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Article

Analytical Solutions for a Generalized Nonlinear Local Fractional Bratu-Type Equation in a Fractal Environment

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Abstract: In the context of fractal space, this study presents a higher-order nonlinear local fractional Bratu-type equation and thoroughly examines this generalized nonlinear equation. Additional analysis and identification of particular special situations of the generalized local fractional Bratu equation is performed. Finally, the Adomian decomposition method is utilized to derive that solution for the generalized Bratu equation of local fractional type. This study contributes to a deeper understanding of these equations and provides a practical computational approach to their solutions.

Keywords: local fractional derivative; generalized Bratu-type equation; Adomin decomposition method

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1. Introduction

The utilization of fractional derivatives represents a pivotal mathematical instrument for the development of sophisticated mathematical and physical models within the intriguing realm of fractal space [1–3]. Yang's pioneering introduction of the local fractional derivative, as documented in his seminal works [4–7], has since captivated the attention of a multitude of researchers. In the realm of applied mathematics and mathematical analysis, the fractal derivative, also known as the Hausdorff derivative, serves as a non-Newtonian extension of the derivative specifically designed for measuring fractals within the framework of fractal geometry. Fractal derivatives have been developed to explore anomalous diffusion, addressing situations where conventional methods overlook the fractal characteristics of the medium. In this context, a fractal measure "t" is scaled by β. Notably, this derivative is considered local, distinguishing it from the more commonly employed fractional derivative. Fractal calculus is structured as a generalization of standard calculus. This derivative has found extensive application in various scientific disciplines, notably in the fields of physics and engineering. Its influence spans critical domains such as nanoengineering, dynamic systems, and microelectronics. The main aspiration of fractional calculus is to extend differentiation and integration to fractional order, a concept that has been around since the 17th century due to the groundbreaking work of Leibnitz, Euler, Lagrange, Abel, Liouville, and many others [8–10]. A growing area of mathematics called fractional calculus (FC) has diverse applications in all connected sectors of science and problems of engineering. Some of the findings were published in books or related review articles. To learn more about local fractional calculus, see [11]. Incorporating fractal calculus into differential equations presents a robust paradigm for modeling intricate systems featuring self-replicating patterns. Within the domain of porous media, a fractional diffusion equation adeptly characterizes the non-local substance transport in materials exhibiting fractal structures. Likewise, a fractional population growth equation captures the dynamics of species expansion in environments marked by fractal patterns. Rooted in fractal calculus, these equations contribute to a nuanced understanding of phenomena in domains like heat conduction, electrical circuits, and wave propagation. This approach proves versatile and insightful, offering a comprehensive means of modeling the complex behaviors inherent in systems displaying fractal characteristics [12–19].

In the past few years, fractional differential equations have been extensively utilized in physics and engineering. Over the past few decades, fractional differential equations have found widespread application in engineering and science. Ordinary differential equations (ODEs) and fractional-order partial differential equations (PDEs) are commonly encountered in various disciplines, including fluid dynamics, biology, and physics. Numerous fields, such as electrical, mechanical, chemical, biological, and economics, particularly in ground transportation, signal image processing, and control theory, involve the use of these equations. Considerable effort has been invested in developing reliable and consistent numerical and analytical methodologies for solving these fractional equations over the last decade or more. For fractional differential equations, the methods of Adomian decomposition and variational iteration stand out among the rest. These methods distinguish themselves by providing approximations for the issues under consideration without relying on linearization or discretization.

Nonlinear differential equations, such as the Bratu-type equation, find application in various fields of science and engineering. Bratu's problem has been utilized in addressing various challenges, including the Chandrashekhar model, the fuel ignition model, the thermal response model, and the framework for the electrospun nanofiber fabrication process. It is employed in chemical kinetics to simulate complex reaction kinetics, in combustion theory to comprehend flame propagation, and in heat transfer to examine temperature distributions in materials. Furthermore, it plays a role in mathematical biology, material science, nuclear physics, electrochemistry, and environmental science, enabling the modeling of complex processes such as population dynamics, diffusion in materials, neutron transport in reactors, electrochemical reactions, and pollutant spread. Beyond specific applications, the Bratu-type equation serves as a fundamental mathematical model, providing insights into nonlinear differential equations and their properties. This makes it a versatile and indispensable tool in scientific research and analysis. The "Nonlinear Local Fractional Bratu-Type Equation" is a mathematically intricate and captivating equation residing within the domain of fractional calculus differential equations. This equation presents a specialized variant of the Bratu equation, well known for its relevance in studying combustion processes and reaction–diffusion phenomena. Within the "Nonlinear Local Fractional Bratu-Type Equation," the incorporation of fractional calculus introduces a unique and labyrinthine facet to the problem. Fractional calculus extends conventional notions of differentiation and integration to noninteger orders, enabling the portrayal of intricate physical processes characterized by memory and anomalous diffusion. Consequently, this equation holds particular significance across a spectrum of scientific and engineering disciplines, including physics, chemistry, biology, and materials science.

The equation typically appears as a differential equation containing a nonlinear term, a fractional derivative, and boundary conditions determined by the specific contextual problem. The solution to this equation provides valuable insights into the behavior of systems governed by fractional diffusion and nonlinear reactions, revealing the intricate interplay between these two phenomena. Ongoing research endeavors focused on the "Non-linear Bratu Equation of Local Fractional Type" captivate mathematicians, physicists, and engineers alike as they grapple with challenges in the realm of non-standard calculus, nonlinear dynamics, and complex systems. A deep understanding of the solutions and attributes of this equation has the potential to advance our understanding of diverse real-world processes, potentially leading to practical applications across domains such as combustion theory, chemical kinetics, and biological modeling.

The Bratu-type equation holds significant utility in the fields of science and engineering. Recognizing its importance, we shift our focus to investigating a generalized local fractional Bratu-type equation [20,21]. In this extension, we employ the Mittag-Leffler function to define a generalized local fractional Bratu-type equation. The inclusion of the Mittag-Leffler function enhances the flexibility of modeling, allowing the customization of equations matching specific behaviors observed in the system under study. This adaptability is crucial to accurately representing the nuances of real-world phenomena. Incorporating the Mittag-Leffler function into the generalization of an equation serves as a valuable approach to enriching the model and addressing the intricacies inherent in fractional calculus. It proves to be a robust tool for characterizing systems with features such as long memory, anomalous diffusion, and other nonlocal or non-Markovian behaviors. Moreover, our approach extends to the n-dimensional generalization of the fractional Bratu-type equation, and the derived model can be easily streamlined to the classical Bratu-type equation by assigning specific values.

$$\frac{d^{2n\beta}\zeta}{d\xi^{2n\beta}} + \wp\, E_\beta\left(\zeta^\beta\right) = 0,\ \xi > 0,\ 0 < \beta < 1, n \in N. \tag{1}$$

In this equation, $\wp$ is a constant and $E_\beta(\zeta^\beta)$ is the Mittag-Leffler function with real parameter β. In addition, ξ and ζ are known and unknown functions.

We also introduce another form of the generalized local fractional Bratu-type equation:

$$\frac{d^{2n\beta}\zeta}{d\xi^{2n\beta}} + \wp\ exp(\zeta) = 0,\ \xi > 0,\ 0 < \beta < 1, n \in N, \tag{2}$$

where $\wp$ is constant. In this equation, $exp(\zeta)$ is the exponential function.

If $n = 1$, then Equation (1), reduces in to

$$\frac{d^{2\beta}\zeta}{d\xi^{2\beta}} + \wp\, E_\beta\left(\zeta^\beta\right) = 0,\ \xi > 0,\ 0 < \beta < 1. \tag{3}$$

If $n = 1$, then Equation (2), reduces in to

$$\frac{d^{2\beta}\zeta}{d\xi^{2\beta}} + \wp\ exp(\zeta) = 0,\ \xi > 0,\ 0 < \beta < 1. \tag{4}$$

Also, if $n = 1$ and $\beta = 1$, then Equation (2), reduces in to the classical Bratu-type equation [6]:

$$\frac{d^2\zeta}{d\xi^2} + \wp\ exp(\zeta) = 0,\ \xi > 0. \tag{5}$$

Establishing the existence and uniqueness of the generalized non-linear local fractional Bratu-type equation formulated is straightforward with the application of the Lipschitz condition. In addition, to address a combustion challenge within a numerical framework, a specific approach is employed. Numerous techniques have been explored to approximate the analytical solution of a nonlinear differential equation. These include the homotopy perturbation method [14,22], the variational iteration method [15,23], the reduced differential transform method [24], the homotopy analysis method [25,26], and the δ-homotopy analysis transform method [27,28], among others.

In this research work, we utilize the Adomian decomposition method—a powerful tool for solving linear or nonlinear differential equations to derive the results of the generalized Bratu-type equation. The Adomian decomposition method is a powerful technique that provides efficient algorithms for approximate analytical solutions and numerical simulations for real-world applications in engineering and applied sciences. This approach is an excellent technique for solving generalized nonlinear fractional differential equations, even though the outcome is expressed in terms of an infinite series. Variational formulation of the local fractional Bratu-type equation is established by the semi-inverse method and its approximate analytical solution is obtained by the Adomian decomposition method.

In addressing the motivation behind this work, it becomes apparent that conventional derivative definitions prove insufficient when tackling space co-ordinate within a fractal environment. To adequately capture this phenomenon, the utilization of Yang's local fractional derivative becomes crucial. This research employs the Adomian decomposition method, recognized as a potent technique for solving both linear and nonlinear differential equations [29], serving as the cornerstone for deriving our findings.

The paper is structured into six sections. Section 1 provides an introduction, while Section 2 covers the Preliminaries. Section 3 presents the Adomian decomposition method (ADM), Section 4 addresses the solution of the generalized local fractional Bratu equation, and Section 5 delves into specific cases, including the numerical solution of the model using the Adomian decomposition method. Concluding remarks are presented in Section 6. The references are discussed at the end of the paper.

2. Preliminaries

In this segment, we introduce essential definitions and foundational principles of the Adomian decomposition method in the contexts of local fractional and calculus of fractional domains.

2.1. Local Fractal Derivative

The local fractional derivative of function $R(t)$ with an order of $0 < \beta < 1$ at the specific point $t = t_0$ is provided in accordance with [4–7]

$$\begin{aligned} D^\beta R_\beta(t_0) &= \tfrac{d^\beta}{dt^\beta} R_\beta(t)|_{t_o} \\ &= \lim_{t \to t_0} \tfrac{\Delta^\beta \left(R_\beta(t) - R_\beta(t_0)\right)}{(t-t_0)^\beta}, \end{aligned}$$

which, for all $\mu > 0,\ 0 < |t - t_0| < \delta$, satisfies condition $\left|R_\beta(t) - R_\beta(t_0)\right| \le \mu^\beta$, where

$$\Delta^\beta \left(R_\beta(t) - R_\beta(t_0)\right) \cong \left[R_\beta(t) - R_\beta(t_0)\right]\Gamma(1+\beta).$$

2.2. Local Fractal Integral

If $\Omega(\ell)$ in fractal space has order $\beta(0,1)$, then the local fractal integral of $\Omega(\ell)$ is defined in the interval (β, λ) as follows:

$$_s I_t{}^{(\beta)} \Omega(\ell) = \frac{1}{\Gamma(1+\beta\,)} \int_s^t \Omega(\theta)(d\theta)^\beta = \frac{1}{\Gamma(1+\beta\,)} \lim_{\Delta\theta \to 0} \sum_{j=0}^{j=I-1} \Omega(\theta_j)\left(\Delta\theta_j\right)^\beta, \tag{6}$$

where $\Delta\theta_j = \theta_{j+1} - \theta_j$, $\Delta\theta = \max\{\Delta\theta_1, \Delta\theta_2, \Delta\theta_3, \ldots\}$ and $\left[\theta_j, \theta_{j+1}\right], j = 0, \ldots, I-1$, $\theta_0 = s, \theta_I = t$, is a division of (s,t).

2.3. Properties of Local Fractional Derivatives

We present the following properties associated with local fractional derivatives (see [30,31]): We let $\zeta_1(\xi), \zeta_2(\xi) \in C_\beta(a,b)$; then,

(i) $_s I_t^{(\beta)}(\zeta_1(\xi) \pm \zeta_2(\xi)) = {}_s I_t^{(\beta)}[\zeta_1(\xi)] \pm {}_s I_t^{(\beta)}[\zeta_2(\xi)]$,
(ii) $_s I_t^{(\beta)}(\zeta_1(\xi) * \zeta_2(\xi)) = \zeta_2(\xi)_s I_t^{(\beta)}[\zeta_1(\xi)] + \zeta_1(\xi)_s I_t^{(\beta)}[\zeta_2(\xi)]$,
(iii) $_s I_t^{(\beta)}\left[\frac{\zeta_1(\xi)}{\zeta_2(\xi)}\right] = \frac{\left(\zeta_2(\xi)_s I_t^{(\beta)}[\zeta_1(\xi)] - \zeta_1(\xi)_s I_t^{(\beta)}[\zeta_2(\xi)]\right)}{(\zeta_2(\xi))^2}$.

The higher-order local fractional order can be expressed as

$$\theta^{(m\beta)}(t) = \overbrace{D_t^\beta \ldots D_t^\beta}^{m\ \ times} \theta(t).$$

2.4. Mittag-Leffler Function

The Swedish mathematician Gosta Mittag-Leffler explored and introduced a function defined as in the expression of power series in 1903 (see [4,8]):

$$E_\beta\left(t^\beta\right) = \sum_{n=0}^{\infty} \frac{t^{n\beta}}{\Gamma(\beta n + 1)}, \tag{7}$$

where $0 < \beta < 1$.

3. Adomian Decomposition Method (ADM)

We have the well-known ADM approach, which is quite useful for solving linear or non-linear differential equations [32,33]:

$$\nu - N\nu = g, \tag{8}$$

where N is a nonlinear operator and ν is an unknown function. The equation defined in (8) is known as a non-linear system. Now, to find the approximate solutions for Equation (8), first, we consider the solution of (8) is unique with the form

$$\nu = \sum_{j=0}^{\infty} \nu_j. \tag{9}$$

It is evident that finding the solution to (9) becomes notably challenging when the given system involves nonlinear terms. To address this challenge, the Adomian decomposition method (ADM) emerges as a pivotal tool. In this approach, the nonlinear term is decomposed as follows:

$$N\nu = \sum_{j=0}^{\infty} B_j, \tag{10}$$

where B_j is known as Adomian polynomials with coefficients $\nu_0, \nu_1, \nu_2, \nu_3, \ldots, \nu_n$, hence

$$B_j = B_j(\nu_0, \nu_1, \nu_2, \ldots, \nu_n). \tag{11}$$

Now, to obtain the values of B_j, we set

$$\nu = \sum_{j=0}^{\infty} q^j \nu_j, \tag{12}$$

and

$$N\nu = \sum_{j=0}^{\infty} q^j B_j, \tag{13}$$

where

$$B_j = \frac{1}{j!} \frac{d^j}{dq^j} N \sum_{j=0}^{\infty} q^j \nu_j \bigg|_{j=0}. \tag{14}$$

Now, putting the value of B_j in (13), from (8), we obtain

$$\sum_{j=0}^{\infty} \nu_j = \sum_{j=0}^{\infty} B_j + g. \tag{15}$$

Now, to obtain the value of ν_j, we define the following relations:

$$\begin{aligned} \nu_0 &= g, \\ \nu_{j+1} &= B_j(\nu_0, \nu_1, \nu_2, \ldots, \nu_j). \end{aligned}$$

For more details on the method, see [32–35].

4. Solution of Generalized Local Fractional Bratu Equation

Theorem 1. *Consider the following generalized Bratu-type equation:*

$$\frac{d^{2n\beta}\zeta}{d\xi^{2n\beta}} + \wp E_\beta\left(\zeta^\beta\right) = 0,\ \xi > 0,\ 0 < \beta < 1,\ n \in N, \tag{16}$$

where $\frac{d^\beta}{d\xi^\beta}$ *is known as Yang's fractional derivative of a local type, with*

$$\left.\zeta(0) = \zeta(1) = 0, \zeta^\beta = \lambda(\text{constant}).\right\} \tag{17}$$

Proof. We have the Adomian decomposition method, which is quite useful in solving nonlinear ordinary or partial differential equations. To solve the differential equation defined in the statement of Theorem 1, first, we consider

$$L_\xi^{2n\beta} = \frac{d^{2n\beta}}{d\xi^{2n\beta}}, \tag{18}$$

and its inverse integral is

$$L_\xi^{-2n\beta}(.) = \underbrace{\int_0^\xi \int_0^\xi \dots \int_0^\xi}_{\{2n\ times\}} (.)\, dx^\beta dx^\beta \dots dx^\beta. \tag{19}$$

The operator form of Equation (16), with Adomian polynomials, is

$$L_\xi^{(2n\beta)}\zeta + \wp \sum_{j=0}^{\infty} B_j = 0. \tag{20}$$

Upon using the inverse operator, we obtain the following recurrence relation:

$$\begin{cases} \zeta_0 = \lambda \frac{\xi^\beta}{\Gamma(\beta+1)}, \\ \zeta_{j+1} = -L_\xi^{-(2n\beta)}\left(\wp \sum_{j=0}^{\infty} B_j\right); \end{cases} \tag{21}$$

hence, putting $j = 0$, in Equation (21), we obtain

$$\zeta_1 = -L_\xi^{-(2n\beta)}(\wp B_0), \tag{22}$$

where $B_0 = E_\beta\left(\zeta_0^\beta\right)$.

And for $j = 1$, Equation (21) offers

$$\zeta_2 = -L_\xi^{-(2n\beta)}(\wp B_1). \tag{23}$$

with $B_1 = \nu_1 E_\beta\left(\zeta_0^\beta\right)$.

In the same manner, for $j = 2$, Equation (21) provides us with

$$\zeta_3 = -L_\xi^{-(2n\beta)}(\wp B_2), \tag{24}$$

where $B_2 = \left(\zeta_2 + \frac{\zeta_1^2}{2}\right) E_\beta\left(\zeta_0^\beta\right)$, and so on.

Then, we can easily find the solution of Equation (16) as

$$\zeta = \zeta_0 + \sum_{j=1}^{\infty} \zeta_j. \tag{25}$$

□

Theorem 2. *Derive the solution of another form of the generalized local fractional Bratu-type equation:*

$$\frac{d^{2n\beta}\zeta}{d\xi^{2n\beta}} + \wp\ exp(\zeta) = 0,\ \xi > 0,\ 0 < \beta < 1,\ n \in N, \tag{26}$$

where $\frac{d^{\beta}}{d\xi^{\beta}}$ *is known as Yang's fractional derivative of local type,and*

$$\left.\zeta(0) = \zeta(1) = 0, \xi^{\beta} = \lambda(\text{constant}).\right\} \tag{27}$$

Proof. Utilizing the Adomian decomposition method, a valuable technique for solving nonlinear ordinary or partial differential equations, we address the solution of the differential equation specified in Theorem 1. Initially, we examine

$$L_{\xi}^{2n\beta} = \frac{d^{2n\beta}}{d\xi^{2n\beta}}, \tag{28}$$

and its inverse integral is

$$L_{\xi}^{-2n\beta}(.) = \underbrace{\int_0^{\xi}\int_0^{\xi}\ldots\int_0^{\xi}}_{\{2n\ times\}} (.)\, dx^{\beta} dx^{\beta} \ldots dx^{\beta}. \tag{29}$$

The operator form of the above equations with Adomian polynomials is

$$L_{\xi}^{(2n\beta)}\zeta + \wp\sum_{j=0}^{\infty} B_j = 0. \tag{30}$$

Upon using the inverse operator, we obtain the following recurrence relation:

$$\begin{cases} \zeta_0 = \lambda \frac{\xi^{\beta}}{\Gamma(\beta+1)}, \\ \zeta_{j+1} = -L_{\xi}^{-(2n\beta)}\left(\wp\sum_{j=0}^{\infty} B_j\right); \end{cases} \tag{31}$$

hence, for $j = 0$, we obtain from (31)

$$\zeta_1 = -L_{\xi}^{-(2n\beta)}(\wp B_0), \tag{32}$$

where $B_0 = e^{\zeta_0}$.

And for $j = 1$, Equation (31) offers

$$\zeta_2 = -L_{\xi}^{-(2n\beta)}(\wp B_1). \tag{33}$$

with $B_1 = \nu_1 e^{\zeta_0}$,

In the same manner for $j = 2$, Equation (31) provides us with

$$\zeta_3 = -L_{\xi}^{-(2n\beta)}(\wp B_2), \tag{34}$$

with $B_2 = \left(\zeta_2 + \frac{\zeta_1^2}{2}\right)e^{\zeta_0}$, and so on. Then, we can easily find the solution of Equation (26) as

$$\zeta = \zeta_0 + \sum_{j=1}^{\infty} \zeta_j. \tag{35}$$

□

5. Particular Cases

Now, we discuss some particular cases of the proposed generalized local fractional Bratu-type equations. In the above-defined Theorems 1 and 2, by putting some specific values of different constants, it is reduced into well-known differential equations.

Special Case (i):

If we put $n = 1$ in the defined (16) of Theorem 1, we obtain

$$\frac{d^{2\beta}\zeta}{d\xi^{2\beta}} + \wp\, E_\beta\left(\zeta^\beta\right) = 0,\ \xi > 0,\ 0 < \beta \le 1, \tag{36}$$

with

$$\left.\zeta(0) = \zeta(1) = 0, \zeta^\beta = \lambda(\text{constant}).\right\} \tag{37}$$

By using the above-defined procedure, we obtain the following results:

$$B_0 = E_\beta\left(\zeta_0^\beta\right), \tag{38}$$

$$B_1 = \nu_1 E_\beta\left(\zeta_0^\beta\right), \tag{39}$$

$$B_2 = \left(\zeta_2 + \frac{\zeta_1^2}{2}\right) E_\beta\left(\zeta_0^\beta\right), \tag{40}$$

$$\vdots$$

Now, using the method defined in the above section, we obtain the following terms:

$$\zeta_0 = \lambda \frac{\xi^\beta}{\Gamma(1+\beta)}; \tag{41}$$

in the same manner,

$$\zeta_1 = -\wp L_\xi^{-2\beta}\left(E_\beta\left(\zeta_0^\beta\right)\right) \tag{42}$$

or

$$\zeta_1 = -\wp \int_0^\xi \int_0^\xi E_\beta\left(\lambda \frac{x^\beta}{\Gamma(1+\beta)}\right) dx^\beta dx^\beta. \tag{43}$$

Upon solving, we obtain

$$\zeta_1 = -\frac{\wp}{\lambda^2}\left[E_\beta\left(\lambda \frac{\xi^\beta}{\Gamma(1+\beta)}\right) - 1 - \lambda \frac{\xi^\beta}{\Gamma(1+\beta)}\right], \tag{44}$$

and using the same approach, we obtain

$$\begin{aligned}\zeta_2 = -\frac{\wp^2}{4\lambda^4}&\left[E_\beta\left(\frac{2\lambda\xi^\beta}{\Gamma(1+\beta)}\right) - \frac{2\lambda\xi^\beta}{\Gamma(1+\beta)} - \right.\\ &\left.\frac{4\lambda\xi^\beta}{\Gamma(1+\beta)} E_\beta\left(\frac{\lambda\xi^\beta}{\Gamma(1+\beta)}\right) + 4E_\beta\left(\frac{\lambda\xi^\beta}{\Gamma(1+\beta)}\right) - 5\right],\end{aligned} \tag{45}$$

and the remaining terms can be acquired using a similar methodology.

Now, by putting the above-obtained coefficients in the following equation, we obtain the required solution,

$$\zeta = \zeta_0 + \sum_{j=1}^{\infty} \zeta_j, \tag{46}$$

or

$$\begin{aligned}\zeta = \lambda \frac{\xi^\beta}{\Gamma(1+\beta)} &- \frac{\wp}{\lambda^2}\left[E_\beta\left(\lambda \frac{\xi^\beta}{\Gamma(1+\beta)}\right) - 1 - \lambda \frac{\xi^\beta}{\Gamma(1+\beta)}\right]\\ -\frac{\wp^2}{4\lambda^4}&\left[E_\beta\left(\frac{2\lambda\xi^\beta}{\Gamma(1+\beta)}\right) - \frac{2\lambda\xi^\beta}{\Gamma(1+\beta)} - \right.\\ &\left.\frac{4\lambda\xi^\beta}{\Gamma(1+\beta)} E_\beta\left(\frac{\lambda\xi^\beta}{\Gamma(1+\beta)}\right) + 4E_\beta\left(\frac{\lambda\xi^\beta}{\Gamma(1+\beta)}\right) - 5\right] + \dots.\end{aligned} \tag{47}$$

Through our investigation, we establish the efficacy of the Adomian decomposition method in solving non-linear differential equations. The results obtained for the generalized local fractional Bratu-type equation offer enhanced clarity and validation of the method's capabilities.

To examine the spatial behavior, we depict Figure 1, illustrating the variations of the dependent variable ζ concerning the independent variable ξ for Case (i) at various values of $\beta = 0.7, 0.8, 0.9$, and 0.99. These results are derived with $\lambda = 1$ and $\wp = 0.5$. The findings reveal a noticeable influence of β on the behavior of ζ. Additionally, it is observed that the value of the space coordinate initially rises and subsequently decreases over a defined time interval.

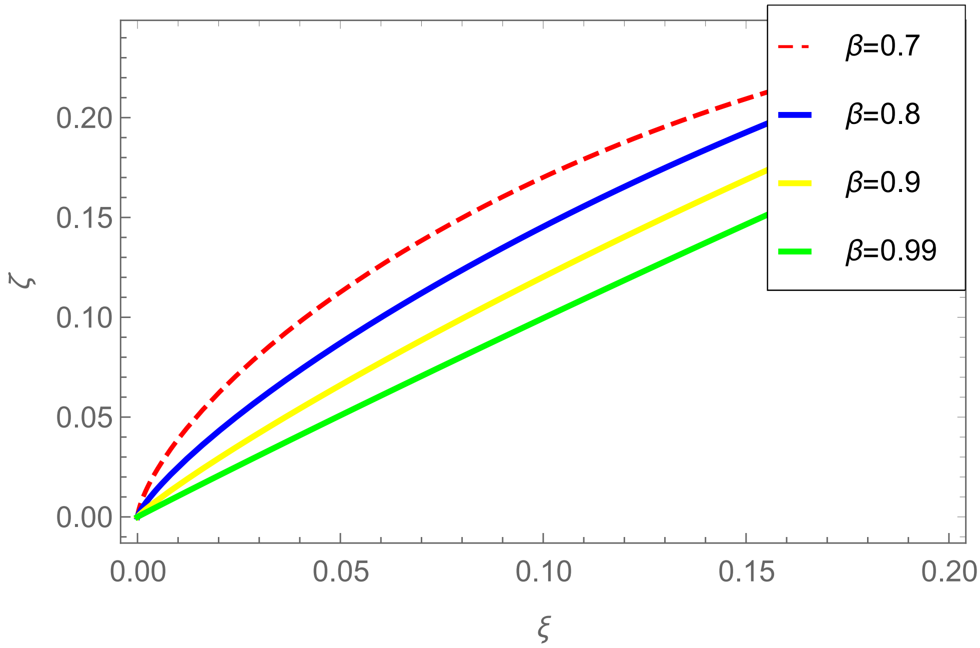

Figure 1. Change in ζ of Case (i) with respect to ξ for different values of β.

Special Case (ii):

If we put $n = 1$ in the defined Equation (26) of Theorem 2, we obtain

$$\frac{d^{2\beta}\zeta}{d\xi^{2\beta}} + \wp exp(\zeta) = 0,\ \xi > 0, 0 < \beta < 1, \tag{48}$$

with

$$\left.\zeta(0) = \zeta(1) = 0,\ \zeta^{\beta} = \lambda(\text{constant}).\right\} \tag{49}$$

$$\zeta_0 = \lambda \frac{\xi^{\beta}}{\Gamma(1+\beta)}, \tag{50}$$

In the same manner,

$$\zeta_1 = -\wp L_{\xi}^{-2\beta}\left(exp\left(\zeta_0^{\beta}\right)\right), \tag{51}$$

or

$$\zeta_1 = -\wp \int_0^{\xi}\int_0^{\xi} exp\left(\lambda \frac{x^{\beta}}{\Gamma(1+\beta)}\right) dx^{\beta} dx^{\beta}. \tag{52}$$

We obtain

$$\zeta_1 = -\frac{\wp}{\lambda^2}\left[exp\left(\lambda \frac{\xi^{\beta}}{\Gamma(1+\beta)}\right) - 1 - \lambda \frac{\xi^{\beta}}{\Gamma(1+\beta)}\right], \tag{53}$$

and using the same approach, we obtain

$$\begin{aligned}\zeta_2 = -\tfrac{\wp^2}{4\lambda^4}\Big[&exp\left(\tfrac{2\lambda\xi^{\beta}}{\Gamma(1+\beta)}\right) - \tfrac{2\lambda\xi^{\beta}}{\Gamma(1+\beta)} \\ &- \tfrac{4\lambda\xi^{\beta}}{\Gamma(1+\beta)} exp\left(\tfrac{\lambda\xi^{\beta}}{\Gamma(1+\beta)}\right) + 4exp\left(\tfrac{\lambda\xi^{\beta}}{\Gamma(1+\beta)}\right) - 5\Big],\end{aligned} \tag{54}$$

and other terms can be obtained by the same approach.

Now, by putting the above-obtained coefficients in the following equation, we obtain the required solution,

$$\zeta = \zeta_0 + \sum_{j=1}^{\infty} \zeta_j, \tag{55}$$

or

$$\begin{aligned} \zeta = {} & \lambda \frac{\xi^\beta}{\Gamma(1+\beta)} - \frac{\wp}{\lambda^2}\left[exp\left(\lambda \frac{\xi^\beta}{\Gamma(1+\beta)}\right) - 1 - \lambda \frac{\xi^\beta}{\Gamma(1+\beta)}\right] \\ & - \frac{\wp^2}{4\lambda^4}\left[exp\left(\frac{2\lambda\xi^\beta}{\Gamma(1+\beta)}\right) - \frac{2\lambda\xi^\beta}{\Gamma(1+\beta)} \right. \\ & \left. - \frac{4\lambda\xi^\beta}{\Gamma(1+\beta)} exp\left(\frac{\lambda\xi^\beta}{\Gamma(1+\beta)}\right) + 4exp\left(\frac{\lambda\xi^\beta}{\Gamma(1+\beta)}\right) - 5\right] + \dots \end{aligned} \tag{56}$$

In our examination, it was demonstrated that the Adomian decomposition method proves highly effective in solving non-linear differential equations. The results obtained for the generalized local fractional Bratu-type equation align with those presented by Yao et al. [21], underscoring the precision and reliability of the method.

To examine the spatial behavior, we depict Figure 2, illustrating the variations of the dependent variable ζ concerning the independent variable ξ for Case (ii) at various values of $\beta = 0.7, 0.8, 0.9$, and 0.99. These results are derived with $\lambda = 1$ and $\wp = 0.5$. The findings reveal a noticeable influence of β on the behavior of ζ. Additionally, it is observed that the value of space coordinate initially rises and subsequently decreases over a defined time interval.

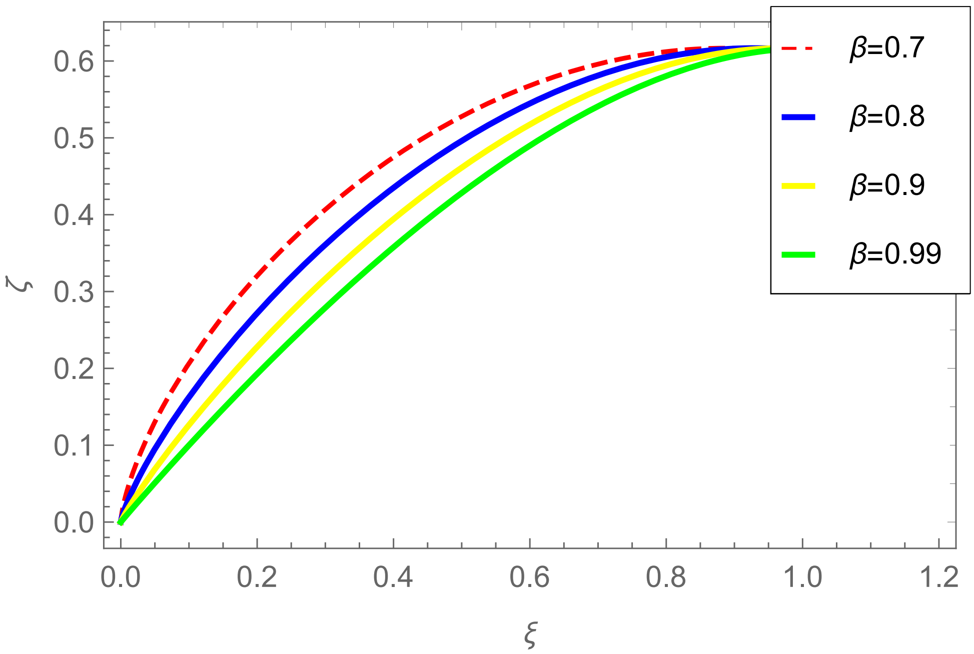

Figure 2. Change in ζ of Case (ii) with respect to ξ for different values of β.

6. Conclusions

In this paper, we introduce the generalized local fractional Bratu-type equation, extending the discussion to encompass the n-dimensional case. We also use the Mittag-Leffler function in this generalization. A more efficient analytical approach is presented for the analysis of the generalized local fractional Bratu-type equation, utilizing the Yang local fractional derivative. Our investigation involves the formulation of two theorems pertaining to the higher-order generalized local fractional Bratu equation, followed by an exploration of its solution using the Adomian decomposition method. Furthermore, we present specific results that underscore the method's effectiveness in approximating analytical solutions for this equation. The incorporation of a more generalized function enhances the utility of the study compared to the original nongeneralized equation.

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Nomenclature

ξ time co-ordinate
ζ space co-ordinate
λ constant
$\wp$ constant

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