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# Finiteness of $\mathcal{N}=4$ Super-Yang–Mills Effective Action in Terms of Dressed $\mathcal{N}=1$ Superfields

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**Abstract:** We argue in favor of the independence on any scale, ultraviolet or infrared, in kernels of the effective action expressed in terms of dressed $\mathcal{N}=1$ superfields for the case of $\mathcal{N}=4$ super-Yang–Mills theory. Under "scale independence" of the effective action of dressed mean superfields, we mean its "finiteness in the off-shell limit of removing all the regularizations". This off-shell limit is scale independent because no scale remains inside these kernels after removing the regularizations. We use two types of regularization: regularization by dimensional reduction and regularization by higher derivatives in its supersymmetric form. Based on the Slavnov–Taylor identity, we show that dressed fields of matter and of vector multiplets can be introduced to express the effective action in terms of them. Kernels of the effective action expressed in terms of such dressed effective fields do not depend on the ultraviolet scale. In the case of dimensional reduction, by using the developed technique, we show how the problem of inconsistency of the dimensional reduction can be solved. Using Piguet and Sibold formalism, we indicate that the dependence on the infrared scale disappears off shell in both the regularizations.

**Keywords:** $R$-operation; gauge symmetry; $\mathcal{N}=4$ supersymmetry; Slavnov–Taylor identity

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## 1. Introduction

The effective action is restricted by consequences of various symmetries of the classical action that at the quantum level take the form of specific identities. One of them is the Slavnov–Taylor (ST) identity [1–6]. This generalizes the Ward–Takahashi identity of quantum electrodynamics to the non-Abelian case and can be derived starting from the property of invariance of the tree-level action with respect to BRST symmetry [7,8]. The ST identity can be formulated as equations involving variational derivatives of the effective action. In the general $\mathcal{N}=1$ supersymmetric theory, it can be written as [9]

$$\begin{gathered}\mathrm{Tr}\left[\int d^8z\,\frac{\delta\Gamma}{\delta V}\frac{\delta\Gamma}{\delta K} - i\int d^6y\,\frac{\delta\Gamma}{\delta c}\frac{\delta\Gamma}{\delta L} + i\int d^6\bar{y}\,\frac{\delta\Gamma}{\delta\bar{c}}\frac{\delta\Gamma}{\delta\bar{L}}\right.\\ \left. -\int d^6y\,\frac{\delta\Gamma}{\delta b}\left(\frac{1}{32}\frac{1}{\alpha}\bar{D}^2D^2V\right) - \int d^6\bar{y}\,\frac{\delta\Gamma}{\delta\bar{b}}\left(\frac{1}{32}\frac{1}{\alpha}D^2\bar{D}^2V\right)\right]\\ -i\int d^6y\,\frac{\delta\Gamma}{\delta\Phi}\frac{\delta\Gamma}{\delta k} + i\int d^6\bar{y}\,\frac{\delta\Gamma}{\delta\bar{k}}\frac{\delta\Gamma}{\delta\bar{\Phi}} = 0.\end{gathered}$$

Here, the standard definition of the measures in superspace is used:

$$d^8z \equiv d^4x\,d^2\theta\,d^2\bar{\theta}, \quad d^6y \equiv d^4y\,d^2\theta, \quad d^6\bar{y} \equiv d^4\bar{y}\,d^2\bar{\theta}.$$

The effective action $\Gamma$ generates one-particle irreducible amplitudes of the quantum fields and contains all the information about the quantum behavior of the theory. It is a functional of the effective fields $V, b, \bar{b}, c, \bar{c}, \Phi, \bar{\Phi}$ and external sources $K, L, \bar{L}, k, \bar{k}$, coupled at tree level to BRST-transformations of the corresponding classical fields [4]. We use two types of UV regularization: regularization by higher derivatives [10,11] and regularization by dimensional reduction [12,13].

We will show that the actual variables of the effective action are dressed effective superfields, that is, they are effective superfields convoluted with some unspecified dressing functions that are parts of propagators. Kernels accompanying the dressed effective fields in the effective action are related to the scattering amplitudes of the particles. A similar problem has been solved in component formalism [14,15]. As has been argued in Refs. [14,15] in that formalism the dressed mean fields appear to be the actual variables of the effective action, leaving the kernels of the action independent of any scale in the limit of removing all the regularizations, ultraviolet or off-shell infrared, in case of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. This statement has been confirmed by the explicit calculation in Ref. [16]. In the present paper, this problem is considered by using $\mathcal{N} = 1$ superfield formalism, which keeps one of the supersymmetries apparent and conserves all the R-symmetries, to which we need to apply anomaly multiplet ideas [17]. The important point of Refs. [14,15,18–23] is the possibility to absorb the two point proper functions in the re-definition of the effective fields.

The on-shell IR divergences are constrained in QFT by the well-known Kinoshita theorem, while they cancel in all observables [24]. However, $\mathcal{N} = 4$ SYM is not a physical theory, and it does not have any observables. How is the finiteness of the kernels of this effective action in the limit of the removing regularization combined with this Kinoshita theorem? Suppose we removed the regularizations in these proper correlators of dressed mean superfields. These kernels become scale-independent after removing these regularizations. The corresponding connected Green functions (connected correlators) of the dressed mean fields contribute to the amplitudes on the mass shell. The amplitudes, which are roughly on-shell values of these connected correlators, may have IR divergences after putting them on-shell. In order to work with the connected correlators on shell, we need a regularization again. This may be any regularization that makes these on-shell values non-singular. These IR divergences may be regularized in a way that has nothing to do with the previous two regularizations, ultraviolet and infrared, used initially for the UV renormalization procedure and for the suppression of the off-shell IR infinities of the $\mathcal{N} = 1$ SUSY gauge theory. The most convenient way from our point of view may be via MB parametrization. We proposed it in Section 6. It is important that this auxiliary regularization may be removed then at the end in the results for “physical quantities” in the “physical processes”, which are $\mathcal{N} = 4$ SUSY models of DIS structure functions [25–28].

These two words, “finiteness” and “scale-independence”, are frequently used concepts in the framework for the AdS-CFT correspondence, where the correlators of the BPS gauge invariant operators may be found or restricted by using the methods of the conformal field theory. In this sense, $\mathcal{N} = 4$ SYM is exactly solvable QFT. The insertions of composite non-BPS operators as external states into the correlators of the components of $\mathcal{N} = 1$ supermultiplets are treated as the only source of UV-divergences [29]. In contrary, in the present paper we talk about the finiteness (and, then as a result of finiteness, about the scale independence) of the connected correlators (they are built from the proper correlators), which contribute to scattering amplitudes in this theory in the Landau gauge [14,15]. Renormalization by the parts of the propagators has been proposed in [18,20].

## 2. ST Identity in $\mathcal{N} = 1$ SUSY Formalism for $\mathcal{N} = 4$ SYM

We consider the $\mathcal{N} = 4$ theory in $\mathcal{N} = 1$ superfield formalism. This model has specific field contents. The Lagrangian of the model in terms of $\mathcal{N} = 1$ superfields is

$$S = \frac{1}{g^2}\frac{1}{128}\mathrm{Tr}\left[\int d^6y\, W_\alpha W^\alpha + \int d^6\bar{y}\, \bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}} + \int d^8z\, e^{-V}\bar{\Phi}_i\, e^V\, \Phi^i + \frac{1}{3!}\int d^6y\, i\epsilon_{ijk}\Phi^i[\Phi^j,\Phi^k] + \frac{1}{3!}\int d^6\bar{y}\, i\epsilon^{ijk}\bar{\Phi}_i[\bar{\Phi}_j,\bar{\Phi}_k]\right].$$

For the $\mathcal{N}=1$ supersymmetry, we use the notation of Ref. [30]. This Lagrangian for $\mathcal{N}=4$ supersymmetry is taken from Ref. [31]. The flavor indices of the matter run in $i=1,2,3$ and the matter superfields are in the adjoint representation of the gauge group, $\Phi^i = \Phi^{ia}\, T^a$. (The vector superfield is in the same representation, $V = V^a T^a$).

Consider for the beginning the general $\mathcal{N}=1$ super-Yang–Mills whose classical action takes the form

$$S = \int d^6y\, \frac{1}{128}\frac{1}{g^2}\mathrm{Tr}\, W_\alpha W^\alpha + \int d^6\bar{y}\, \frac{1}{128}\frac{1}{g^2}\mathrm{Tr}\, \bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}} + \int d^8z\, \bar{\Phi}\, e^V\, \Phi + \int d^6y\left[Y^{ijk}\Phi_i\Phi_j\Phi_k + M^{ij}\Phi_i\Phi_j\right] + \int d^6\bar{y}\left[\bar{Y}_{ijk}\bar{\Phi}^i\bar{\Phi}^j\bar{\Phi}^k + \bar{M}_{ij}\bar{\Phi}^i\bar{\Phi}^j\right]. \tag{1}$$

We do not specify the representation of the matter fields here. It is some general reducible representation of the gauge group with a set of irreducible representations. The Yukawa couplings $Y^{ijk}$ and $M^{ij}$ appear at some general triple vertex and mass terms in four dimensions. The path integral describing the quantum theory is defined as

$$\begin{aligned} Z[J,\eta,\bar{\eta},\rho,\bar{\rho},j,\bar{j},K,L,\bar{L},k,\bar{k}] = \int dV\, dc\, d\bar{c}\, db\, d\bar{b}\, d\Phi\, d\bar{\Phi}\, \exp i\Big[S \\ +2\,\mathrm{Tr}\left(\int d^8z\, JV + i\int d^6y\, \eta c + i\int d^6\bar{y}\, \bar{\eta}\bar{c} + i\int d^6y\, \rho b + i\int d^6\bar{y}\, \bar{\rho}\bar{b}\right) \\ +\left(\int d^6y\, \Phi\, j + \int d^6\bar{y}\, \bar{j}\, \bar{\Phi}\right) \\ +2\,\mathrm{Tr}\left(i\int d^8z\, K\delta_{\bar{c},c}V + \int d^6y\, Lc^2 + \int d^6\bar{y}\, \bar{L}\bar{c}^2\right) \\ +\int d^6y\, k\, c\, \Phi + \int d^6\bar{y}\, \bar{\Phi}\, \bar{c}\, \bar{k}\Big], \end{aligned} \tag{2}$$

The derivation of the ST identity in general supersymmetric theory can be found for example in Ref. [23]. The result is

$$\begin{aligned} \mathrm{Tr}\Big[\int d^8z\, \frac{\delta\Gamma}{\delta V}\frac{\delta\Gamma}{\delta K} - i\int d^6y\, \frac{\delta\Gamma}{\delta c}\frac{\delta\Gamma}{\delta L} + i\int d^6\bar{y}\, \frac{\delta\Gamma}{\delta\bar{c}}\frac{\delta\Gamma}{\delta\bar{L}} \\ - \int d^6y\, \frac{\delta\Gamma}{\delta b}\left(\frac{1}{32}\frac{1}{\alpha}\bar{D}^2D^2V\right) - \int d^6\bar{y}\, \frac{\delta\Gamma}{\delta\bar{b}}\left(\frac{1}{32}\frac{1}{\alpha}D^2\bar{D}^2V\right)\Big] \\ -i\int d^6y\, \frac{\delta\Gamma}{\delta\Phi}\,\frac{\delta\Gamma}{\delta k} + i\int d^6\bar{y}\, \frac{\delta\Gamma}{\delta\bar{k}}\,\frac{\delta\Gamma}{\delta\bar{\Phi}} = 0. \end{aligned} \tag{3}$$

Regularization is necessary to analyze the identities. First, we consider the regularization by dimensional reduction. Under this regularization, the algebra of the Lorentz indices is conducted in four dimensions, but integration is conducted in $4-2\epsilon$ dimensions in the momentum or in the position space. As has been shown in Refs. [14,15], such a regularization is self-consistent at least in $\mathcal{N}=4$ supersymmetric Yang–Mills theory in component formalism at all orders of the perturbation theory. In the next paragraphs, we will show such a regularization procedure can be applied at all orders of the perturbation theory in superfield formalism too if this regularization is combined with Piguet and Sibold formalism of Refs. [32,33]

### 3. From Double-Ghost $Lcc$ Correlator to Other Correlators

Let us analyze this identity in the following way. One can start by considering the monomial $Lcc$ of the effective action. Due to supersymmetry, superficial divergences are absent in chiral vertices [34,35]. This theorem is a direct consequence of the Grassmannian integration and has been described, e.g., in Ref. [34]. However, there could be finite contributions. They remain finite in the limit of removing the ultraviolet regularization. For example, at one loop level, one can find among others the following kernel structure for the correlator $Lcc$ [14]:

$$\int d^4\theta\, d^4x_1 d^4x_2 d^4x_3 \frac{1}{(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^4} f^{bca} \times \\ \times \left(D^2\, L^a(x_1,\theta,\bar{\theta})\right) c^b(x_2,\theta,\bar{\theta}) c^c(x_3,\theta,\bar{\theta}).$$

An efficient form to parameterize this contribution via MB transformation is given in Section 6.

The Landau gauge is a specific case in gauge theories because we do not need to renormalize the gauge fixing parameter. Absence of the gauge parameter is enough condition to avoid this possible source of appearance of the scale dependence in kernels of dressed mean superfields through the renormalization of the gauge parameter. From the effective action $\Gamma$, we can extract the two point ghost proper correlator $G(z-z')$,

$$G^{\dagger} = G,$$

and a two point connected ghost correlator $G^{-1}(z-z')$, which is related to the previous one in the following way:

$$\int d^8z'\, G(z_1-z')\, G^{-1}(z_2-z') = \delta^{(8)}(z_2-z_1).$$

This definition is valid in each order of perturbation theory. One can absorb this two point proper function into a non-local redefinition of the effective fields $K$ and $V$ in the following manner [20]:

$$\tilde{V} \equiv \int d^8z'\, V(z')\, G^{-1}(z-z'), \tag{4}$$
$$\tilde{K} \equiv \int d^8z'\, K(z')\, G(z-z').$$

One can see that the part of the ST identity without the gauge fixing term is covariant with respect to such a re-definition of the effective fields. We will call the construction (4) dressed effective (or mean) superfields. Proceeding at one loop level in terms of the dressed effective superfields, one can see from the ST identity that the divergence of the $\tilde{K}c\tilde{V}$ vertex must be canceled by the divergence of the $Lcc$ vertex. However, the $\mathcal{N}=1$ $Lcc$ vertex does not diverge at one loop level in momentum space due to supersymmetry ( For simplicity, we consider the Landau gauge).This means that the ST identity clearly shows that the $\tilde{K}c\tilde{V}$ is also finite [14,15], that is, it does not diverge in the limit of removing the regularization. The rest of the UV divergence in the propagator of the dressed gauge fields $\tilde{V}\tilde{V}$ can be removed by redefining the gauge coupling constant. In theories with zero beta function, in the case of this paper it is $\mathcal{N}=4$ supersymmetric Yang–Mills theory, this last divergence is absent. The other graphs are finite since proper correlators that can be constructed from the $\tilde{K}c\tilde{V}$ (or $Lcc$), and $\tilde{V}\tilde{V}$ correlators by means of the ST identity. This is a direct consequence of the ST identity and $R$-operation [14,36]. We can repeat this argument in each order of perturbation theory, which is completely the same as in Refs. [14,15] in component formalism.

Why may we copy this proof from the component formalism of Refs. [14,15]? This proof described in the previous paragraph is based on the structure of the gauge part and on the BRST symmetry of the classical action extended by additional BRST-invariant monomials [4]. The BRST symmetry is the origin of the ST identity [4,18,20]. This is why the proof in the previous paragraph may be copied from the nonsupersymmetric case. The only difference is that in the Wess–Zumino gauge the components of the same multiplet are treated separately in this proof. Their renormalization constants do not coincide (this may be seen for example in Chern-Simons theory Ref. [37]); however, it does not present any obstacle. The superficial divergence of the $Lcc$ vertex is absent in both the formalisms (superfield and component) but by the different reasons. In the component formalism it happens due to the transversality of the Landau gauge, which does not allow this double-ghost vertex to diverge superficially [14,15].

Thus, the whole gauge part of the effective action can be considered as the functional of the effective fields $\tilde{V}$, $\tilde{K}$, $L$, $c$, $\bar{c}$, $\tilde{b}$ and $\tilde{\bar{b}}$. Due to the antighost equation [20], the antighost field $b$ is always dressed in the same manner as the auxiliary field $K$ is dressed. Kernels of this effective action are functions of the gauge coupling, mutual distances and in general of the ultraviolet scale, because the divergences in subgraphs must be removed by the renormalization of the gauge coupling. But if the beta function is zero the kernels have no UV scale dependence. In the position space in component formalism the infrared divergences can be analysed in the same way like ultraviolet divergences were analysed in Ref. [36] in momentum space by means of $R$-operation [16]. However, it is difficult to repeat this argument in the position space in superfield formalism since the propagator of the vector superfield is dangerous in the infrared region. In view of this difficulty, we use the formalism developed in Refs. [32,33], where the problem of off-shell infrared divergences of superfield formalism has been solved.

## 4. Off-Shell IR Divergence in Superfield Formalism

The infrared regulator has been introduced by means of the following trick of renormalization of the vector gauge field $V$ :

$$V \to V + \mu\theta^2\bar{\theta}^2 V,$$

where $\mu$ is the infrared regulator mass. Propagators of the lowest components of the gauge superfield obtain a shift by the infrared regulator mass, which is enough to make the Feynman graphs safe in the infrared region of momentum space. It appears one can construct the classical action that satisfies a generalized ST identity which involves BRST counterparts of the gauge parameters. Then, a general solution to the generalized ST identity has been found at the classical level. As a consequence of that solution, the path integral that corresponds to this solution possesses the property of independence of v.e.v.s of gauge invariant quantities on the gauge parameters [33]. The independence of the physical quantities on the infrared scale was obtained in the same way. In addition, the new external superfield $u$ can be introduced so that a shift of its highest component is proportional to the infrared scale $\mu$. That field also participates in the generalized ST identity.

One can analyze the generalized ST identity of Ref. [33], which is obtained after the modification of the standard one (3) by including additional external fields and gauge parameters. The appearance of the additional insertions of the external field $u$ or spurions $\mu\theta^2\bar{\theta}^2$ into supergraphs does not change the nonrenormalization theorems. This property has been used in Refs. [38–40] to derive the relation between softly broken and rigid renormalization constants in $\mathcal{N} = 1$ supersymmetric theory. Thus, it cannot bring any changes to our conclusions about the $Lcc$ vertex from the point of our analysis since all divergent subgraphs remain divergent and all convergence properties of subgraphs remain unchanged. In that sense, subgraphs are finite after renormalization by the dressing functions, but all the vertex as a whole is also finite superficially due to the property of Grassmannian integration, which is not broken by the insertions of the external superfields [38–40].

There is also another way to explain the independence of the physical quantities on the infrared scale $\mu$. The point is that the the factors $(1+\mu\theta^2\bar{\theta}^2)$ coming from vertices will be canceled with factors $(1+\mu\theta^2\bar{\theta}^2)^{-1}$ coming from propagators. However, the propagators are IR-finite with the $\mu$ addition and thus the theory is regularized in the infrared. The same trick can be applied to demonstrate the independence of the correlators of dressed mean superfields in the Landau gauge of the infrared scale $\mu$.

As we have written at the end of the previous section, in the position space in component formalism the infrared divergences can be analyzed in the same way ultraviolet divergences were analyzed in Ref. [36] in momentum space by means of $R$-operation [16,41]. The off-shell IR divergence, which has its origin in the propagator of a vector superfield, is a typical problem for the superfield formalism [32,33]. In the component formalism, this problem does not exist, and this statement may be checked by calculating the index of divergence at infinity in position space in complete analogy with $R$-operation of the overlapping divergences in momentum space [16,41].

## 5. Regularization of UV Divergence

In this paper, we have significantly used the vanishing of the gauge beta function in $\mathcal{N}=4$ super-Yang–Mills theory. The vanishing of the beta function in first three orders of the perturbation theory has been established in Refs. [42–44]. Originally, in the background field technique, it has been shown in Ref. [45] that in $\mathcal{N}=2$ supersymmetric Yang–Mills theory the beta function vanishes beyond one loop. The same result has been derived in Ref. [46] by using the background field technique with unconstrained $\mathcal{N}=2$ superfields. The arguments of [45,46] are based on the fact that $\mathcal{N}=2$ supersymmetry prohibits any counterterms to the gauge coupling except for one loop contribution. In Ref. [35], the fact that $\mathcal{N}=2$ supersymmetric YM theory does not have contributions to the beta function beyond one loop has been argued based on currents of R-symmetry, which are in the same supermultiplet with the energy-momentum tensor. The proportionality of the trace anomaly of the energy momentum tensor to the beta function in general nonsupersymmetric Yang–Mills theories has been proved in Ref. [47]. Since R-symmetry does not have an anomaly in $\mathcal{N}=4$ theory, the same is true for the anomaly of the conformal symmetry, which is proportional to the beta function. At one loop level, the beta function is zero with this field contents [42,43]. Moreover, explicit calculation has been conducted at two loops in terms of $\mathcal{N}=1$ superfields [48], and it has been shown that the beta function of $\mathcal{N}=2$ theory is zero at two loops.

The dimensional reduction was known to be inconsistent [49]. We proposed a solution to this problem in component formalism in Ref. [15] for $\mathcal{N}=4$ super-Yang–Mills theory. We can repeat similar arguments in the case of superfield formalism. The only new feature here is the appearance of the infrared scale $\mu$ in subgraphs, as it has been explained above. The point is that the vertex $Lcc$ is always convergent superficially in superfield formalism. In $\mathcal{N}=4$ super-Yang-Mills theory ultraviolet divergences in the subgraphs of the $Lcc$ vertex should cancel each other at the end. The insertion of the operator of the conformal anomaly into vacuum expectation values of operators of gluonic fields at different points in spacetime is proportional to the beta function of the gauge coupling [47]. Due to the algebra of the four-dimensional supersymmetry, the beta function should be zero [17]. Algebra of the supersymmetry operators in the Hilbert space created by dressed fields can be considered as four-dimensional in Lorentz indices as well as in spinor indices since the limit $\epsilon\to 0$ is non-singular at one-loop order, two-loop order and higher orders as we have seen in the previous paragraphs. Thus, we can consider each correlator as purely four-dimensional, solving order-by-order the problem of dimensional discrepancy of convolutions in Lorentz and spinor indices. The dependence on the infrared scale $\mu$ is canceled by itself from the contributions of vertices and propagators, as it has been shown above.

Another regularization scheme for $\mathcal{N} = 1$ supersymmetric Yang–Mills theory exists, which is the higher derivatives regularization scheme [10,11]. According to Ref. [11], $\mathcal{N} = 2$ supersymmetry can be maintained by the regulator piece of the Lagrangian in HDR. The scheme is discussed in detail in Ref. [4] for the nonsupersymmetric case. A direct supersymmetric generalization of the regularization by higher derivatives has been constructed in Ref. [10]. This generalization has also been considered in detail in the paper [11], in particular in the background field technique. As has been explicitly shown in Refs. [10,11], at one loop level HDR regularizes all the supergraphs in a gauge invariant manner and this repeats the corresponding construction in the nonsupersymmetric version of HDR [4]. However, when applied to explicit examples, this approach is known to yield incorrect results in the Landau gauge [50]. A number of proposals have been put forward to treat this problem [51,52]. As was shown in Ref. [52], the contradiction, noticed in [50], is related to the singular character of the Landau gauge. In all other covariant gauges, the method works and to also include the Landau gauge one has to add one more Pauli–Villars field to obtain the correct result [51,52]. Having used this regularization, a new scheme has been proposed in Refs. [53–55]. Calculations in the higher derivative regularization in terms of superfields can be found in Refs.[53–58]. To perform the calculation, it is proposed to break the gauge symmetry by using some HDR scheme with the usual derivatives instead of covariant derivatives and then to restore the ST identity for the effective action by using some non-invariant counterterms. This problem has been solved in Refs. [53–55]. All the arguments, given above for the regularization by the dimensional reduction in favor of finiteness of the kernels of dressed mean superfields are valid also for the regularization by higher derivatives. The only difference is that to remove the regularization by higher derivatives we take the limit $\Lambda \to \infty$ instead of $\epsilon \to 0$ as it was for the case of dimensional reduction, where $\Lambda$ is the regularization scale of HDR.

As we have already mentioned, the regularization by higher derivatives exists for $\mathcal{N} = 2$ superfield formalism, $\mathcal{N} = 1$ superfield formalism, or for the components of the supermultiplets in the Wess–Zumino gauge. In all these cases, HDR may keep $\mathcal{N} = 4$ supersymmetry. This happens because the $\mathcal{N} = 2$ SUSY regulators of [11] may be rewritten in terms of $\mathcal{N} = 1$ superfields and further in components in the Wess–Zumino gauge. Ward identities related with $\mathcal{N} = 4$ supersymmetry (in particular with the multiplet of anomalies considered in the first paragraph of this section) are valid in each of these formalisms under the higher derivative regularization. The vanishing of the gauge beta function may be proved in each of these cases based on this connection between the trace anomaly and R-symmetry for $\mathcal{N} = 4$ SYM. Although the renormalization constants do not coincide for the different members of the same multiplet in the Wess–Zumino gauge [44], all the Ward identities are valid in both the regularizations (DRED and HDR). After removing the regularizations, the proper correlators of the dressed mean superfields become exactly four-dimensional in terms of the dressed components or in terms of the dressed $\mathcal{N} = 1$ superfields.

## 6. Three-Point Green Function of Dressed Mean Fields

Conformal symmetry imposes strong restrictions on the Green functions [59]. It is true for any conformal field theory in any dimensions. Fradkin and co-authors have constructed conformal invariant QED [59,60]. It was expected that QED in the infrared limit possesses the conformal invariance because the coupling becomes fixed in the IR limit due to the renormalization group equations. The renormalization procedure in the conformal QED has been analyzed in [59,60]. Conformal invariance would be expected in QCD in the UV limit too due to the renormalization group equations and asymptotic freedom. Thus, the conformal invariance has a sense in QCD in the UV limit and in QED in the IR limit. The three-point vector Green functions may be fixed in both the cases up to coefficients, the four-point Green functions are restricted too, but not so strongly. There is a lot of freedom remaining in the Green functions of higher ranges. In Ref. [60], in the framework of the conformal QED, the connected Green functions of the gauge vector field and spinor fields are constrained in position space by the conformal symmetry.

We may apply these arguments of the Ref. [60] to the three-point connected Green function of the dressed mean vector superfields in order to find its form in position space up to coefficients. Because the effective action of the dressed mean fields does not depend on any scale, conformal transformations of the fields of the classical action at the level of the effective action are converted to the corresponding conformal transformations of the dressed mean fields. The generators of the conformal group will act on the dressed mean fields of the effective action. The trick is very similar to that of how the ST identity imposed on the effective action has been obtained from the BRST symmetry of the classical action by means of the Legendre transformations [4,18], or an even simpler example may be given, the invariance with respect to Lorentz symmetry may be obtained for the effective action via the Legendre transformation of the connected diagram generator. However, the conformal symmetry of the classical action restricts the proper vertices in an implicit way, but the constraints imposed by this symmetry on the connected Green functions are explicit.

In order to use the constraints from the conformal field theory, the fields or superfields should be dynamical, otherwise we cannot transform the conformal symmetry of the classical action to the constraints on the vacuum expectation values of the fields (on the connected Green functions). By the same trick, the Green functions of the softly broken and rigid supersymmetric theories were related in [61]. This is the case of the dressed mean superfields from the Lagrangian of $\mathcal{N} = 4$ SYM, they are dynamical. We may apply this trick to the connected correlators, but not to a proper vertex. This is why we cannot fix the $Lcc$ vertex based on the conformal group transformations, the $L$ field is not dynamical, it is an external auxiliary source coupled to the composite double-ghost operator and not to the dynamical field from the integration measure of the path integral.

Thus, the conformal invariance imposes constraints on the connected Green function of the dressed mean fields for all the orders in the gauge coupling. Nothing except for transformation laws of fields or operators with respect to the conformal group is necessary [59,60] to establish these restrictions. The three-point connected Green function is fixed up to coefficients. The coefficient is a series in terms of the gauge coupling. Can it be checked explicitly by the direct calculations in position space? To calculate the connected Green function in position space at the tree level, it is necessary to integrate three propagators over coordinates of the common vertex. As we have mentioned in [62], for QED it has been conducted in Refs. [63–65] in which three classical propagators were integrated in the common vertex. The result is a structure predicted by the conformal group [63–65]. In QCD, the calculation at the tree level of the connected three propagators over the common vertex has been conducted in Ref. [41]. The result also satisfies the form dictated by the conformal group. It is known that the structure of the integral of three propagators over the common vertex should contain the Davydychev integral $J(1,1,1)$ in position space [16]. The conformal symmetry of the theory is the only reason that prohibits its appearance in the result for this integral of three gluon propagators in the Landau gauge in position space after summing all the contributions [41].

In the previous paragraph of this section, we wrote about the calculations in the component formalism at the tree level in QED and in $\mathcal{N} = 4$ SYM for the connected Green function of three dressed gluons. The results are consistent with the constraints imposed by the conformal group. This conformal structure of the connected Green function should survive at the higher loops, however nobody has checked this by explicit calculations. In the literature there is no result (to our knowledge) about the calculation of the three point off-shell gluon function at two loops in QCD or $\mathcal{N} = 4$ SYM for a proper vertex or a connected Green correlator. In Refs. [16,62], we wrote that the three-point connected Green functions of dressed gluons may be found explicitly at all the loops due to conformal symmetry in the Landau gauge in $\mathcal{N} = 4$ SYM. We also made a conjecture in [15] that all the gauge dependence in other gauges should be in the dressing functions of the dressed mean fields in this theory. Calculation for the $Lcc$ vertex has been conducted explicitly at the two loop level in the component formalism [16,41,66].

Now, the question is why does the structure of the $Lcc$ vertex look so random [16,41,66] while the structure of the connected three-gluon Green function of the dressed mean gluons is so simple and conformal? The $Lcc$ vertex is related by the ST identity to the proper (one particle irreducible) three-gluon vertex diagram which has highly nontrivial structure and contains non-conformal $J(1,1,1)$ contributions [67] already at the one-loop level in momentum space for QCD. By construction of the dressed mean field correlators, this contribution should be summed with the one-loop self energy contribution to the propagators to obtain the complete three-point connected Green function of the dressed gluons in the component formalism. As the result, these non-conformal $J(1,1,1)$ contributions should vanish in this total one-loop result for this Green function in momentum space for $\mathcal{N}=4$ SYM. This cancellation may be possible because $J(1,1,1)$ is a highly non-trivial combination of the Appell functions and its derivatives that resulted in a combination of Euler polylogarithms [68]. In momentum space, this conformal Green function consists of a product of powers of the momenta of the propagator legs only. Thus, the conformal structure is the only structure that survives in the connected three-point Green function of the dressed gluons at the one loop level in position space and in momentum space. This structure does not contain non-conformal $J(1,1,1)$ contributions. Again, the only reason for cancellation of these non-conformal $J(1,1,1)$ contributions at the one-loop level is the constraints imposed by the conformal symmetry in position space on this connected correlator. It is consistent with the results of Refs.[62,69–71] where it has been proven that three-point Green functions in the massless theories are invariant with respect to Fourier transformation.

In the review [59], it is described how to use conformal invariance in order to find restrictions on the correlation functions of gauge invariant operators in a conformal field theory for all the loops. It appears to be a useful tool for work with operator product expansion in a conformal field theory. Later, this tool has been applied to AdS-CFT correspondence [72,73]. However, in this paper we consider the connected Green function of the dressed mean superfields in the Landau gauge. The one-loop structure for this $Lcc$ vertex in the component formalism is simple and coincides for any YM theory. It can be found in Refs. [14–16] .

The representation in terms of component fields is

$$\int d^4x_1 d^4x_2 d^4x_3 \frac{ig^2N}{2^8\pi^6} f^{abc} L^a(x_1) c^b(x_2) c^c(x_3)\, V^{(1)}(x_1,x_2,x_3),$$

where

$$V^{(1)}(x_1,x_2,x_3) = \frac{-1}{[12]^2[23]^2} + \frac{2}{[12]^2[31]^2} + \frac{-1}{[23]^2[31]^2} + \frac{-1}{[12][23][31]^2} + \frac{2}{[12][23]^2[31]} + \frac{-1}{[12]^2[23][31]}. \tag{5}$$

We assume the concise notation of Ref. [16], where $[Ny]=(x_N-y)^2$ and analogously for $[Nz]$, and $[yz]=(y-z)^2$, that is, $N=1,2,3$ stands for $x_N=x_1,x_2,x_3$, respectively, which are the external points of the triangle diagram in Figure 1 in position space. As we have mentioned in Ref. [16], this representation (5) does not match the representation for the connected scalar three-point function imposed by the conformal symmetry.

Moreover, the one-loop expression (5) in position space is not written in a form convenient to solve the ST identity. We need a different representation to apply it in the ST identity. For this purpose, we first consider a scalar triangle diagram of Ref. [74] and note that any term in the one-loop contribution in $Lcc$ vertex (5) may be considered as such a scalar triangle diagram depicted in Figure 2 in position space with arbitrary indices $\alpha_1,\alpha_2,\alpha_3$ on the propagators.

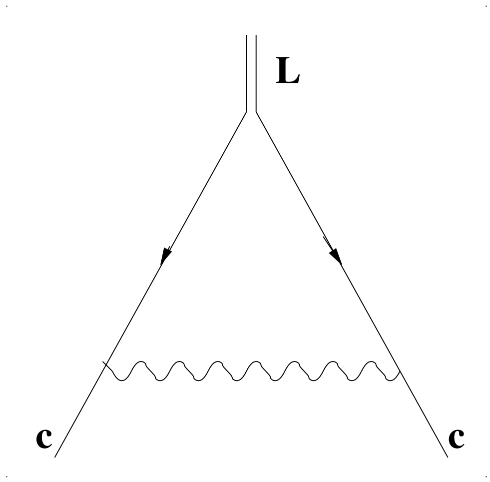


**Figure 1.** The only one-loop contribution to the $Lcc$ vertex.

Let us first analyze the diagram in Figure 2 in momentum space. The $d$-dimensional momenta $p_1$, $p_2$, $p_3$ enter this diagram in Figure 3 and are related by momentum conservation

$$p_1 + p_2 + p_3 = 0.$$

In momentum space, the diagram in Figure 3 corresponds to the integral

$$J(\nu_1, \nu_2, \nu_3) = \int Dk \, \frac{1}{[(k+q_1)^2]^{\nu_1}[(k+q_2)^2]^{\nu_2}[(k+q_3)^2]^{\nu_3}}. \quad (6)$$

The running momentum $k$ in the triangle diagram in Figure 3 is the integration variable in the integral (6). The notation is chosen in such a way that the index of propagator $\nu_1$ stands on the line opposite to the vertex of triangle into which the momentum $p_1$ enters. The notation $q_1$, $q_2$ and $q_3$ are taken from Ref. [75]. As it may be seen from the diagram in Figure 3,

$$p_1 = q_3 - q_2, \quad p_2 = q_1 - q_3, \quad p_3 = q_2 - q_1.$$

The integral measure in momentum space is defined as in [16] by

$$Dk \equiv \pi^{-\frac{d}{2}} d^d k.$$

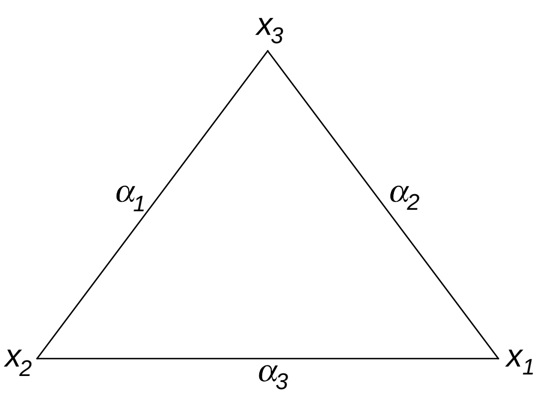


**Figure 2.** One-loop massless scalar triangle in position space. Integration in position space should be carried out over internal points only (See Ref. [76]); however, this diagram does not have internal points at all. The points $x_1$, $x_2$, $x_3$ are external points.

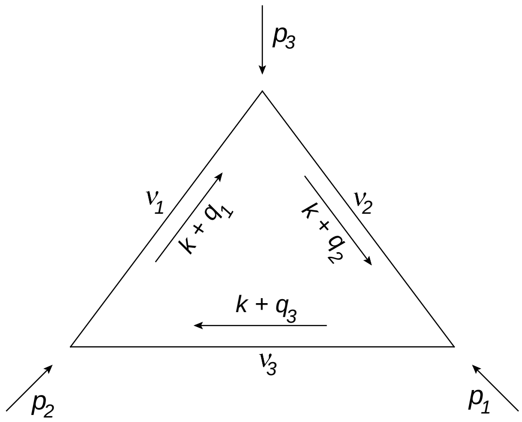


**Figure 3.** One-loop massless scalar triangle in momentum space.

Such a definition of the integration measure in momentum space helps to avoid powers of $\pi$ in formulas for the momentum integrals. The Mellin–Barnes representation of the integral $J(\nu_1,\nu_2,\nu_3)$ may be obtained via Feynman parameters [75] and has the form

$$J(\nu_1,\nu_2,\nu_3) = \frac{1}{\Pi_i\Gamma(\nu_i)\Gamma(d-\Sigma_i\nu_i)}\frac{1}{(p_3^2)^{\Sigma\nu_i - d/2}}\oint_C dz_2\,dz_3\left(\frac{p_1^2}{p_3^2}\right)^{z_2}\left(\frac{p_2^2}{p_3^2}\right)^{z_3}\{\Gamma(-z_2)\Gamma(-z_3)$$
$$\Gamma(-z_2-\nu_2-\nu_3+d/2)\Gamma(-z_3-\nu_1-\nu_3+d/2)\Gamma(z_2+z_3+\nu_3)\Gamma(\Sigma\nu_i-d/2+z_3+z_2)\}$$
$$\equiv \frac{1}{(p_3^2)^{\Sigma\nu_i-d/2}}\oint_C dz_2\,dz_3 x^{z_2}y^{z_3}D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3].$$

We have used here the definition of the Mellin–Barnes transform $D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3]$ from our Ref. [76],

$$D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3] = \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2-\nu_2-\nu_3+d/2)\Gamma(-z_3-\nu_1-\nu_3+d/2)}{\Pi_i\Gamma(\nu_i)}$$
$$\times\frac{\Gamma(z_2+z_3+\nu_3)\Gamma(\Sigma\nu_i-d/2+z_3+z_2)}{\Gamma(d-\Sigma_i\nu_i)}.$$

With all these definitions made, we consider the triangle scalar diagram in Figure 2 in position space, repeating in part the integral transformations used in Ref. [74].

$$\frac{1}{[12]^{\alpha_3}[23]^{\alpha_1}[31]^{\alpha_2}} = \frac{\pi^{-3d/2}4^{-\Sigma_i\alpha_i}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\times$$
$$\int dq_1dq_2dq_3\frac{e^{iq_3(x_1-x_2)}e^{iq_1(x_2-x_3)}e^{iq_2(x_3-x_1)}}{(q_3^2)^{d/2-\alpha_3}(q_1^2)^{d/2-\alpha_1}(q_2^2)^{d/2-\alpha_2}}$$
$$= \frac{\pi^{-3d/2}4^{-\Sigma_i\alpha_i}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int dq_1dq_2dq_3\frac{e^{ix_1(q_3-q_2)}e^{ix_2(q_1-q_3)}e^{ix_3(q_2-q_1)}}{(q_3^2)^{d/2-\alpha_3}(q_1^2)^{d/2-\alpha_1}(q_2^2)^{d/2-\alpha_2}}$$
$$= \frac{\pi^{-3d/2}4^{-\Sigma_i\alpha_i}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\times$$

$$
\begin{aligned}
&\int dp_1 dp_2 dp_3 dq_3 \delta(p_1+p_2+p_3) \frac{e^{ix_1p_1}e^{ix_2p_2}e^{ix_3p_3}}{[q_3^2]^{d/2-\alpha_3}[(p_2+q_3)^2]^{d/2-\alpha_1}[(p_1-q_3)^2]^{d/2-\alpha_2}} \\
&= \frac{\pi^{-3d/2}4^{-\Sigma_i\alpha_i}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{(2\pi)^d\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \\
&\int dp_1 dp_2 dp_3 dq_3 dx_5 \frac{e^{i(x_1-x_5)p_1}e^{i(x_2-x_5)p_2}e^{i(x_3-x_5)p_3}}{[q_3^2]^{d/2-\alpha_3}[(p_2+q_3)^2]^{d/2-\alpha_1}[(p_1-q_3)^2]^{d/2-\alpha_2}} \\
&= \frac{\pi^{-d}4^{-\Sigma_i\alpha_i}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{(2\pi)^d\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \\
&\int dp_1 dp_2 dp_3 dx_5 e^{i(x_1-x_5)p_1}e^{i(x_2-x_5)p_2}e^{i(x_3-x_5)p_3} J(d/2-\alpha_1, d/2-\alpha_2, d/2-\alpha_3) \\
&= \frac{\pi^{-d}4^{-\Sigma_i\alpha_i}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{(2\pi)^d\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \\
&\int dp_1 dp_2 dp_3 dx_5 \oint_C dz_2 dz_3 \frac{e^{i(x_1-x_5)p_1}e^{i(x_2-x_5)p_2}e^{i(x_3-x_5)p_3}}{(p_3^2)^{d-\Sigma_i\alpha_i+z_2+z_3}(p_1^2)^{-z_2}(p_2^2)^{-z_3}} D^{(z_2,z_3)}[d/2-\alpha_1, d/2-\alpha_2, d/2-\alpha_3] \\
&= \frac{\pi^{d/2}4^{d/2}\Gamma(d/2-\alpha_1)\Gamma(d/2-\alpha_2)\Gamma(d/2-\alpha_3)}{(2\pi)^d\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \times \\
&\int dx_5 \oint_C dz_2 dz_3 \frac{D^{(z_2,z_3)}[d/2-\alpha_1, d/2-\alpha_2, d/2-\alpha_3]}{[35]^{\Sigma_i\alpha_i-z_2-z_3-d/2}[15]^{d/2+z_2}[25]^{d/2+z_3}} \times \\
&\frac{\Gamma(\Sigma_i\alpha_i-z_2-z_3-d/2)\Gamma(d/2+z_2)\Gamma(d/2+z_3)}{\Gamma(d-\Sigma_i\alpha_i+z_2+z_3)\Gamma(-z_2)\Gamma(-z_3)}.
\end{aligned}
$$

Any term in (5) may be represented in this form, and for the all one-loop $Lcc$ vertex in the component formalism, we may write

$$
\begin{aligned}
&\int d^4x_1 d^4x_2 d^4x_3 \frac{ig^2N}{2^8\pi^6} f^{abc} L^a(x_1)c^b(x_2)c^c(x_3)\, V^{(1)}(x_1,x_2,x_3) \\
&= \int d^4x_1 d^4x_2 d^4x_3 \frac{ig^2N}{2^8\pi^6} f^{abc} L^a(x_1)c^b(x_2)c^c(x_3) \int dx_5 \oint_C dz_2 dz_3 \frac{\mathcal{M}(z_2,z_3)}{[15]^{\Delta-z_2-z_3}[25]^{z_2}[35]^{z_3}}.
\end{aligned} \tag{7}
$$

Here, $\Delta$ is the dimension of this $Lcc$ vertex. In the case that we consider ($\mathcal{N}=4$ SYM in $d=4$) $\Delta=6$. The Equation (7) is the parametrization for the $Lcc$ vertex we need to solve the ST identity by the method developed in Refs. [18,19]. This parametrization is valid for any number of loops. This auxiliary vertex is related to the three-gluon proper vertex via ST identity in the way we described in Section 3. Thus, in the component formalism the conformal structure of the connected Green function of three dressed gluons transforms implicitly to the structure of the $Lcc$ vertex because the integral relation between the connected Green function of dressed gluons and the corresponding proper function of dressed gluons is highly nontrivial in all the loops. In turn, the Bethe–Salpeter equation puts strong restrictions on the Mellin–Barnes image $\mathcal{M}(z_2,z_3)$ of the $Lcc$ vertex in the parametrization (7) in an explicit way [62,76].

The $Lcc$ is finite in the limit of removing the regularizations as we have shown in Refs. [14,15] and checked explicitly in Refs. [16,41,66] in the component formalism. We have written in the Introduction that scale independence is a consequence of finiteness. Now, we do not have any scale because the amplitudes are on-shell values of the connected Green functions of dressed fields and therefore IR divergences after putting the moments on-shell that may appear. In order to work with these connected correlators on shell, we need a regularization again. This may be completely new regularization, for example, we may shift the complex variables of integration in the Mellin planes by some value $\epsilon$ and obtain

$$\int d^4x_1 d^4x_2 d^4x_3 \frac{ig^2N}{2^8\pi^6} f^{abc} L^a(x_1) c^b(x_2) c^c(x_3) \int dx_5 \oint_C dz_2 dz_3 \frac{\mathcal{M}(z_2+\epsilon, z_3+\epsilon)}{[15]^{\Delta-z_2-z_3-2\epsilon}[25]^{z_2+\epsilon}[35]^{z_3+\epsilon}}$$

instead of (7). Such a regularization will regularize the IR divergences on-shell. We do not need any regularization of such a kind at the tree level. In terms of this vertex, all other vertices may be fixed [18,19,62,76,77].

## 7. Chiral Superfields

So far the pure gauge sector has been considered. Let us look now at the matter two point functions. Schematically, one can write the two point vertex as

$$\int d^8z d^8z' \bar{\Phi}(z) G_\Phi(z-z') \Phi(z').$$

The function $G_\Phi(z-z')$ can be divided into two equal parts $\tilde{G}_\Phi$,

$$\int d^8z' \tilde{G}_\Phi(z_1-z') \tilde{G}_\Phi(z'-z_2) = G_\Phi(z_1-z_2).$$

This is a product in momentum space. Now we define the new fields $\tilde{\Phi}$,

$$\int d^8z' \tilde{G}_\Phi(z-z') \Phi(z') \equiv \tilde{\Phi}(z),$$

and represent the effective action in terms of these fields. In particular, the divergent part of the function $\tilde{G}_\Phi$ can be absorbed into the redefinition of the Yukawa couplings and masses. However, in $\mathcal{N}=4$ supersymmetric theory, masses are absent and the Yukawa coupling (which coincides with the gauge coupling) is not renormalized due to the structure of the Yukawa terms in the classical action. Thus, in terms of the dressed effective superfields $\tilde{\Phi}$ and $\tilde{V}$, the effective action does not have any dependence on the UV and IR scales for $\mathcal{N}=4$ supersymmetric theory.

Because the superficial divergence in the Yukawa coupling for $\mathcal{N}=1$ supersymmetry is absent due to Grassmannian integration in the superspace [35], all the renormalization from the self-energy of the chiral superfields should be absorbed by the renormalization of the Yukawa coupling. But in this special case of $\mathcal{N}=4$, SYM the gauge coupling and Yukawa coupling is the same and gauge coupling does not receive any renormalization in this theory due to the vanishing of the gauge beta function, which means that the chiral superfields in this model should not be renormalized in order to absorb infinities, that is, the self-energy of the chiral superfields may be finite only. On the contrary, in the component formalism self-energies of components of the chiral multiplet are divergent [29].

Absence of the renormalization of the self-energy for the chiral superfields is not the only example when the bubble diagram does not contribute. Another example would be the massless bubble diagrams with external on-shell momenta of Ref. [78], which are divergent both in UV and IR, but they vanish due to cancellation between UV and IR poles in $d=4-2\epsilon$. It implies the IR and UV divergences are related to each other and can be determined from one another [78].

## 8. Conclusions

In conclusion, all the correlator $Lcc$ with all the possible contributions included turn out to be totally finite in $\mathcal{N}=4$ super-Yang–Mills theory in the Landau gauge, and this property can be used to find it exactly in all the orders of the perturbation theory. The vertex $Lcc$ in spite of being scale independent cannot be found by conformal symmetry since the external auxiliary superfield $L$ does not propagate (it is not in the measure of the path integral). Three point connected Green functions of supermultiplets containing physical fields (like vector supermultiplet or matter supermultiplet) could be fixed by conformal symmetry up to some coefficient depending on the gauge coupling and number of colors.

In QCD, the matter fields are not in adjoint representation, the level of the symmetry is much lower and the beta function does not vanish. The rest of the singularity in the vector propagator can be absorbed into the gauge coupling to organize the bare coupling. The bare coupling together with the logarithm of ratio of the distance to the scale, leads to the running (effective) coupling. It means that in $d$ dimensions, the massless nonsupersymmetric gauge theory is a conformal gauge theory in terms of the running effective coupling (formed from the bare coupling) and dressed mean fields [66]. Another way to break supersymmetry and obtain the results for QCD from supersymmetric theories is via diffeomorphisms in the superspace [61].

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