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Critical Behavior of (2 + 1)-Dimensional QED: 1/N Expansion

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Abstract: We present recent results on dynamical chiral symmetry breaking in (2 + 1)-dimensional QED with N four-component fermions. The results of the $1/N$ expansion in the leading and next-to-leading orders were found exactly in an arbitrary nonlocal gauge.

Keywords: (2 + 1)-dimensional QED; dynamical chiral symmetry breaking; $1/N$ expansion; non-local gauge

1. Introduction

In this conference report, we present in a self-contained way the results of Refs. [1,2], where the critical behavior of Quantum Electrodynamics in $2+1$ dimensions (QED$_3$) have been studied. Contrary to previous reports [3–5], we here follow the Addendum of Ref. [2], which contains a strong upgrade of the exact results of [2] thereby proving the complete gauge-independence of the value of the critical fermion number, N_c, which is such that dynamical chiral symmetry breaking (DχSB) in QED$_3$ takes place only for $N < N_c$. Indeed, following Ref. [2] and after long discussions with Valery Gusynin, the expansion prescription used in Ref. [2] (and reported on in [3,4]) was modified in the Addendum. The expansion was initially based on (an NLO correction to) the gap equation and was modified to (an NLO correction to) the parameter α of its solution (see Equation (6) below). This subtle change in the interpretation of the NLO corrections does not affect at all the LO results of Appelquist et al. [6] but significantly modifies the NLO results (see below Section 4) leading to gauge-invariant N_c values after the so-called Nash resummation (see below for more).

The model is described by the Lagrangian:

$$L = \overline{\Psi}(i\hat{\partial} - e\hat{A})\Psi - \frac{1}{4}F^2_{\mu\nu}\,, \tag{1}$$

where Ψ is taken to be a four component complex spinor. In the case of N fermion flavours, the QED$_3$ has a $U(2N)$ symmetry. The parity-invariant term $m\overline{\Psi}\Psi$ with fermion mass m breaks this symmetry up to $U(N) \times U(N)$. In the massless case, loop expansions suffer from infrared divergences. The latter are softened when analyzing the model in a $1/N$ expansion [7–9]. Since the theory is super-renormalizable, then the mass scale is given by the dimensional coupling constant: $a = Ne^2/8$, which remains fixed as $N \to \infty$. Early studies of this model (see Refs. [6,10]) showed that physics is damped rapidly at the momentum scales $p \gg a$ and that the fermion mass term, which violates the flavour symmetry, is dynamically generated at scales that are orders of magnitude smaller than the internal scale a. Since then, the DχSB in QED$_3$ and the N dependence of the mass of the dynamic fermion have been the subject of extensive research, see, e.g., [1–31].

One of the central problems is related to the critical number N_c of fermions, which is such that DχSB takes place only for $N < N_c$. The exact definition of N_c is crucial for understanding the phase structure of QED$_3$ with far-reaching consequences from particle physics to condensed matter physics systems with relativistic low-energy excitations [32–35]. It turns out that the values that can be found in the literature range from $N_c \to \infty$ [10–15] corresponding to DχSB for all N values, up to $N_c \to 0$ in the case when the DχSB sign is not found [16–18].

Central to our work is the approach of Appelquist et al. [6], which found that $N_c = 32/\pi^2 \approx 3.24$ by solution of the Schwinger-Dyson (SD) gap equation using the $1/N$ expansion in leading order (LO) approximation. Lattice modeling in accordance with a finite nonzero value of N_c can be found in [22–25].

Shortly after [6], Nash approximately included the next-to-leading order (NLO) corrections and managed to partially resum the renormalization constant of the wave function at the level of the gap equation; he found [26]: $N_c \approx 3.28$.

Recently, in [1], NLO corrections could be calculated exactly in the Landau gauge, obtaining $N_c \approx 3.17$ (see Erratum to [1]), i.e., a value that is very close to the value of Nash in [26]. More recently, in Ref. [2] the results of [1] were generalized to an arbitrary nonlocal gauge [36,37]. In addition, Ref. [2] (see also its Appendix) showed that resumming the renormalization of the wave function gives a gauge-independent critical number of fermion flavors, $N_c = 2.8469$, the value of which coincides with the results obtained in [29].

The purpose of this paper is to present the main arguments of the papers [1,2] (and corresponding Addendum and Erratum, respectively) leading to exact DχSB results in arbitrary nonlocal gauge [36,37]. This achievement represents a significant improvement in terms of the approximate Nash NLO results that were made mostly in the Feynman gauge. In this regard, considerable interest is currently being devoted to studying the gauge dependence of several models, see [31,38,39]. The use of the Landau gauge in [1] was motivated by recent results on QED$_3$ [31] that revealed the gauge-independence of N_c upon using the Ball-Chiu vertex [40]. In fact, after resummation of the renormalization constant of the wave function, we find that the LO and NLO terms in the gap equation become gauge-invariant and match the results of [29].

2. SD Equations

With the conventions of Ref. [1], the inverse fermion propagator has the following form: $S^{-1}(p) = [1 + A(p)]\,(i\hat{p} + \Sigma(p))$ where $A(p)$ is the fermion wave function and $\Sigma(p)$ is a dynamically generated parity-preserving mass, which is assumed to be the same for all fermions. The SD equation for the fermion propagator can be decomposed into scalar and vector components as follows:

$$\tilde{\Sigma}(p) = \frac{2a}{N}\,\mathrm{Tr}\int \frac{d^3k}{(2\pi)^3}\,\frac{\gamma^\mu D_{\mu\nu}(p-k)\Sigma(k)\Gamma^\nu(p,k)}{[1+A(k)]\,(k^2+\Sigma^2(k))}\,, \tag{2a}$$

$$A(p)p^2 = -\frac{2a}{N}\,\mathrm{Tr}\int \frac{d^3k}{(2\pi)^3}\,\frac{D_{\mu\nu}(p-k)\hat{p}\gamma^\mu\hat{k}\Gamma^\nu(p,k)}{[1+A(k)]\,(k^2+\Sigma^2(k))}\,, \tag{2b}$$

where $\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)]$, $D_{\mu\nu}(p)$ is the photon propagator in the non-local ξ-gauge:

$$D_{\mu\nu}(p) = \frac{P^\xi_{\mu\nu}(p)}{p^2\,[1+\Pi(p)]}, \quad P^\xi_{\mu\nu}(p) = g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\,, \tag{3}$$

$\Pi(p)$ is the polarization operator and $\Gamma^\nu(p,k)$ is the vertex function. We shall study Equations (2) for arbitrary values of the gauge-fixing parameter ξ. All calculations will be performed using standard perturbation theory rules for massless Feynman diagrams, as in [41,42], see also recent reviews [43,44]. For the most complex diagrams, the Gegenbauer polynomial technique will be used following [45].

3. LO

The $1/N$ expansion at the LO accuracy amounts to the following substitutions: $A(p) = 0$, $\Pi(p) = a/|p|$ and $\Gamma^\nu(p,k) = \gamma^\nu$, where the fermion mass was neglected in the calculation of $\Pi(p)$. The LO diagram contributing to the gap Equation (2a), see Figure 1, reads:

$$\Sigma(p) = \frac{8(2+\xi)a}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{(k^2+\Sigma^2(k))\,[(p-k)^2 + a\,|p-k|]} \,. \tag{4}$$

Following [6], we consider the limit of large a and linearize Equation (4) which gives

$$\Sigma(p) = \frac{8(2+\xi)}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2\,|p-k|} \,. \tag{5}$$

The mass function can be parameterized as [6]:

$$\Sigma(k) = B\,(k^2)^{-\alpha}\,, \tag{6}$$

where B is arbitrary and the index α must be found in a self-consistent way. Using this ansatz, Equation (5) reads:

$$\Sigma^{(\mathrm{LO})}(p) = \frac{4(2+\xi)B}{N} \frac{(p^2)^{-\alpha}}{(4\pi)^{3/2}} \frac{2\beta}{\pi^{1/2}}\,, \tag{7}$$

from which the LO gap equation is obtained:

$$1 = \frac{(2+\xi)\beta}{L} + O(L^{-2}), \quad \text{or} \quad \beta^{-1} = \frac{(2+\xi)}{L} + O(L^{-2})\,, \tag{8}$$

where

$$\beta = \frac{1}{\alpha\,(1/2-\alpha)} \quad \text{and} \quad L \equiv \pi^2 N\,. \tag{9}$$

Note that the two equations in (8) are completely equal to each other. Solving them, we obtain:

$$\alpha_\pm = \frac{1}{4}\left(1 \pm \sqrt{1 - \frac{16(2+\xi)}{L}}\right)\,, \tag{10}$$

which reproduces the solution given by Appelquist et al. [6]. The gauge-dependent critical number of fermions: $N_c \equiv N_c(\xi) = 16(2+\xi)/\pi^2$, such that $\Sigma(p) = 0$ for $N > N_c$ and $\Sigma(0) \simeq \exp\left[-2\pi/(N_c/N-1)^{1/2}\right]$, for $N < N_c$. Thus, DχSB arises when α becomes complex, that is, for $N < N_c$.

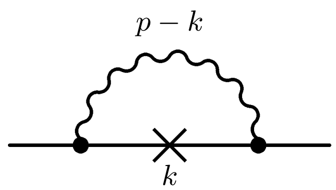

Figure 1. LO diagram to dynamically generated mass $\Sigma(p)$. The crossed line indicates a mass insert.

The gauge-dependent fermion wave function can be computed in a similar way. At LO, Equation (2b) simplifies as:

$$A(p)p^2 = -\frac{2a}{N}\mathrm{Tr} \int \frac{d^Dk}{(2\pi)^D} \frac{P^\xi_{\mu\nu}(p-k)\hat{p}\gamma^\mu\hat{k}\gamma^\nu}{k^2|p-k|}\,, \tag{11}$$

where the integral is dimensionally regularized with $D = 3 - 2\varepsilon$. Taking the trace and calculating the integral on the r.h.s. outputs:

$$A(p) = \frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\, C_1(\xi) + \mathrm{O}(\varepsilon)\,, \quad C_1(\xi) = +\frac{2}{3\pi^2 N}\left((2-3\xi)\left[\frac{1}{\varepsilon} - 2\ln 2\right] + \frac{14}{3} - 6\xi\right), \tag{12}$$

where the $\overline{MS}$ parameter $\overline{\mu}$ has the standard form $\overline{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ using the Euler constant γ_E. Note that in the $\xi = 2/3$-gauge, the value of $A(p)$ is finite and $C_1(\xi = 2/3) = +4/(9\pi^2 N)$. From Equation (12), the LO wave-function renormalization constant can be extracted: $\lambda_A = \mu(d/d\mu)A(p) = 4(2 - 3\xi)/(3\pi^2 N)$, that matches the result of of [46,47].

4. NLO

Now consider the NLO contributions that can be parameterized as:

$$\Sigma^{(\mathrm{NLO})}(p) = \left(\frac{8}{N}\right)^2 B\, \frac{(p^2)^{-\alpha}}{(4\pi)^3}\, (\Sigma_A + \Sigma_1 + 2\,\Sigma_2 + \Sigma_3)\,, \tag{13}$$

where each contribution to the linearized gap equation is represented graphically in Figure 2. When these contributions are added to the LO result, Equation (7), the gap equation has the following general form:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{\overline{\Sigma}_A(\xi) + \overline{\Sigma}_1(\xi) + 2\,\overline{\Sigma}_2(\xi) + \overline{\Sigma}_3(\xi)}{L^2}\,, \tag{14}$$

where $\overline{\Sigma}_i = \pi\Sigma_i$, $(i = 1, 2, 3.A)$.

Contribution Σ_A, see (A) in Figure 2, comes from the LO value of $A(p)$ and is singular. Using dimensional regularization for an arbitrary parameter ξ, it takes the form:

$$\overline{\Sigma}_A(\xi) = 4\,\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,\beta\left[\left(\frac{4}{3}(1-\xi) - \xi^2\right)\left[\frac{1}{\varepsilon} + \Psi_1 - \frac{\beta}{4}\right] + \left(\frac{16}{9} - \frac{4}{9}\xi - 2\xi^2\right)\right], \tag{15}$$

where

$$\Psi_1 = \Psi(\alpha) + \Psi(1/2-\alpha) - 2\Psi(1) + \frac{3}{1/2-\alpha} - 2\ln 2\,, \tag{16}$$

and Ψ is the digamma function.

The contribution of diagram (1) in Figure 2 is finite (the shaded blob contains the diagrams shown in Figure 3) and reads:

$$\overline{\Sigma}_1(\xi) = -2(2+\xi)\,\beta\,\hat{\Pi}, \qquad \hat{\Pi} = \frac{92}{9} - \pi^2\,, \tag{17}$$

where the gauge dependence comes from the fact that we are working in a nonlocal gauge, and $\hat{\Pi}$ arises from the two-loop polarization operator in the dimension $D = 3$ [27,28,48–51].

The contribution of diagram (2) in Figure 2 is again singular. Dimensionally regularizing it gives:

$$\begin{aligned}
\overline{\Sigma}_2(\xi) &= -2\,\frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}}\,\beta\left[\frac{(2+\xi)(2-3\xi)}{3}\left(\frac{1}{\varepsilon} + \Psi_1 - \frac{\beta}{4}\right) + \frac{\beta}{4}\left(\frac{14}{3}(1-\xi) + \xi^2\right)\right.\\
&\quad\left. + \frac{28}{9} + \frac{8}{9}\xi - 4\xi^2\right] + (1-\xi)\,\hat{\Sigma}_2\,,\\
\hat{\Sigma}_2(\alpha) &= (4\alpha-1)\beta\Big[\Psi'(\alpha) - \Psi'(1/2-\alpha)\Big] + \frac{\pi}{2\alpha}\,\tilde{I}_1(\alpha) + \frac{\pi}{2(1/2-\alpha)}\tilde{I}_1(\alpha+1)\,,
\end{aligned} \tag{18}$$

where Ψ' is the trigamma function and $\tilde{I}_1(\alpha)$ is a dimensionless integral that was defined in [1].

The singularities in $\overline{\Sigma}_A(\xi)$ and $\overline{\Sigma}_2(\xi)$ cancel each other, so their sum is finite. Defining: $\overline{\Sigma}_{2A}(\xi) = \overline{\Sigma}_A(\xi) + 2\overline{\Sigma}_2(\xi)$, the latter reads:

$$\overline{\Sigma}_{2A}(\xi) = 2(1-\xi)\hat{\Sigma}_2(\alpha) - \left(\frac{14}{3}(1-\xi) + \xi^2\right)\beta^2 - 8\beta\left(\frac{2}{3}(1+\xi) - \xi^2\right). \tag{19}$$

Finally, the contribution of diagram (3) in Figure 2 is finite and reads:

$$\begin{aligned}\overline{\Sigma}_3(\xi) &= \hat{\Sigma}_3(\alpha,\xi) + \left(3+4\xi-2\xi^2\right)\beta^2, \ \hat{\Sigma}_3(\alpha,\xi) = \frac{1}{4}\left(1+8\xi+\xi^2+2\alpha(1-\xi^2)\right)\pi\tilde{I}_2(\alpha) \\ &+ \frac{1}{2}\left(1+4\xi-\alpha(1-\xi^2)\right)\pi\tilde{I}_2(1+\alpha) + \frac{1}{4}\left(-7-16\xi+3\xi^2\right)\pi\tilde{I}_3(\alpha),\end{aligned} \tag{20}$$

where the dimensionless integrals $\tilde{I}_2(\alpha)$ and $\tilde{I}_3(\alpha)$ were defined in [1].

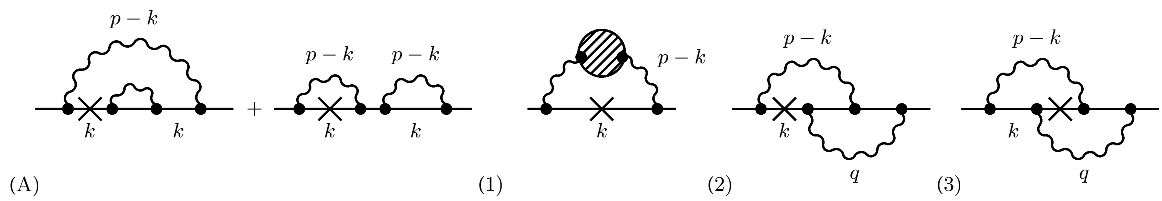

Figure 2. NLO diagrams for dynamically generated mass $\Sigma(p)$. The symbol (A) shows the contribution of the LO fermion wave function and symbols (1), (2) and (3) correspond to the different topologies of the NLO corrections themselves. The shaded blob contains the sum of the diagrams shown in Figure 3.

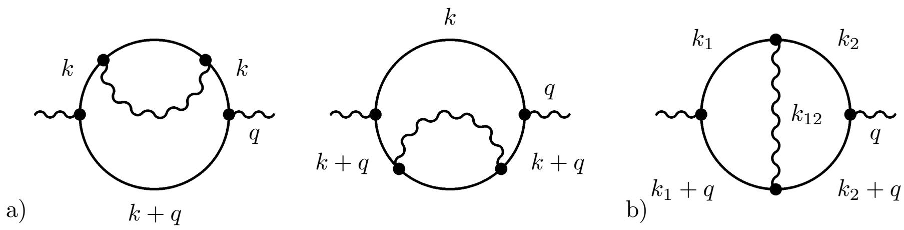

Figure 3. The diagrams contributing to the shaded blob is shown in Figure 2. Symbols (**a**) and (**b**) correspond to the different topologies of the corrections to the polarization operator.

Combining all the above results, the gap Equation (14) can be written explicitly as:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8S(\alpha,\xi) - 2(2+\xi)\hat{\Gamma}\mathrm{I}\beta + \left(-\frac{5}{3} + \frac{26}{3}\xi - 3\xi^2\right)\beta^2 - 8\beta\left(\frac{2}{3}(1-\xi) - \xi^2\right)\right], \tag{21}$$

where $S(\alpha,\xi) = \left(\hat{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\hat{\Sigma}_2(\alpha)\right)/8$.

4.1. Extraction of the Most "Important" Terms

Following Ref. [26], we would like to resum the term LO along with some of the NLO contributions containing terms $\sim\beta^2$. To do this, we will now rewrite the gap Equation (21) in a more suitable form.This is equivalent to extracting the terms $\sim\beta$ and $\sim\beta^2$ from the complex parts of the fermion self-energy, Equations (18) and (20). Such calculations give:

$$\hat{\Sigma}_2(\alpha) = \beta(3\beta-8) + \tilde{\Sigma}_2(\alpha), \ \overline{\Sigma}_3(\xi) = -4\xi(4+\xi)\beta + \tilde{\Sigma}_3(\alpha,\xi). \tag{22}$$

Then, using the results (22), the gap Equation (21) can be written as:

$$1 = \frac{(2+\xi)\beta}{L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - 2(2+\xi)\hat{\Gamma}\mathrm{I}\beta + \left(\frac{2}{3} - \xi\right)(2+\xi)\beta^2 + 4\beta\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right], \tag{23}$$

where $\tilde{S}(\alpha,\xi) = \Big(\tilde{\Sigma}_3(\alpha,\xi) + 2(1-\xi)\tilde{\Sigma}_2(\alpha)\Big)/8$. At this point, Equations (21) and (23) are strictly equivalent to each other and give the same values for $N_c(\xi)$.

4.2. Gap Equation

Following the Addendum to [2], we now proceed to the calculation of the NLO correction to the parameter β^{-1} of the solution of the SD equation. From (23), we have:

$$\beta^{-1} = \frac{2+\xi}{L} + \frac{1}{L^2}\Big[\frac{8}{\beta}\tilde{S}(\beta,\xi) - 2(2+\xi)\hat{\Pi} + \left(\frac{2}{3}-\xi\right)(2+\xi)\,\beta + 4\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big] + O(L^{-3})\,. \quad (24)$$

It is clear from this equation that the first term in brackets is of the order $\sim 1/L$ (as can be seen from the iterative solution of the Equation (24)) and, therefore, its contribution is of the order $\sim 1/L^3$ and should be neglected in the present study. So, with NLO accuracy, we get that:

$$\beta^{-1} = \frac{2+\xi}{L} + \frac{1}{L^2}\Big[\left(\frac{2}{3}-\xi\right)(2+\xi)\,\beta - 2(2+\xi)\hat{\Pi} + 4\left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\Big] + O(L^{-3})\,. \quad (25)$$

Now we are able to calculate β^{-1} from Equation (25) as a combination of the terms $\sim 1/L$ and $\sim 1/L^2$. This, however, is not so important in this analysis. Since we are interested in the critical mode, we can obtain L_c in a simple way from (25) (or equally from the Equation (21) using the condition $\tilde{S}(\beta,\xi) = 0$) by setting $\beta = 16$ and preserving the conditions $O(1/L^2)$. This gives:

$$L_c^2 - 16(2+\xi)L_c + 32\left[(2+\xi)\hat{\Pi} + 2\xi\left(\frac{20}{3} + 3\xi\right)\right] = 0\,. \quad (26)$$

Solving this equation, we have two standard solutions:

$$L_{c,\pm} = 8\left(2+\xi \pm \sqrt{d_1(\xi)}\right), \quad d_1(\xi) = 4 - \frac{8}{3}\xi - 2\xi^2 - \frac{2+\xi}{2}\hat{\Pi}\,. \quad (27)$$

Combining these values with the one of $\hat{\Pi}$ in the Equation (17), we obtain:

$$N_c(\xi = 0) = 3.17, \quad N_c(\xi = 2/3) = 2.91\,, \quad (28)$$

where the "$-$" solution is unphysical, and there is no solution in the Feynman gauge ($\xi = 1$). The range of ξ-values for which a solution exists corresponds to $\xi_- \le \xi \le \xi_+$, where $\xi_+ = 0.82$ and $\xi_- = -2.24$.

4.3. Resummation

Equation (23) is a convenient starting point for a resummation of the wave function renormalization constant. To second order, the expansion of the latter reads [51]:

$$\lambda_A = \frac{\lambda^{(1)}}{L} + \frac{\lambda^{(2)}}{L^2} + \cdots, \quad \lambda^{(1)} = 4\left(\frac{2}{3}-\xi\right), \quad \lambda^{(2)} = -8\left(\frac{8}{27} + \left(\frac{2}{3}-\xi\right)\hat{\Pi}\right). \quad (29)$$

As can be seen from Equation (23), the NLO term $\sim\beta^2$ is proportional to the LO renormalization constant of the wave function. This term, together with the LO term in the gap equation, can be considered as terms of order one and zero, respectively, in the expansion in λ_A. Following Nash, one can then resum the complete expansion of λ_A at the level of the gap equation (see Ref. [2]), leading to:

$$1 = \frac{8\beta}{3L} + \frac{\beta}{4L^2}\left(\lambda^{(2)} - 4\lambda^{(1)}\left(\frac{14}{3} + \xi\right)\right) + \frac{\Delta(\alpha,\xi)}{L^2}\,, \quad (30)$$

where $\Delta(\alpha,\xi) = 8\tilde{S}(\alpha,\xi) - 4\beta\,(\xi^2 + 4\xi + 8/3) + 2\beta\,(2+\xi)\,\hat{\Pi}$.

Interestingly, the LO term in the Equation (30) is now gauge independent. Using the Equation (29), Equation (30) can now be be written as:

$$1 = \frac{8\beta}{3L} + \frac{1}{L^2}\left[8\tilde{S}(\alpha,\xi) - \frac{16}{3}\beta\left(\frac{40}{9} + \hat{\Pi}\right)\right] + O(L^{-3})\,, \tag{31}$$

which demonstrates a strong suppression of the gauge dependence, since ξ-dependent terms exist, but they enter the equation only through the remainder $\tilde{S}$, which is very small in numerical terms.

By analogy with the previous subsection, we now compute the NLO correction to the parameter β^{-1} of the solution of the SD equation. From (31), this gives:

$$\beta^{-1} = \frac{8}{3L} + \frac{1}{L^2}\left[\frac{8}{\beta}\tilde{S}(\alpha,\xi) - \frac{16}{3}\left(\frac{40}{9} + \hat{\Pi}\right)\right] + O(L^{-3})\,. \tag{32}$$

From this equation it again becomes clear that the first term in brackets is of the order of $\sim 1/L$ (which can be seen by solving Equation (32) iteratively) and, therefore, its contribution is $\sim 1/L^3$ and should be ignored in this analysis. This observation was shown to us by Valery Gusynin. So, we have:

$$\beta^{-1} = \frac{8}{3L} - \frac{1}{L^2}\,\frac{16}{3}\left(\frac{40}{9} + \hat{\Pi}\right) + O(L^{-3})\,, \tag{33}$$

which is now completely gauge-independent.

Now consider Equation (33) (or, equivalently, Equation (31) with the condition $\tilde{S}(\beta,\xi) = 0$) at the critical point $\alpha = 1/4$ ($\beta = 16$), preserving all the terms $O(1/L^2)$. This gives:

$$L_c^2 - \frac{128}{3}L_c + \frac{256}{3}\left(\frac{40}{9} + \hat{\Pi}\right) = 0\,. \tag{34}$$

Solving Equation (34), we have two standard solutions:

$$L_{c,\pm} = \frac{64}{3}\left(1 \pm \sqrt{d_2(\xi)}\right)\,,\quad d_2(\xi) = 1 - \frac{3}{16}\left(\frac{40}{9} + \hat{\Pi}\right) = \frac{1}{6} - \frac{3}{16}\hat{\Pi} \tag{35}$$

and we have for the "+" solution (the "−" one is nonphysical):

$$\overline{L}_c = 28.0981, \qquad \overline{N}_c = 2.85\,. \tag{36}$$

The results of Equation (36) are completely consistent with the recent results of [29].

5. Conclusions

We have presented a study of DχSB in QED$_3$, including an exact computation of $1/N^2$ corrections to the SD equation and considering the full ξ-dependence of the gap equation. Following Nash, the renormalization constant of the wave function was resummed at the level of the gap equation, which led to a gauge-invariant critical number of fermion flavours, $\overline{N}_c = 2.85$, in full accordance with the result of Ref. [29].

As noted in [49,52–54], the limit of the large N for the photon propagator in QED$_3$ has exactly the same dependence on momentum as in the so-called reduced QED [55–59]. One of the differences is that the gauge fixing parameter in reduced QED is half as much as in QED$_3$. This difference can be accounted for using our current QED$_3$ results along with multi-loop results obtained in [48,49,52–54]. The case of reduced QED and its relation to the generation of a dynamic gap in graphene, which is the subject of active current research, see, e.g., reviews [60–62], were considered in our article [63].

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