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Article

The Properties of Meromorphic Multivalent q-Starlike Functions in the Janowski Domain

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Abstract: Many researchers have defined the q-analogous of differential and integral operators for analytic functions using the concept of quantum calculus in the geometric function theory. In this study, we conduct a comprehensive investigation to identify the uses of the Sălăgean q-differential operator for meromorphic multivalent functions. Many features of functions that belong to geometrically defined classes have been extensively studied using differential operators based on q-calculus operator theory. In this research, we extended the idea of the q-analogous of the Sălăgean differential operator for meromorphic multivalent functions using the fundamental ideas of q-calculus. With the help of this operator, we extend the family of Janowski functions by adding two new subclasses of meromorphic q-starlike and meromorphic multivalent q-starlike functions. We discover significant findings for these new classes, including the radius of starlikeness, partial sums, distortion theorems, and coefficient estimates.

Keywords: quantum (or q-) calculus; q-derivative operator; Sălăgean q-differential operator; meromorphic multivalent q-starlike functions; Janowski functions

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1. Introduction and Definitions

Currently, researchers have given more attention to the study of q-calculus due to its applications in the fields of physics and mathematics. Before Ismail et al. [1] looked into the q-extension of the class of starlike functions, Jackson [2,3] was the first to consider some applications of q-calculus and define the q-analogue of the derivative and integral. After that, several scholars carried out great studies in geometric function theory (GFT). The q-Mittag–Leffler functions were specifically researched by Srivastava and others, and the authors of [4] also studied the class of q-starlike functions and looked into a third Hankel determinant. A recent survey-cum-expository review conducted by Srivastava [5] is also beneficial for researchers studying these subjects. In this review study, Srivastava [5] discussed applications of the fractional q-derivative operator in geometric function theory and provided some mathematical justifications. In their paper [6], Arif et al. defined and explored the q-derivative operator for multivalent functions, and [7] Zang et al. defined a generalized conic domain and then investigated a novel subclass of q-starlike functions using the definition of subordination and q-calculus operator theory. Recently, many well-known mathematicians have used q-calculus and studied some subclasses of analytic functions and their properties (see, for example, [8,9]). Recently, several authors published a series of studies [10–12] focusing on the classes of q-starlike functions connected to Janowski functions [13] from various angles.

The above works serve as the main inspiration for this article, which will first define a new q-analog of the Sălăgean differential operator for meromorphic multivalent functions.

By taking this operator into consideration, a new subclass of meromorphic multivalent functions related to Janowski functions is defined and studied, along with its geometric properties such as sufficient coefficient estimates, partial sums, distortion theorems, and the radius of starlikeness.

The set $\mathcal{M}(p)$ contains all meromorphic multivalent functions h that are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{\varsigma : \varsigma \in \mathbb{C} \quad \text{and} \quad 0 < |\varsigma| < 1\},$$

and have the following series of representation:

$$h(\varsigma) = \frac{1}{\varsigma^p} + \sum_{i=0}^{\infty} a_{i+p}\varsigma^{i+p}, \quad (p \in \mathbb{N} = \{1, 2, ...\}). \tag{1}$$

In particular, if $p = 1$, then

$$h(\varsigma) = \frac{1}{\varsigma} + \sum_{i=1}^{\infty} a_i \varsigma^i, \tag{2}$$

In other words, we have

$$\mathcal{M}(1) = \mathcal{M},$$

which is the set of meromorphic univalent functions that are analytic in the punctured open unit disk.

A function $h \in \mathcal{MS}^*(p)$ is called a meromorphic multivalent starlike function if $h \in \mathcal{M}(p)$ satisfies the inequality

$$Re\left(-\frac{\varsigma h'(\varsigma)}{h(\varsigma)}\right) > 0.$$

A function $h \in \mathcal{MS}^*(p, \alpha)$ is called a meromorphic multivalent starlike functions of the order α $(0 \leq \alpha < 1)$ if $h \in \mathcal{M}(p)$ satisfies the inequality

$$Re\left(-\frac{\varsigma h'(\varsigma)}{h(\varsigma)}\right) > \alpha, \ \ ((0 \leq \alpha < p)$$

In particular, we have

$$\mathcal{MS}^*(p, 0) = \mathcal{MS}^*(p).$$

A function $h \in \mathcal{MC}(p)$ is called a meromorphic multivalent convex function if $h \in \mathcal{M}(p)$ satisfies the inequality

$$Re\left(-\left(1 + \frac{\varsigma h''(\varsigma)}{h'(\varsigma)}\right)\right) > 0. \ \ (0 \leq \alpha < p)$$

A function $h \in \mathcal{MC}(p, \alpha)$ is called a meromorphic multivalent convex function of the order α $(0 \leq \alpha < p)$ if $h \in \mathcal{M}(p)$ satisfies the inequality

$$Re\left(-\left(1 + \frac{\varsigma h''(\varsigma)}{h'(\varsigma)}\right)\right) > \alpha.$$

In particular, we have

$$\mathcal{MC}(p, 0) = \mathcal{MC}(p).$$

The basic ideas of these classes started in 1959 when Cluin [14] studied meromorphic schlicht functions. In 1963, Pommerenke [15] defined a class of meromorphic starlike functions and investigated coefficient estimates, and in [16], Royste studied meromorphic starlike multivalent functions for the first time and also found the same type of coefficient problems for the class of meromorphic starlike multivalent functions. In 1970, Miller [17] defined a class of meromorphic convex functions and investigated some generalized coefficient problems and other useful characteristics of meromorphic convex functions.

Cho and Owa [18] examined the partial sum for meromorphic p-valent functions, while Aouf et al. [19] determined a class of meromorphic p-valent functions and investigated the partial sums for meromorphic p-valent functions. In 2004, Srivastava [20] suggested some new classes of meromorphic multivalent functions and described some helpful features of meromorphic functions. Frasin and Maslina [21] investigated positive coefficients for a class of meromorphic functions.

A function $\varphi(z)$ is said to be in the class $P[F,\mathcal{K}]$ if it is analytic in $\mathbb{U}^*$ with $\varphi(z) = 1$ and

$$\varphi(z) \prec \frac{1+Fz}{1+\mathcal{K}z},$$

Equivalently, we can write

$$\left|\frac{\varphi(z)-1}{F-\mathcal{K}\varphi(z)}\right| < 1.$$

Recalling certain definitions of the q-calculus operator theory would be helpful because they are essential for understanding this article. Unless otherwise stated, we assume the following throughout the article:

$$q \in (0,1), \quad -1 \leq \mathcal{K} < F \leq 1, \quad \text{and} \quad p \in \mathbb{N}.$$

Definition 1 ([22]). *The q-number $[\zeta]_q$ is defined by*

$$[\zeta]_q = \begin{cases} \frac{1-q^\zeta}{1-q}, & (\zeta \in \mathbb{C}), \\ \sum_{k=0}^{i-1} q^k, & (\zeta = i \in \mathbb{N}), \end{cases}$$

and for any non-negative integer i, we have

$$[i]_q! = \begin{cases} [i]_q[i-1]_q[i-2]_q ... [2]_q[1]_q, & i \geq 1, \\ 1, & i = 0. \end{cases}$$

Definition 2 ([2,3]). *Let $\mathcal{A}$ be the set of all analytic functions h in the open unit disk*

$$\mathbb{U} = \{\varsigma : \varsigma \in \mathbb{C} \quad \text{and} \quad |\varsigma| < 1\}$$

and have the following series representation.

$$h(\varsigma) = \varsigma + \sum_{i=2}^{\infty} a_i \varsigma^i.$$

The q-derivative (or q-difference) D_q is defined by

$$\left(D_q h\right)(\varsigma) = \begin{cases} \frac{h(\varsigma)-h(q\varsigma)}{(1-q)\varsigma}, & (\varsigma \neq 0), \\ h'(0), & (\varsigma = 0). \end{cases} \tag{3}$$

Equation (3) shows that if h is differentiable at ς, then

$$\lim_{q\to 1-} \left(D_q h\right)(\varsigma) = h'(\varsigma).$$

For $h \in \mathcal{A}$, and from Equation (3), we have

$$(D_q h)(\varsigma) = 1 + \sum_{i=2}^{\infty} [i]_q a_i \varsigma^{i-1}.$$

Definition 3 ([23])**.** *The Sălăgean q-differential operator for $h \in \mathcal{A}$ is defined by*

$$\begin{aligned} \mathcal{S}_q^0 h(\varsigma) &= h(\varsigma),\ \mathcal{S}_q^1 h(\varsigma) = \varsigma D_q h(\varsigma) = \frac{h(q\varsigma) - h(\varsigma)}{q-1}, \cdots, \\ \mathcal{S}_q^m h(\varsigma) &= \varsigma D_q \Big(\mathcal{S}_q^{m-1} h(\varsigma) \Big) = h(\varsigma) * \left(\varsigma + \sum_{i=2}^{\infty} [i]_q^m \varsigma^i \right), \\ &= \varsigma + \sum_{i=2}^{\infty} [i]_q^m a_i \varsigma^i. \end{aligned}$$

Mahmood et al. extended the concept of the q-difference operator for $h \in \mathcal{M}$ and constructed a new subclass $\mathcal{MS}_q^*[F, \mathcal{K}]$ of meromorphic functions using the analogue of Definition 2:

Definition 4 ([24])**.** *For $h \in \mathcal{M}$, the q-derivative (or q-difference) D_q is defined by*

$$(D_q h)(\varsigma) = \frac{h(\varsigma) - h(q\varsigma)}{(1-q)\varsigma}. \tag{4}$$

For $h \in \mathcal{M}$, and from Equation (4), we have

$$(D_q h)(\varsigma) = \frac{-1}{q\varsigma^2} + \sum_{i=1}^{\infty} [i]_q a_i \varsigma^{i-1},\ \forall \varsigma \in \mathcal{U}^*. \tag{5}$$

Using Equations (1) and (4), we extend the idea of the Sălăgean q-differential operator for meromorphic functions as follows:

Definition 5. *Let $h \in \mathcal{M}$. Then, the Sălăgean q-differential operator for a meromorphic function is given by*

$$\begin{aligned} \mathcal{S}_q^0 h(\varsigma) &= h(\varsigma),\ \mathcal{S}_q^1 h(\varsigma) = D_q h(\varsigma) = \frac{h(q\varsigma) - h(\varsigma)}{(q-1)\varsigma}, \\ &\cdots \\ \mathcal{S}_q^m h(\varsigma) &= D_q \Big(\mathcal{S}_q^{m-1} h(\varsigma) \Big) \\ \mathcal{S}_q^m h(\varsigma) &= \frac{-1}{q\varsigma^2} + \sum_{i=1}^{\infty} [i]_q^m a_i \varsigma^{i-1}. \end{aligned} \tag{6}$$

Definition 6. *Let h be a meromorphic multivalent function given by Equation (1). Then, the Sălăgean q-differential operator is given by*

$$\begin{aligned} \mathcal{S}_{q,p}^0 h(\varsigma) &= h(\varsigma),\ \mathcal{S}_{q,p}^1 h(\varsigma) = D_q h(\varsigma) = \frac{h(\varsigma) - h(q\varsigma)}{(1-q)\varsigma}, \\ &\cdots \\ \mathcal{S}_{q,p}^m h(\varsigma) &= D_q \Big(\mathcal{S}_{q,p}^{m-1} h(\varsigma) \Big) \\ \mathcal{S}_{q,p}^m h(\varsigma) &= \frac{-1}{q^p \varsigma^{p+1}} + \sum_{i=0}^{\infty} [i+p]_q^m a_{i+p} \varsigma^{i+p-1}. \end{aligned} \tag{7}$$

Remark 1. *By taking $p = 1$ in Equation (7), then we have the Sălăgean q-differential operator for $h \in \mathcal{M}$, which is given by Equation (6).*

In the case of the recently introduced Sălăgean q-differential operator $h \in \mathcal{M}$, we introduce a novel subclass of meromorphic q-starlike functions connected to Janowski functions.

Definition 7. *A function $h \in \mathcal{M}$ belongs to the class $\mathcal{MS}_q^*[m, F, \mathcal{K}]$ if*

$$\left| \frac{(\mathcal{K}-1)\left(-\frac{\varsigma(\mathcal{S}_q^m h)(\varsigma)}{h(\varsigma)}\right)-(F-1)}{(\mathcal{K}+1)\left(-\frac{\varsigma(\mathcal{S}_q^m h)(\varsigma)}{h(\varsigma)}\right)-(F+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

We provide a novel subclass of meromorphic q-starlike functions connected to Janowski functions in the context of the recently introduced Sălăgean q-differential operator $h \in \mathcal{M}(p)$.

Definition 8. *A function $h \in \mathcal{M}(p)$ belongs to the class $\mathcal{MS}_{q,p}^*[m, F, \mathcal{K}]$ if*

$$\left| \frac{(\mathcal{K}-1)\left(-\frac{\varsigma(\mathcal{S}_{q,p}^m h)(\varsigma)}{h(\varsigma)}\right)-(F-1)}{(\mathcal{K}+1)\left(-\frac{\varsigma(\mathcal{S}_{q,p}^m h)(\varsigma)}{h(\varsigma)}\right)-(F+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Remark 2. *It can be easily observed that*

$$\mathcal{MS}_{q,1}^*(1, F, \mathcal{K}) = \mathcal{MS}_q^*(F, \mathcal{K}),$$

which was introduced and studied by Mahmood et al. [24].

Remark 3. *It is clear that*

$$\lim_{q \to 1-} \mathcal{MS}_{q,1}^*[m, F, \mathcal{K}] = \mathcal{MS}^*[F, \mathcal{K}],$$

which was introduced and studied by Ali et al. [25].

Remark 4. *For $q \to 1-$, $m = 1$, $F = 1$, and K$= -1$, then*

$$\lim_{q \to 1-} \mathcal{MS}_{q,1}^*[1, -1] = \mathcal{MS}^*,$$

where $\mathcal{MS}^*$ denotes the class of meromorphic starlike function.

The sufficient condition for $h \in \mathcal{MS}_{q,p}^*[m, F, \mathcal{K}]$ is examined in Theorem 1, which can be used as a supporting result to research further findings. We will also look into the relationship between a function h of the type (Equation (1)) and the partial sums of its series

$$h_k(\varsigma) = \frac{1}{\varsigma^p} + \sum_{i=0}^{k} a_{i+p}\varsigma^{i+p}, \qquad (k \in \mathbb{N}), \tag{8}$$

when the coefficients are sufficiently small.

2. Main Results

2.1. Sufficient Condition

Theorem 1. *If a function $h \in \mathcal{M}(p)$ of the form in Equation (1) satisfies the following condition, then $h \in \mathcal{MS}^*_{q,p}[m,F,\mathcal{K}]$:*

$$\begin{aligned} &\sum_{i=0}^{\infty} 2\Big([i+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[i+p]_q^m-(F-1)\Big|q^p|a_{i+p}| \\ \leq\ & |(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p). \end{aligned} \tag{9}$$

Proof. Supposing that Equation (9) is satisfied, then it is enough to prove that

$$\left|\frac{(\mathcal{K}-1)\left(-\frac{\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)}{h(\varsigma)}\right)-(F-1)}{(\mathcal{K}+1)\left(-\frac{\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)}{h(\varsigma)}\right)-(F+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

Now, we have

$$\begin{aligned} &\left|\frac{(\mathcal{K}-1)\left(-\frac{\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)}{h(\varsigma)}\right)-(F-1)}{(\mathcal{K}+1)\left(-\frac{\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)}{h(\varsigma)}\right)-(F+1)}-\frac{1}{1-q}\right| \\ =\ &\left|\frac{(\mathcal{K}-1)\left(-\frac{\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)}{h(\varsigma)}\right)-(F-1)}{(\mathcal{K}+1)\left(-\frac{\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)}{h(\varsigma)}\right)-(F+1)}-\frac{1+q-q}{1-q}\right| \end{aligned}$$

$$\begin{aligned} &=\left|\frac{-(\mathcal{K}-1)\varsigma\left(\mathcal{S}^m_{q,p}h\right)(\varsigma)-(F-1)h(\varsigma)}{-(\mathcal{K}+1)\varsigma\left(\mathcal{S}^m_{q,p}h\right)(\varsigma)-(F+1)h(\varsigma)}-1-\frac{q}{1-q}\right| \\ &\leq\left|\frac{-(\mathcal{K}-1)\varsigma\left(\mathcal{S}^m_{q,p}h\right)(\varsigma)-(F-1)h(\varsigma)}{-(\mathcal{K}+1)\varsigma\left(\mathcal{S}^m_{q,p}h\right)(\varsigma)-(F+1)h(\varsigma)}-1\right|+\frac{q}{1-q} \end{aligned}$$

$$\begin{aligned} &=\ 2\left|\frac{\varsigma\left(\mathcal{S}^m_{q,p}h\right)(\varsigma)+h(\varsigma)}{-(\mathcal{K}+1)\varsigma(\mathcal{S}^m_{q,p}h)(\varsigma)-(F+1)h(\varsigma)}\right|+\frac{q}{1-q} \\ &=\ 2\left|\frac{\left(1-\frac{1}{q^p}\right)+\sum\limits_{i=0}^{\infty}\left(1+[i+p]_q^m\right)a_{i+p}\varsigma^{i+p}}{(\mathcal{K}+1)\frac{1}{q^p}-(F+1)-\sum\limits_{i=0}^{\infty}\left((\mathcal{K}+1)[i+p]_q^m-(F-1)\right)a_{i+p}\varsigma^{i+p}}\right|+\frac{q}{1-q} \end{aligned}$$

$$\begin{aligned} &=\ 2\left|\frac{\frac{(q^p-1)}{q^p}+\sum\limits_{i=0}^{\infty}\left(1+[i+p]_q^m\right)a_{i+p}\varsigma^{i+p}}{(\mathcal{K}+1)\frac{1}{q^p}-(F+1)-\sum\limits_{i=0}^{\infty}\left((\mathcal{K}+1)[i+p]_q^m+(F+1)\right)a_{i+p}\varsigma^{i+p}}\right|+\frac{q}{1-q} \\ &\leq\ 2\left(\frac{|q^p-1|+\sum\limits_{i=0}^{\infty}\left(1+[i+p]_q^m\right)q^p\left|a_{i+p}\right|}{|(\mathcal{K}+1)-(F+1)q^p|-\sum\limits_{i=0}^{\infty}\left|\{(\mathcal{K}+1)[i+p]_q^m-(F-1)\}q^p\right|\left|a_{i+p}\right|}\right)+\frac{q}{1-q}. \end{aligned} \tag{10}$$

The inequality in Equation (10) is bounded by $\frac{1}{1-q}$ if

$$\begin{aligned}&\sum_{i=0}^{\infty} 2\Big([i+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[i+p]_q^m-(F-1)\Big|q^p|a_{i+p}|\\ <\quad & |(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p).\end{aligned}$$

Thus, this completes the proof of Theorem 1. □

Corollary 1. *If a function $h\in\mathcal{M}(p)$ of the form in Equation (1) belongs to the class $\mathcal{MS}^*_{q,p}[m,F,\mathcal{K}]$, then*

$$a_{i+p}\le\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([i+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[i+p]_q^m-(F-1)\Big|q^p},\quad (i\in\mathbb{N}).\tag{11}$$

This equality will satisfy the function

$$h_i(\varsigma)=\frac{1}{\varsigma^p}+\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([i+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[i+p]_q^m-(F-1)\Big|q^p}\varsigma^{i+p-1}.$$

Theorem 2. *If a function $h\in\mathcal{M}$ of the form given in Equation (2) satisfies the following condition, then $h\in\mathcal{MS}^*_q[m,F,\mathcal{K}]$:*

$$\begin{aligned}&\sum_{i=0}^{\infty} 2\Big([i+1]_q^m+1\Big)+\Big|(\mathcal{K}+1)[i+1]_q^m-(F-1)\Big|q|a_{i+p}|\\ \le\quad & |(\mathcal{K}+1)-(F+1)q|+2(1-q).\end{aligned}\tag{12}$$

By taking $p=1$ and $m=1$ in Theorem 1, then we have following known result, which was introduced in [24]:

Corollary 2 ([24]). *If a function $h\in\mathcal{M}$ of the form in Equation (1) satisfies the following condition, then $h\in\mathcal{MS}^*_q[F,\mathcal{K}]$:*

$$\sum_{i=1}^{\infty}\Lambda(i,F,\mathcal{K},q)|a_i|\le\Upsilon(F,\mathcal{K},q),$$

where

$$\Lambda(i,F,\mathcal{K},q)=2\Big([i]_q+1\Big)+\Big|(\mathcal{K}+1)[i]_q-(F-1)\Big|q$$

and

$$\Upsilon(F,\mathcal{K},q)=|(\mathcal{K}+1)-(F+1)q|+2(1-q).$$

2.2. Distortion Inequalities

Theorem 3. *If $h\in\mathcal{MS}^*_{q,p}[m,F,\mathcal{K}]$, then*

$$\begin{aligned}&\frac{1}{r^p}-\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([1+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[1+p]_q^m-(F-1)\Big|q^p}r^p\\ \le\quad & |h(\varsigma)|\le\frac{1}{r^p}+\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([1+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[1+p]_q^m-(F-1)\Big|q^p}r^p.\end{aligned}$$

This equality holds for the function

$$h(\varsigma)=\frac{1}{\varsigma^p}+\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([1+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[1+p]_q^m-(F-1)\Big|q^p}\varsigma^p\quad at\ \varsigma=ir.$$

Proof. Let $h \in \mathcal{MS}^*_{q,p}[m,F,\mathcal{K}]$. Then, in light of Theorem 1, we have

$$\begin{aligned}
&2\Big([1+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[1+p]_q^m-(F-1)\Big|q^p\sum_{i=1}^{\infty}|a_{i+p}| \\
\leq\ & \sum_{i=1}^{\infty}2\Big([i+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[i+p]_q^m-(F-1)\Big|q^p|a_{i+p}| \\
<\ & |(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p),
\end{aligned}$$

which yields

$$\begin{aligned}
|h(\varsigma)| &\leq \frac{1}{r^p}+\sum_{i=1}^{\infty}|a_{i+p}|r^{i-p}\leq\frac{1}{r^p}+r^p\sum_{i=1}^{\infty}|a_{i+p}| \\
&\leq \frac{1}{r^p}+\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([1+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[1+p]_q^m-(F-1)\Big|q^p}r^p.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|h(\varsigma)| &\geq \frac{1}{r^p}-\sum_{i=1}^{\infty}|a_{i+p}|r^{i-p} \\
&\geq \frac{1}{r^p}-r^p\sum_{i=1}^{\infty}|a_{i+p}| \\
&\geq \frac{1}{r^p}-\frac{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}{2\Big([1+p]_q^m+1\Big)+\Big|(\mathcal{K}+1)[1+p]_q^m-(F-1)\Big|q^p}r^p.
\end{aligned}$$

Thus, this completes the proof of Theorem 3. □

Theorem 4. *If a function h of the form in Equation (2) belongs to the class $\mathcal{MS}^*_q[m,F,\mathcal{K}]$, then*

$$\begin{aligned}
&\frac{1}{r}-\frac{|(\mathcal{K}+1)-(F+1)q|+2(1-q)}{2\Big([2]_q^m+1\Big)+\Big|(\mathcal{K}+1)[2]_q^m-(F-1)\Big|q}r \\
\leq\ & |h(\varsigma)|\leq\frac{1}{r}+\frac{|(\mathcal{K}+1)-(F+1)q|+2(1-q)}{2\Big([2]_q^m+1\Big)+\Big|(\mathcal{K}+1)[2]_q^m-(F-1)\Big|q}r.
\end{aligned}$$

This equality holds for the function

$$h(\varsigma)=\frac{1}{\varsigma}+\frac{|(\mathcal{K}+1)-(F+1)q|+2(1-q)}{2\Big([2]_q^m+1\Big)+\Big|(\mathcal{K}+1)[2]_q^m-(F-1)\Big|q}\varsigma \quad \text{at } \varsigma=ir.$$

Proof. Here, we omit the proof of Theorem 4. It is similar to that of the proof of Theorem 3. □

For $p=1$ and $m=1$ in Theorem 3, then we have the known corollary given in [24]:

Corollary 3 ([24])**.** *If $h\in\mathcal{MS}^*_q[F,\mathcal{K}]$, then*

$$\begin{aligned}
&\frac{1}{r}-\frac{|(\mathcal{K}+1)-(F+1)q|+2(1-q)}{2([2]_q+1)+|(\mathcal{K}+1)-(F-1)|q}r \\
\leq\ & |h(\varsigma)|\leq\frac{1}{r}+\frac{(\mathcal{K}+1)-(F+1)(1-q)}{2([2]_q+1)+|(\mathcal{K}+1)[2]_q-(F-1)|q}r.
\end{aligned}$$

This equality holds for the function

$$h(\varsigma) = \frac{1}{\varsigma} + \frac{|(\mathcal{K}+1)-(F+1)q| + 2(1-q)}{2([2]_q+1) + |(\mathcal{K}+1)[2]_q - (F-1)|q}\varsigma \quad at\ \varsigma = ir.$$

Theorem 5. *If $h \in \mathcal{MS}^*_{q,p}[m,F,\mathcal{K}]$, then*

$$\begin{aligned} &\frac{1}{r^{p+1}} - \frac{(p+1)|(\mathcal{K}+1)-(F+1)q^p| + 2(1-q^p)}{2\left([1+p]_q^m+1\right) + \left|(\mathcal{K}+1)[1+p]_q^m - (F-1)\right|q^p} \\ \leq\ & |h(\varsigma)| \leq \frac{1}{r^{p+1}} + \frac{(p+1)|(\mathcal{K}+1)-(F+1)q^p| + 2(1-q^p)}{2\left([1+p]_q^m+1\right) + \left|(\mathcal{K}+1)[1+p]_q^m - (F-1)\right|q^p}, \quad (|\varsigma| = r). \end{aligned}$$

Proof. Here, we omit the proof of Theorem 5. Its proof is similar to that of the proof Theorem 3. □

For $p = 1$ and $m = 1$, then we have a known corollary introduced in [24]:

Corollary 4 ([24]). *If $h \in \mathcal{MS}^*_q[F,\mathcal{K}]$, then*

$$\begin{aligned} &\frac{1}{r^2} - \frac{2|(\mathcal{K}+1)-(F+1)q| + 2(1-q)}{2([2]_q+1) + |(\mathcal{K}+1)[2]_q - (F-1)|q} \\ \leq\ & |h'(\varsigma)| \leq \frac{1}{r^2} + \frac{2|(\mathcal{K}+1)-(F+1)q| + 2(1-q)}{2([2]_q+1) + |(\mathcal{K}+1)[2]_q - (F-1)|q}, \quad (|\varsigma| = r). \end{aligned}$$

2.3. Partial Sums for the Function Class $\mathcal{MS}^*_{q,p}[m,F,\mathcal{K}]$

In this section, we study the ratio of a function of the form in Equation (1) to its sequence of partial sums

$$h_k(\varsigma) = \frac{1}{\varsigma^p} + \sum_{i=0}^{k} a_{i+p}\varsigma^{i+p}$$

when the coefficients of h are sufficiently small to satisfy the condition in Equation (9). We will investigate the sharp lower bounds for

$$Re\left(\frac{h(\varsigma)}{h_k(\varsigma)}\right), \left(\frac{h_k(\varsigma)}{h(\varsigma)}\right), Re\left(\frac{\mathcal{S}^m_{q,p}h(\varsigma)}{\mathcal{S}^m_{q,p}h_k(\varsigma)}\right) \text{ and } Re\left(\frac{\mathcal{S}^m_{q,p}h_k(\varsigma)}{\mathcal{S}^m_{q,p}h(\varsigma)}\right).$$

The sequence of partial sums of h_k is denoted by

$$h_k(\varsigma) = \frac{1}{\varsigma^p} + \sum_{i=0}^{k} a_{i+p}\varsigma^{i+p}.$$

Theorem 6. *If a function $h \in \mathcal{M}(p)$ of the form in Equation (1) satisfies the condition in Equation (9), then*

$$Re\left(\frac{h(\varsigma)}{h_k(\varsigma)}\right) \geq 1 - \frac{1}{\chi_{k+p+1}} \quad (\forall \varsigma \in \mathbb{U}) \tag{13}$$

and

$$Re\left(\frac{h_k(\varsigma)}{h(\varsigma)}\right) \geq \frac{\chi_{k+p+1}}{1+\chi_{k+p+1}}, \quad (\forall \varsigma \in \mathbb{U}), \tag{14}$$

where

$$\chi_{k+p} = \frac{2\left([k+p]_q^m+1\right) + \left|(\mathcal{K}+1)[k+p]_q^m - (F-1)\right|q^p}{|(\mathcal{K}+1)-(F+1)q^p| + 2(1-q^p)}. \tag{15}$$

Proof. For the proof of the inequality in Equation (13), we set

$$\begin{aligned}
&\chi_{k+p+1}\left[\frac{h(\varsigma)}{h_j(\varsigma)} - \left(1 - \frac{1}{\chi_{k+p+1}}\right)\right] \\
&= \frac{1 + \sum\limits_{i=0}^{k} a_{i+p}\varsigma^{i+p-1} + \chi_{k+p+1}\sum\limits_{i=k+1}^{\infty} a_{i+p}\varsigma^{i+p+1}}{1 + \sum\limits_{i=0}^{k} a_{i+p}\varsigma^{i+p+1}} \\
&= \frac{1 + q_1(\varsigma)}{1 + q_2(\varsigma)}.
\end{aligned}$$

If we fix

$$\frac{1 + q_1(\varsigma)}{1 + q_2(\varsigma)} = \frac{1 + w(\varsigma)}{1 - w(\varsigma)},$$

then after some simplification, we obtain

$$w(\varsigma) = \frac{q_1(\varsigma) - q_2(\varsigma)}{2 + q_1(\varsigma) + q_2(\varsigma)}.$$

We find that

$$w(\varsigma) = \frac{\chi_{k+p+1}\sum\limits_{i=k+1}^{\infty} a_{i+p}\varsigma^{i+p-1}}{2 + 2\sum\limits_{i=0}^{k} a_{i+p}\varsigma^{i+p+1} + \chi_{k+p+1}\sum\limits_{i=k+1}^{\infty} a_{i+p}\varsigma^{i+p+1}}$$

and

$$|w(\varsigma)| \leq \frac{\chi_{k+p+1}\sum\limits_{i=k+1}^{\infty} |a_{i+p}|}{2 - 2\sum\limits_{i=0}^{k} |a_{i+p}| - \chi_{k+p+1}\sum\limits_{i=k+1}^{\infty} |a_{i+p}|}.$$

Now, one can see that

$$|w(\varsigma)| \leq 1$$

if and only if

$$2\chi_{k+p+1}\sum_{i=k+1}^{\infty} |a_{i+p}| \leq 2 - 2\sum_{i=0}^{k} |a_{i+p}|,$$

which implies that

$$\sum_{i=0}^{k} |a_{i+p}| + \chi_{k+p+1}\sum_{i=k+1}^{\infty} |a_{i+p}| \leq 1. \tag{16}$$

Finally, to prove Equation (13), it is enough to show that the L.H.S. of Equation (16) is bounded above by $\sum\limits_{i=0}^{\infty} \chi_{i+p}|a_{i+p}|$, which is equal to

$$\sum_{i=0}^{k} (1 - \chi_{i+p})|a_{i+p}| + \sum_{i=k+1}^{\infty} \left(\chi_{k+p+1} - \chi_{i+p}\right)|a_{i+p}| \geq 0. \tag{17}$$

Hence, the proof of the inequality in Equation (13) is complete.

For the proof of the inequality in Equation (14), we fix

$$\begin{aligned}
&\left(1+\chi_{k+p}\right)\left(\frac{h_k(\varsigma)}{h(\varsigma)}-\frac{\chi_{k+p}}{1+\chi_{k+p}}\right)\\
&=\frac{1+\sum\limits_{i=0}^{k}a_{i+p}\varsigma^{i+p-1}-\chi_{k+p+1}\sum\limits_{i=k+1}^{\infty}a_{i+p}\varsigma^{i+p-1}}{1+\sum\limits_{i=0}^{\infty}a_{i+p}\varsigma^{i+p-1}}\\
&=\frac{1+w(\varsigma)}{1-w(\varsigma)},
\end{aligned}$$

where

$$|w(\varsigma)|\leq\frac{\left(1+\chi_{k+p+1}\right)\sum\limits_{i=k+1}^{\infty}|a_{i+p}|}{2-2\sum\limits_{i=0}^{k}|a_{i+p}|-\left(\chi_{k+p+1}-1\right)\sum\limits_{i=k+1}^{\infty}|a_{i+p}|}\leq 1. \tag{18}$$

The inequality in Equation (18) is equivalent to

$$\sum_{i=0}^{k}|a_{i+p}|+\chi_{k+p+1}\sum_{i=k+1}^{\infty}|a_{i+p}|\leq 1. \tag{19}$$

Finally, we can find that the L.H.S. in Equation (19) is bounded above by $\sum\limits_{i=0}^{\infty}\chi_{i+p}|a_{i+p}|$, and thus we have completed the inequality in Equation (14). Hence, the proof of Theorem 6 is complete. □

Theorem 7. *If $h\in\mathcal{M}(p)$ of the form in Equation (1) satisfies the condition in Equation (9), then*

$$Re\left(\frac{\mathcal{S}_{q,p}^{m}h(\varsigma)}{\mathcal{S}_{q,p}^{m}h_{p,k}(\varsigma)}\right)\geq 1-\frac{[k+p]_q^m}{\chi_{k+p+1}},\qquad(\forall\varsigma\in\mathbb{U})$$

and

$$Re\left(\frac{\mathcal{S}_{q,p}^{m}h_{p,k}(\varsigma)}{\mathcal{S}_{q,p}^{m}h(\varsigma)}\right)\geq\frac{\chi_{k+p+1}}{\chi_{k+p+1}+[k+p]_q^m},\qquad(\forall\varsigma\in\mathbb{U}),$$

where χ_{k+p} is given by Equation (15).

Proof. Here we omit the proof of Theorem 7. It is similar to that of Theorem 6. □

2.4. Partial Sums for the Function Class $MS_q^*[m,F,\mathcal{K}]$

We will study the ratio of a function of the form in Equation (1) to its sequence of partial sums

$$h_k(\varsigma)=\frac{1}{\varsigma}+\sum_{i=0}^{k}a_{i+1}\varsigma^{i+1}$$

when the coefficients of h are sufficiently small to satisfy the condition in Equation (9). We will investigate the sharp lower bounds for

$$Re\left(\frac{h(\varsigma)}{h_k(\varsigma)}\right),\ \left(\frac{h_k(\varsigma)}{h(\varsigma)}\right),\ Re\left(\frac{\mathcal{S}_q^{m}h(\varsigma)}{\mathcal{S}_q^{m}h_k(\varsigma)}\right)\ \text{and}\ Re\left(\frac{\mathcal{S}_q^{m}h_k(\varsigma)}{\mathcal{S}_q^{m}h(\varsigma)}\right).$$

The sequence of partial sums of h_k is denoted by

$$h_k(\varsigma) = \frac{1}{\varsigma} + \sum_{i=0}^{k} a_{i+1}\varsigma^{i+1}.$$

Theorem 8. *If we let $h \in \mathcal{M}$ of the form in Equation (2) satisfy the condition in Equation (12), then*

$$Re\left(\frac{h(\varsigma)}{h_k(\varsigma)}\right) \geq 1 - \frac{1}{\chi_{k+2}} \qquad (\forall \varsigma \in \mathbb{U})$$

and

$$Re\left(\frac{h_k(\varsigma)}{h(\varsigma)}\right) \geq \frac{\chi_{k+2}}{1+\chi_{k+2}} \qquad (\forall \varsigma \in \mathbb{U}),$$

where

$$\chi_{k+1} = \frac{2(1-\alpha)\left([k+1]_q^m + 1\right) + \left|(\mathcal{K}+1)[k+1]_q^m - (F-1)\right|q}{|(\mathcal{K}+1) - (F+1)q| + 2(1-q)}. \tag{20}$$

Proof. Here, we omit the proof for Theorem 8. It is similar to that of the proof for Theorem 7. □

Theorem 9. *If we let $h \in \mathcal{M}$ of the form in Equation (2) satisfy the condition in Equation (12), then*

$$Re\left(\frac{\mathcal{S}_q^m h(\varsigma)}{\mathcal{S}_q^m h_k(\varsigma)}\right) \geq 1 - \frac{[k+1]_q^m}{\chi_{k+2}}, \qquad (\forall \varsigma \in \mathbb{U})$$

and

$$Re\left(\frac{\mathcal{S}_q^m h_k(\varsigma)}{\mathcal{S}_q^m h(\varsigma)}\right) \geq \frac{\chi_{k+2}}{\chi_{k+2} + [k+1]_q^m}, \qquad (\forall \varsigma \in \mathbb{U}),$$

where χ_{k+1} is given by Equation (20).

Proof. Here, we omit the proof for Theorem 9. It is similar to that of the proof for Theorem 6. □

2.5. Radius of Starlikeness

In the next result, we obtain the radius of starlikeness for the class $\mathcal{MS}_{q,p}^*[m, F, \mathcal{K}]$:

Theorem 10. *Let the function h with Equation (1) belong to the class $\mathcal{MS}_{q,p}^*[m, F, \mathcal{K}]$. If*

$$\inf_{i \geq 1}\left[\frac{(1-\alpha)2\left([i+p]_q^m + 1\right) + \left|(\mathcal{K}+1)[i+p]_q^m - (F-1)\right|q^p}{(i+p+1-\alpha)|(\mathcal{K}+1) - (F+1)q^p| + 2(1-q^p)}\right]^{\frac{1}{i+p}} = r$$

is positive, then the function h is p-valently meromorphically starlike to the order α in $|\varsigma| \leq r$.

Proof. To prove the above result, we have to show that

$$\left|\frac{\varsigma h'(\varsigma)}{h(\varsigma)} + 1\right| \leq 1 - \alpha, \qquad (0 \leq \alpha < 1) \quad \text{and} \quad |\varsigma| \leq r_1.$$

From the above inequality, we have

$$\begin{aligned}\left|\frac{\varsigma h'(\varsigma)}{h(\varsigma)} + 1\right| &= \left|\frac{\sum_{i=0}^{\infty}(i+p+\alpha)a_{i+p}\varsigma^{i+p}}{\frac{1}{\varsigma^p} + \sum_{i=0}^{\infty} a_{i+p}\varsigma^{i+p}}\right| \\ &\leq \frac{\sum_{i=0}^{\infty}(i+p+\alpha)|a_{i+p}||\varsigma|^{i+p}}{1 - \sum_{i=0}^{\infty}|a_{i+p}||\varsigma|^{i+p}}.\end{aligned} \tag{21}$$

Hence, Equation (21) holds true if

$$\sum_{i=0}^{\infty}(i+p+\alpha)|a_{i+p}||\varsigma|^{i+p} \leq (1-\alpha)\left(1-\sum_{i=0}^{\infty}|a_{i+p}||\varsigma|^{i+p}\right). \tag{22}$$

Now, we can set the inequality in Equation (22) as follows:

$$\sum_{i=0}^{\infty}\left(\frac{i+p+1-\alpha}{1-\alpha}\right)|a_{i+p}||\varsigma|^{i+p} \leq 1. \tag{23}$$

With the help of Equation (9), the inequality in Equation (23) is true if

$$\begin{aligned}&\left(\frac{i+p+1-\alpha}{1-\alpha}\right)|\varsigma|^{i+p}\\ \leq\ &\frac{(1-\alpha)2\left([i+p]_q^m+1\right)+\left|(\mathcal{K}+1)[i+p]_q^m-(F-1)\right|q^p}{|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}.\end{aligned} \tag{24}$$

By solving Equation (24) for $|\varsigma|$, we have

$$|\varsigma| \leq \left(\frac{(1-\alpha)2\left([i+p]_q^m+1\right)+\left|(\mathcal{K}+1)[i+p]_q^m-(F-1)\right|q^p}{(i+p+1-\alpha)|(\mathcal{K}+1)-(F+1)q^p|+2(1-q^p)}\right)^{\frac{1}{i+p}}. \tag{25}$$

This completes the proof. □

3. Discussion

This section serves as an introduction to the conclusions section, we will specifically highlight the relevance of our primary findings and their applications. With a primary motive to consolidate the study of the famous convex function with starlike and convex functions, Govindaraj and Sivasubramanian in [23] involved the q-calculus operator and defined the Sălăgean q-differential operator for analytic functions. However, the meromorphic functions and meromorphic multivalent functions could not be defined with the other geometrically defined subclasses of $\mathcal{M}$ and $\mathcal{M}(p)$ using the same meromorphic q-analogue of the Sălăgean differential operator. For the functions in $\mathcal{M}$ and $\mathcal{M}(p)$, we smartly established a Sălăgean q-differential operator in this study so that normalization could be preserved.

When considering the Sălăgean q-differential operator for $h \in \mathcal{M}$, the family of functions $\mathcal{MS}_q^*[m,F,\mathcal{K}]$ (see Definition 7) is defined to include q-starlike functions, and the other family of functions $\mathcal{MS}_{q,p}^*[m,F,\mathcal{K}]$ (see Definition 8) is defined by using the Sălăgean q-differential operator for $h \in \mathcal{M}(p)$.

Another notable difference from earlier research is the fact that we found criteria for the classes of $\mathcal{MS}_q^*[m,F,\mathcal{K}]$ and $\mathcal{MS}_{q,p}^*[m,F,\mathcal{K}]$ that are more broadly applicable. Hence, if we let $p=1$ and $m=1$, then some of our results in Section 2 will reduce to results for the class of q-starlike functions introduced in [24]. The approach used by different authors in this paper in arriving at solutions to the challenges of the classes is the same. However, several novel and traditional results can be obtained as a special case of our main findings.

4. Conclusions

The extension and unification of various well-known classes of functions were the main objectives of this paper. In this article, we used the q-calculus operator theory, introduced the Sălăgean q-differential operator for meromorphic multivalent functions and defined two new subclasses of meromorphic multivalent functions in the Janowski domain. We investigated some interesting properties, such as coefficient estimates, partial sums, distortion theorems, and the radius of starlikeness. The technique and ideas of this paper may stimulate further research in the theory of multivalent meromorphic functions and further generalized classes of meromorphic functions can be defined and investigated for several other useful properties such as Hankal determinants, Feketo–Sezego problems, coefficient inequalities, growth problems, and many others.

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