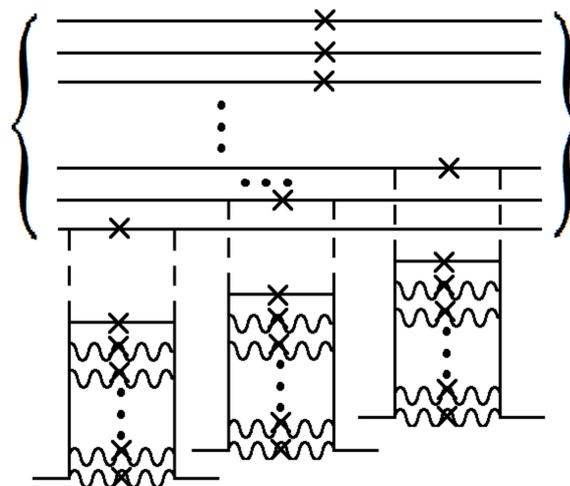


Figure 9. Squared modulus of the amplitude of the absorption of several protons by a black hole. The lines are as in Figure 6.

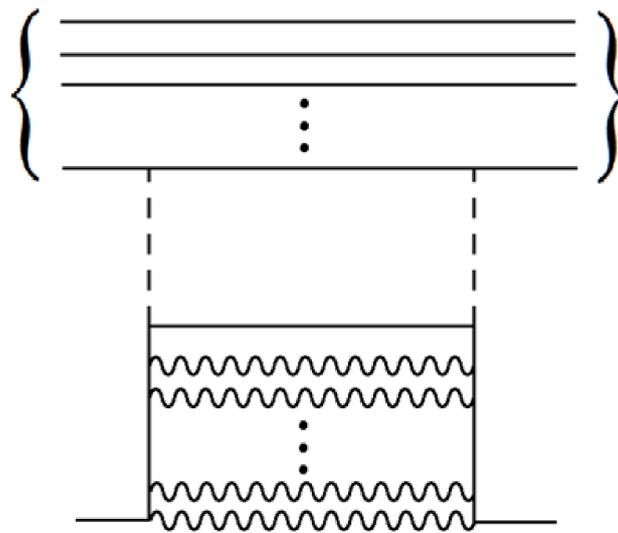


Figure 10. The elastic scattering amplitude of a proton on a black hole. The lines are as in Figure 6.

In the Appendix A it is shown that the AGK theorem is valid for such diagrams.

An averaging is necessary over particles located within the Schwarzschild radius. Such averaging can be carried out using the inclusive technique [9–12]. The inclusive cross-section is obtained from the cut amplitudes, fixing the observed particle, integrating over the momentum of others, and summing over all the remaining particles. Therefore, for inclusive cross-sections, cuts between amplitudes do not provide any contributions. Equations (A30)–(A39) (see also (9)) imply that all diagrams for single-particle inclusive cross-sections are cut down. Only the diagram such as one shown in Figure 10, remains. Since both gravitons are connected to the same black hole particle, the input of this diagram will be proportional to the sum of all black hole particles, i.e., to the mass of the black hole.

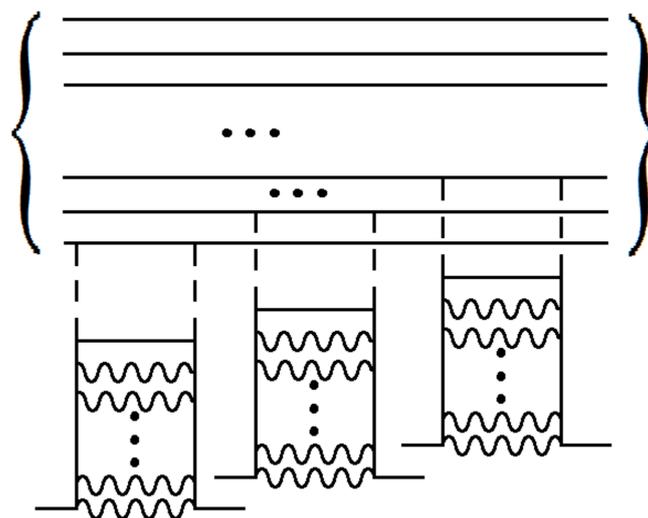


Figure 11. The elastic scattering amplitude of several protons on a black hole. The lines are as in Figure 6.

The total energy carried away by the emitted particle is equal to the energy of the particle multiplied by the inclusive cross-section of the production of such a particle and integrated over all possible energy values. The luminosity is proportional to the energy carried away by a photon in a certain frequency range of the photon. It can be determined as follows:

$$L \propto \int_{w_1}^{w_2} \frac{dw}{w} \int d^2k_{\perp} \cdot w f(w, k_{\perp}^2), \tag{10}$$

where $f(w, k_{\perp}^2)$ is the inclusive cross-section of the photon radiation by a proton in the field of a black hole. Considering that:

$$f(w, k_{\perp}^2) \propto M_{BH},$$

we will have for luminosity:

$$L = a M_{BH},$$

where the parameter a depends only on the amplitude of the graviton proton scattering.

3. Comparison with Experimental Data

The data are found to provide the dependence of the black hole mass on the black hole luminosity such as [1]:

$$\log_{10} M_{BH} = (8.95 \pm 0.11) + (1.11 \pm 0.18) \log_{10}(L_V/10^{11}), \tag{11}$$

and [2]:

$$\log_{10} M_{BH} = 9.16 + 1.16 \log_{10}(L_V/10^{11}). \tag{12}$$

This goes along with the correspondences:

$$\begin{aligned} L_V &= 0.87 \cdot 10^3 M_{BH}^{0.9} \\ L_V &= 1.27 \cdot 10^3 M_{BH}^{0.86}. \end{aligned} \tag{13}$$

Such correspondences are very close to the one we obtained (1). In [2], it is also noted that the black holes of the galaxies NGC 3842 and NGC4889 do not fit into the phenomenological formulas that describe the black holes given in Table 4 of [2].

Using the quantum mechanical formula derived by us, we divided the data [2] into five groups as given in Tables 1–5 and shown in Figures 12–16 along with the dependence (1).

Table 1. The first group of galaxies [2]: $L_V/M_{BH} < 25$.

Galaxy	M_{BH}	$M_{BH\min}$	$M_{BH\max}$	$\log_{10} L_V$	L_V/M_{BH}
N4889	2.1×10^{10}	5.5×10^9	3.7×10^{10}	11.42	12.5
N3842	9.7×10^9	7.2×10^9	1.27×10^{10}	11.20	16.3
N4486 (M87)	6.3×10^9	5.9×10^9	6.6×10^9	11.10	20.0
N4649 (M60)	4.7×10^9	3.7×10^9	5.8×10^9	10.99	21.2
N1332	1.45×10^9	1.25×10^9	1.65×10^9	10.14	9.5
N4291	9.2×10^8	6.3×10^8	1.21×10^9	10.20	17.2
N5845	5.4×10^8	3.7×10^8	7.1×10^8	9.84	12.8

Table 2. The second group of galaxies [2]: $25 < L_V/M_{BH} < 40$.

Galaxy	M_{BH}	M_{BHmin}	M_{BHmax}	$\log_{10}L_V$	L_V/M_{BH}
N6086	3.8×10^9	2.6×10^9	5.5×10^9	11.18	39.7
IC1459	2.8×10^9	1.6×10^9	3.9×10^9	10.96	35.8
N3115	9.6×10^8	6.7×10^8	1.5×10^9	10.40	26.1
N4026	2.1×10^8	1.7×10^8	2.8×10^8	10.35	34.5

Table 3. The third group of galaxies [2]: $90 < L_V/M_{BH} < 150$.

Galaxy	M_{BH}	M_{BHmin}	M_{BHmax}	$\log_{10}L_V$	L_V/M_{BH}
A1836-BCG	3.9×10^9	3.3×10^9	4.3×10^9	11.26	47
N1399	1.3×10^9	6.4×10^8	1.8×10^9	10.78	46
N524	8.3×10^8	7.9×10^8	9.2×10^8	10.67	56
N3608	4.7×10^8	3.7×10^8	5.7×10^8	10.35	47.8
N3379(M105)	4.6×10^8	3.4×10^8	5.7×10^8	10.37	50.9
N3377	1.9×10^8	9.0×10^7	2.9×10^8	9.98	50.3

Table 4. The fourth group of galaxies [2]: $90 < L_V/M_{BH} < 150$.

Galaxy	M_{BH}	M_{BHmin}	M_{BHmax}	$\log_{10}L_V$	L_V/M_{BH}
A3565-BCG	1.4×10^9	1.2×10^9	1.7×10^9	11.24	124
N4374 (M84)	8.5×10^8	7.7×10^8	9.4×10^8	10.91	96.4
N1399	5.1×10^8	4.4×10^8	5.8×10^8	10.78	118
N3585	3.4×10^8	2.8×10^8	4.9×10^8	10.65	131
N4697	2.0×10^8	1.8×10^8	2.2×10^8	10.45	141
N5576	1.8×10^8	1.4×10^8	2.1×10^8	10.44	143
N821	1.8×10^8	1.0×10^8	2.6×10^8	10.43	149
N221 (M32)	3.1×10^6	2.5×10^6	3.7×10^6	8.66	147

Table 5. The fifth group of galaxies [2]: $150 < L_V/M_{BH} < 350$.

Galaxy	M_{BH}	M_{BHmin}	M_{BHmax}	$\log_{10}L_V$	L_V/M_{BH}
N4261	5.5×10^8	4.3×10^8	6.6×10^8	11.02	190
N7052	4.0×10^8	2.4×10^8	6.8×10^8	10.87	185
N5128	3.0×10^8	2.8×10^8	3.4×10^8	10.66	152
N4473	1.0×10^8	5.0×10^7	1.5×10^8	10.39	245
N4459	7.4×10^7	6.0×10^7	8.8×10^7	10.36	310
N1023	4.6×10^7	4.1×10^7	5.1×10^7	10.18	330
N2549	1.4×10^7	1.0×10^7	1.47×10^7	9.6	284
N4486A	1.3×10^7	9.0×10^6	1.8×10^7	9.41	198
N4757	1.0×10^7	4.0×10^6	1.6×10^7	9.42	263

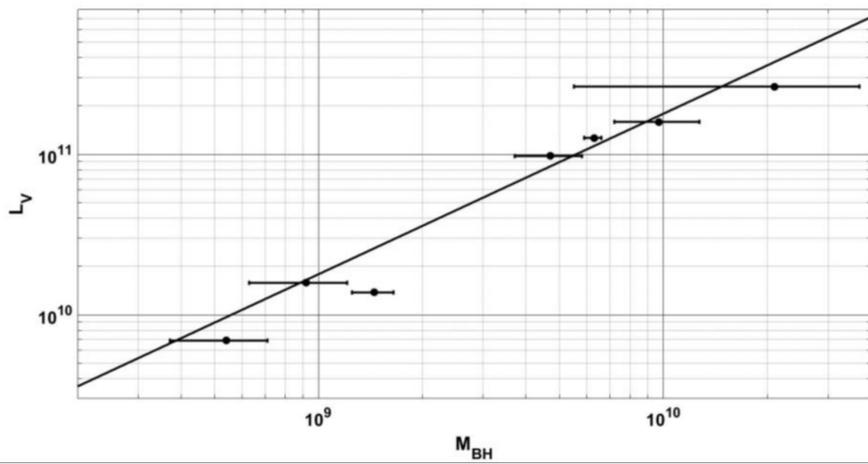


Figure 12. Dependence of the luminosity of a black hole on its mass (Table 1). The data are taken from [2]. $L_V = 17.87 M_{BH}$.

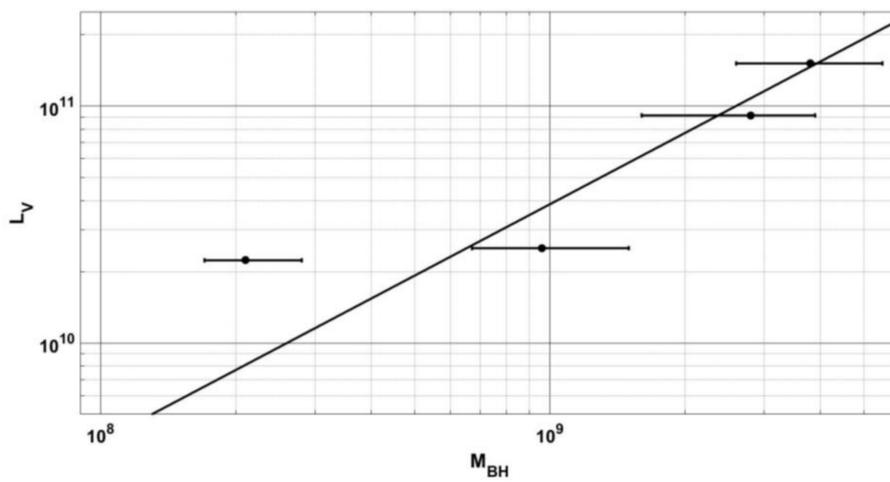


Figure 13. Dependence of the luminosity of a black hole on its mass (Table 2). The data are taken from [2]. $L_V = 38.50 M_{BH}$.

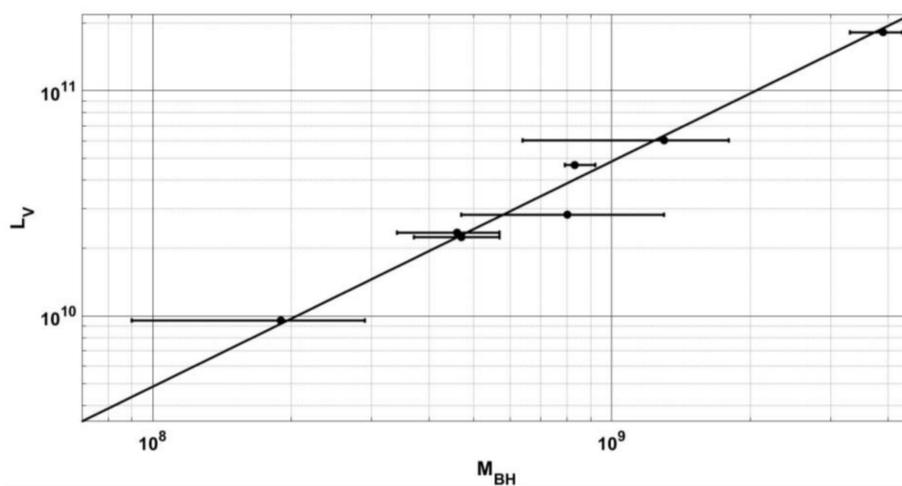


Figure 14. Dependence of the luminosity of a black hole on its mass (Table 3). The data are taken from [2]. $L_V = 48.58 M_{BH}$.

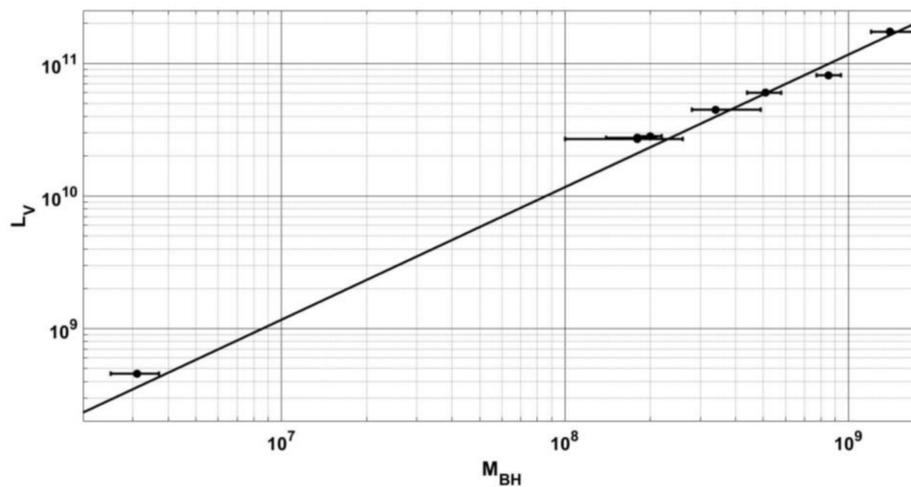


Figure 15. Dependence of the luminosity of a black hole on its mass (Table 4). The data are taken from [2]. $L_V = 116.56 M_{BH}$.

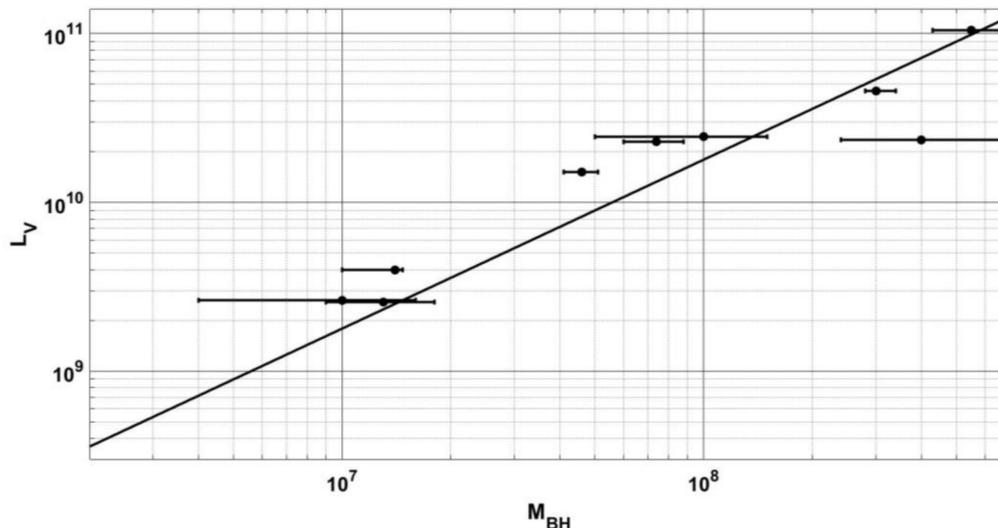


Figure 16. Dependence of the luminosity of a black hole on its mass (Table 5). The data are taken from [2]. $L_V = 179.17 M_{BH}$.

We believe that the substance of the universe consists of 91% of hydrogen and 9% of helium. We have considered only one charged particle. In reality, hydrogen and helium atoms fall on a black hole. In hydrogen, there are two charged particles: a proton and an electron. In helium, there are four charged particles and two neutrons. Therefore, we must take into account the bremsstrahlung of all these particles together.

We can assume that the first group includes galaxies in which the substance is distributed around a black hole, like in the whole Universe. Therefore, galaxies NGC3842 and NGC4889 do not differ from other galaxies. In other groups, the density of the substance near the black hole is increased. Assuming that the remaining parameters of the graviton-proton interaction are the same for all groups, then the substance density is increased by two, three, six, and ten times. However, in this matter, more studies are required, which are planned for the future.

4. Conclusions

It was shown a black hole can emit photons via a mechanism of Beckenstein—Hawking radiation [13–16]. In this paper, we made the case that there is another possible process for emitting photons from a black hole.

We obtained a linear dependence of the luminosity of a black hole on the mass of a black hole. It occurs because all inelastic processes of photon emission by several atoms falling on a black hole are completely cut down by interference. Only the inclusive amplitude of photon emission by one atom falling on a black hole remains. The quantum mechanical formula we derived actually coincides with the phenomenological formulas described earlier in other papers. We believe that the difference in the constant of proportionality in (1) is due to differences in the density of substance near the black hole.

We also believe that the dependence of the galaxy velocity dispersion σ on the mass of the black hole:

$$(\sigma / km s^{-1}) = 4.8 \cdot M_{BH}^{0.2}$$

most likely cannot be derived from a quantum-mechanical consideration.

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Appendix A. Derivation of AGK Relations

We present the S -matrix as the product of several unitary operators. Each of these operators acts only on its own states with certain quantum numbers.

$$S = S_1 \cdot S_2 \cdot \dots \cdot S_n. \tag{A1}$$

This, according to Mandelstam [17], immediately ensures the absence of planar diagrams. First, let us consider the case when the S -matrix is the product of two operators:

$$S = S_1 \cdot S_2. \tag{A2}$$

Let us determine a T -matrix that describes the interaction:

$$\begin{aligned} S &= 1 + iT, \\ S_l &= 1 + iT_l. \end{aligned} \tag{A3}$$

In this case:

$$S = S_1 \cdot S_2 = (1 + iT_1)(1 + iT_2) = 1 + i(T_1 + T_2) - T_1 T_2, \tag{A4}$$

i.e.,

$$iT = i(T_1 + T_2) - T_1 T_2. \tag{A5}$$

From the unitarity bound, it follows that:

$$i(T^+ - T) = TT^+. \tag{A6}$$

We express the left side of this equality through T_1 and T_2 :

$$i(T^+ - T) = i(T_1^+ - T_1) + i(T_2^+ - T_2) \tag{A7}$$

$$+ T_1 T_2 + T_1^+ T_2^+. \tag{A8}$$

The right hand side of (A6) becomes:

$$TT^+ = T_1 T_1^+ + T_2 T_2^+ \tag{A9}$$

$$+ T_1 T_2^+ + T_2 T_1^+ \tag{A10}$$

$$+ i(T_1 - T_1^+)T_2 T_2^+ + i(T_2 - T_2^+)T_1 T_1^+ \tag{A11}$$

$$+ T_1 T_1^+ T_2 T_2^+. \tag{A12}$$

Now, after we have considered unitary operators, let us turn to the scattering amplitudes.

Let us introduce the two-particle state $|p_1^{(1)}, p_2^{(1)}\rangle, |p_1^{(2)}, p_2^{(2)}\rangle, |p_1^{(1)'}, p_2^{(1)'}\rangle, |p_1^{(2)'}, p_2^{(2)'}\rangle$ and the complete system:

$$\sum_l \int d\tau_{kl} |k_l^{(j)}\rangle \langle k_l^{(j)}| = 1. \tag{A13}$$

Here, $d\tau_{kl}$ is the phase volume of the complete system, and j is the corresponding quantum number.

We multiply the Equations (20) and (21) by $|p_1^{(1)}, p_2^{(1)}\rangle, |p_1^{(2)}, p_2^{(2)}\rangle$ on the right and by $\langle p_1^{(1)'}, p_2^{(1)'}|, \langle p_1^{(2)'}, p_2^{(2)'}|$ on the left. In addition, we insert the complete systems between the operators $T_1 T_1^+$ and $T_2 T_2^+$.

The first line of (A7)–(A8) will be equal to the first line of (A9)–(A12):

$$\begin{aligned} 2\text{Im}T_1(s_1, t_1) &= \sum_l \int d\tau_{kl} |T_{1,2l}|^2, \\ 2\text{Im}T_2(s_2, t_2) &= \sum_l \int d\tau_{kl} |T_{2,2l}|^2. \end{aligned} \tag{A14}$$

Here, $s_j = (p_1^{(j)} + p_2^{(j)})^2$ is the square of the total energy in the center-of-mass system, and $t_j = (p_1^{(j)} - p_1^{(j)'})^2$ is the square of the momentum transferred. $T_j(s_j, p_j)$ is the two-particle amplitude, and $T_{j,2l}$ is the transition amplitude of two particles into l particles.

Then, Equation (A8) takes the form:

$$T_1 T_2 + T_1^+ T_2^+ = 2\text{Re}T_1(s_1, t_1)\text{Re}T_2(s_2, t_2) - 2\text{Im}T_1(s_1, t_1)\text{Im}T_2(s_2, t_2). \tag{A15}$$

This expression represents the total input to the amplitude of elastic scattering with the exchange of two amplitudes, $T_1(s_1, t_1)$ and $T_2(s_2, t_2)$.

Equation (A10) is equal to:

$$T_1 T_2^+ + T_2 T_1^+ = 2\text{Re}T_1(s_1, t_1)\text{Re}T_2(s_2, t_2) + 2\text{Im}T_1(s_1, t_1)\text{Im}T_2(s_2, t_2). \tag{A16}$$

In Equation (A11), we insert the complete systems into $T_1 T_1^+, T_2 T_2^+$ and, using (A14), we get:

$$\begin{aligned} &i(T_1 - T_1^+)T_2 T_2^+ + i(T_2 - T_2^+)T_1 T_1^+ \\ &= 2\text{Im}T_1(s_1, t_1) \cdot 2\text{Im}T_2(s_2, t_2) - 2\text{Im}T_2(s_2, t_2)2\text{Im}T_1(s_1, t_1). \end{aligned} \tag{A17}$$

For Equation (A12), using (A14), we find:

$$T_1 T_1^+ \cdot T_2 T_2^+ = 2\text{Im}T_1(s_1, t_1) 2\text{Im}T_2(s_2, t_2). \tag{A18}$$

We obtained the coefficients following from the first AGK ratio for the exchange of two amplitudes $T_1(s_1, t_1), T_2(s_2, t_2)$. Summation of Equation (A16)–(A18) gives us (A15):

$$2\text{Re}T_1(s_1, t_1) \text{Re}T_2(s_2, t_2) - 2\text{Im}T_1(s_1, t_1) \text{Im}T_2(s_2, t_2). \tag{A19}$$

Let us find the input to the inclusive cross-section from the exchange of two amplitudes. To obtain it, we have to take one particle out of convolution (A14). Equation (A16) does not have such convolutions. In Equation (A17), from the convolution $T_2 T_2^+$ in the first two terms, we choose one particle. At the same time, $T_2 T_2^+$ is replaced by the corresponding inclusive cross-section, $f_2(\vec{k})$, where

\vec{k} is the momentum of the selected particle. In the last two terms $T_1 T_1^+$, is replaced by an inclusive cross-section, $f_1(\vec{k})$. Therefore, instead of Equation (A17), we have:

$$-4\text{Im}T_1(s_1, t_1) \cdot f_2(\vec{k}) - 4\text{Im}T_2(s_2, t_2) \cdot f_1(\vec{k}).$$

In Equation (A12), we replace both $T_1 T_1^+$ and $T_2 T_2^+$.

$$4\text{Im}T_1(s_1, t_1) \cdot f_2(\vec{k}) + 4\text{Im}T_2(s_2, t_2) \cdot f_1(\vec{k}).$$

Both inputs go cut down, regardless of the magnitudes of the total energies s_1 and s_2 , as it follows from the AGK ratios.

Let us consider the unitarity bound with three S -matrices.

$$S = S_1 \cdot S_2 \cdot S_3 = (1 + iT_1)(1 + iT_2)(1 + iT_3) \\ = 1 + i(T_1 + T_2 + T_3) - T_1 T_2 - T_1 T_3 - T_2 T_3 - iT_1 T_2 T_3. \tag{A20}$$

The result is:

$$iT = i(T_1 + T_2 + T_3) - T_1 T_2 - T_1 T_3 - T_2 T_3 - iT_1 T_2 T_3. \tag{A21}$$

We omit the inputs corresponding to the convolutions of one or two T -matrices. Then:

$$i(T - T^+) = i(T_1^+ T_2^+ T_3^+ - T_1 T_2 T_3), \tag{A22}$$

$$TT^+ = i(T_1^+ T_2 T_3 - T_1 T_2^+ T_3^+) + i(T_2^+ T_1 T_3 - T_2 T_1^+ T_3^+) + i(T_3^+ T_1 T_2 - T_3 T_1^+ T_2^+) \tag{A23}$$

$$+ T_1 T_1^+ (T_2 - T_2^+) (T_3^+ - T_3) + T_2 T_2^+ (T_1 - T_1^+) (T_3^+ - T_3) + T_3 T_3^+ (T_1 - T_1^+) (T_2^+ - T_2) \tag{A24}$$

$$+ i[T_1 T_1^+ T_2 T_2^+ (T_3 - T_3^+) + T_1 T_1^+ T_3 T_3^+ (T_2 - T_2^+) + T_2 T_2^+ T_3 T_3^+ (T_1 - T_1^+)] \tag{A25}$$

$$+ T_1 T_1^+ T_2 T_2^+ T_3 T_3^+. \tag{A26}$$

Using the previously introduced symbols, moving from the T -matrix to the scattering amplitudes, we obtain for Equation (A22):

$$i(T_1 T_2 T_3 - T_1^+ T_2^+ T_3^+) = 2\text{Im}T_1 \text{Im}T_2 \text{Im}T_3 \\ - 2[\text{Re}T_1 \text{Re}T_2 \text{Im}T_3 + \text{Im}T_1 \text{Re}T_2 \text{Re}T_3 + \text{Re}T_1 \text{Im}T_2 \text{Re}T_3] \tag{A27}$$

and for the right hand side of Equation (A23):

$$i(T_1^+ T_2 T_3 - T_1 T_2^+ T_3^+) = -2\text{Re}T_1 \text{Re}T_2 \text{Im}T_3 - 2\text{Re}T_1 \text{Re}T_3 \text{Im}T_2 + 2\text{Re}T_2 \text{Re}T_3 \text{Im}T_1 \\ - 2\text{Im}T_1 \text{Im}T_2 \text{Im}T_3,$$

$$i(T_2^+ T_1 T_3 - T_2 T_1^+ T_3^+) = -2\text{Re}T_1 \text{Re}T_2 \text{Im}T_3 - 2\text{Re}T_2 \text{Re}T_3 \text{Im}T_1 + 2\text{Re}T_1 \text{Re}T_3 \text{Im}T_2 \\ - 2\text{Im}T_1 \text{Im}T_2 \text{Im}T_3,$$

$$i(T_3^+ T_1 T_2 - T_3 T_1^+ T_2^+) = -2\text{Re}T_1 \text{Re}T_3 \text{Im}T_2 - 2\text{Re}T_2 \text{Re}T_3 \text{Im}T_1 + 2\text{Re}T_1 \text{Re}T_2 \text{Im}T_3 \\ - 2\text{Im}T_1 \text{Im}T_2 \text{Im}T_3.$$

Then the r.h.s. of Equation (A23) reads:

$$- 2(\text{Re}T_1 \text{Re}T_2 \text{Im}T_3 + \text{Im}T_1 \text{Re}T_2 \text{Re}T_3 + \text{Re}T_1 \text{Im}T_2 \text{Re}T_3) - 6\text{Im}T_1 \text{Im}T_2 \text{Im}T_3. \tag{A28}$$

In Equation (A24), the convolution line passes either over $T_1 T_1^+$:

$$T_1 T_1^+ (T_2 - T_2^+) (T_3^+ - T_3) = 2\text{Im}T_1 \cdot 2\text{Im}T_2 \cdot 2\text{Im}T_3,$$

or over $T_2 T_2^+$:

$$T_2 T_2^+ (T_1 - T_1^+) (T_3^+ - T_3) = 2\text{Im}T_2 \cdot 2\text{Im}T_1 \cdot 2\text{Im}T_3,$$

or over $T_3 T_3^+$:

$$T_3 T_3^+ (T_1 - T_1^+) (T_2^+ - T_2) = 2\text{Im}T_3 \cdot 2\text{Im}T_1 \cdot 2\text{Im}T_2.$$

Total Equation (A24) gives:

$$24\text{Im}T_1(s_1, t_1)\text{Im}T_2(s_2, t_2)\text{Im}T_3(s_3, t_3). \tag{A29}$$

Equation (A25) corresponds to the simultaneous convolution of two pairs of T-matrices:

$$i[T_1 T_1^+ T_2 T_2^+ (T_3 - T_3^+) + T_1 T_1^+ T_3 T_3^+ (T_2 - T_2^+) + T_2 T_2^+ T_3 T_3^+ (T_1 - T_1^+)] = -24\text{Im}T_1(s_1, t_1)\text{Im}T_2(s_2, t_2)\text{Im}T_3(s_3, t_3). \tag{A30}$$

Finally, Equation (A26) corresponds to the simultaneous convolution of all three pairs of T-matrices.

$$T_1 T_1^+ T_2 T_2^+ T_3 T_3^+ = 8\text{Im}T_1(s_1, t_1)\text{Im}T_2(s_2, t_2)\text{Im}T_3(s_3, t_3). \tag{A31}$$

The summation of Equations (A28)–(A30) gives a contribution (A27), which corresponds to the exchange contribution of three two-particle amplitudes into the amplitude of elastic scattering.

Continuing the similar procedure of convolutions with a large number of the T-matrices, we obtain:

$$F_n^m = (-1)^{n-m} C_n^m \prod_{j=1}^n 2\text{Im}T_j(s_j, t_j), \tag{A32}$$

$$F_n^0 = (-1)^n C_n^m \prod_{j=1}^n 2\text{Im}T_j(s_j, t_j) + 2\text{Im} \left[(-i) \prod_{j=1}^n i T_j(s_j, t_j) \right]. \tag{A33}$$

Note that from (A32) follow the relations (9):

$$\begin{aligned} \sum_{m=1}^n m F_n^m(s, t) &= 0, \\ \dots\dots\dots \\ \sum_{m=l}^n m(m-1) \dots (m-l+1) F_n^m(s, t) &= 0. \end{aligned} \tag{A34}$$

We show that single-particle and two-particle inclusive cross-sections are cut down.

We cannot select and exclude single-particle inclusive cross-sections from Equation (A28). In Equation (A29), we have:

$$8f_1(k)\text{Im}T_2\text{Im}T_3 + 8\text{Im}T_1 f_2(k)\text{Im}T_3 + 8\text{Im}T_1\text{Im}T_2 f_3(k). \tag{A35}$$

From Equation (A30) one gets:

$$\begin{aligned} -8f_1(k)\text{Im}T_2\text{Im}T_3 - 8\text{Im}T_1 f_2(k)\text{Im}T_3 - 8f_1(k)\text{Im}T_2 \text{Im}T_3 \\ -8\text{Im}T_1\text{Im}T_2 f_3(k) - 8\text{Im}T_1 f_2(k)\text{Im}T_3 - 8\text{Im}T_1\text{Im}T_2 f_3(k). \end{aligned} \tag{A36}$$

From Equation (A31) we have:

$$8f_1(k)\text{Im}T_2\text{Im}T_3 + 8\text{Im}T_1 f_2(k)\text{Im}T_3 + 8\text{Im}T_1\text{Im}T_2 f_3(k). \quad (\text{A37})$$

In summation, all these terms go cut down.

For a two-particle inclusive cross-section, Equation (A30) implies:

$$-8f_1(k)f_2(k)\text{Im}T_3 - 8f_1(k)\text{Im}T_2 f_3(k) - 8\text{Im}T_1 f_2(k)f_3(k). \quad (\text{A38})$$

Equation (A31) implies:

$$8f_1(k)f_2(k)\text{Im}T_3 + 8f_1(k)\text{Im}T_2 f_3(k) + 8\text{Im}T_1 f_2(k)f_3(k). \quad (\text{A39})$$

In summation, two of these inputs go cut down.

The obtained results are based on the use of the *S*-matrix unitarity method — the requirements are that the probability of all possible processes must be equal to 1. In this method, it is not essential which space-time metric is used—the sum of the probabilities of all possible processes must be equal to 1 in any metric.

In this article, we considered the proton and the hydrogen atom as whole objects for simplicity. In the following articles, with explicit calculation of inclusive photon production cross-sections, we will consider the interaction of a photon with quarks and an electron. We believe that only one graviton will interact with any of the proton quarks. Therefore, we do not expect changes in the dependency (1).

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