fractal and fractional

Article

Quasilinear Fractional Order Equations and Fractional Powers of Sectorial Operators

Vladimir E. Fedorov [1], Marko Kostić [2,*] and Tatyana A. Zakharova [1]

[1] Department of Mathematical Analysis, Mathematics Faculty, Chelyabinsk State University, Kashirin Brothers Str. 129, 454001 Chelyabinsk, Russia; kar@csu.ru (V.E.F.); tanya_1997_smirnova@mail.ru (T.A.Z.)

[2] Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia

* Correspondence: marco.s@verat.net; Tel.: +381-641-289-790

Abstract: The fractional powers of generators for analytic operator semigroups are used for the proof of the existence and uniqueness of a solution of the Cauchy problem to a first order semilinear equation in a Banach space. Here, we use an analogous construction of fractional powers A^γ for an operator A such that $-A$ generates analytic resolving families of operators for a fractional order equation. Under the condition of local Lipschitz continuity with respect to the graph norm of A^γ for some $\gamma \in (0,1)$ of a nonlinear operator, we prove the local unique solvability of the Cauchy problem to a fractional order quasilinear equation in a Banach space with several Gerasimov–Caputo fractional derivatives in the nonlinear part. An analogous nonlocal Lipschitz condition is used to obtain a theorem of the nonlocal unique solvability of the Cauchy problem. Abstract results are applied to study an initial-boundary value problem for a time-fractional order nonlinear diffusion equation.

Keywords: fractional differential equation; fractional Gerasimov–Caputo derivative; the Cauchy problem; sectorial operator; fractional power of operator; initial-boundary value problem

MSC: 35R11; 34A08

Citation: Fedorov, V.E.; Kostić, M.; Zakharova, T.A. Quasilinear Fractional Order Equations and Fractional Powers of Sectorial Operators. *Fractal Fract.* **2023**, *7*, 385. https://doi.org/10.3390/fractalfract7050385

Academic Editor: Agnieszka B. Malinowska

Received: 12 April 2023
Revised: 30 April 2023
Accepted: 3 May 2023
Published: 5 May 2023

1. Introduction

Differential equations with fractional derivatives have attracted increasing interest among researchers over the last few decades, both from a theoretical point of view [1–6] and because of their importance for the study of many applied problems (see, e.g., [7–11]). In this paper, we study the quasilinear equation

$$D^\alpha z(t) + Az(t) = B(t, D^{\alpha_1} z(t), D^{\alpha_2} z(t), \ldots, D^{\alpha_n} z(t)) \tag{1}$$

in a Banach space $\mathcal{Z}$ with Gerasimov–Caputo fractional derivatives $D^\beta z$ with $\beta > 0$ and Riemann–Liouville fractional integrals $D^\beta z$ with $\beta \leq 0$. Here, $m-1 < \alpha \leq m \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$. A linear closed operator $-A$ in $\mathcal{Z}$ belongs to the class $\mathcal{A}_\alpha(\theta_0, a_0)$, which is introduced into the consideration in [12] for the study of linear Equation (1) (with $B \equiv 0$) by the methods of the resolving families of operators. The corresponding inhomogeneous linear equation ($B \equiv f(t)$) was researched in [13,14] in the cases of Hölderian or continuous forms in the graph norm of the operator A function f correspondingly. Equation (1) with a linear operator B, which is called a multi-term equation, was studied in [15] for the equation with bounded operators, and in [16] in the case of unbounded operators at the lower order fractional derivatives. The issues of the unique solvability for a class of nonlinear equations of the form (1) with an operator $-A \in \mathcal{A}_\alpha(\theta_0, a_0)$ in the linear part and with a nonlinear operator B, which is continuous in the graph norm of the operator

A and Lipschitz continuous with respect to the phase variables, is studied in [17]. Obtained results were used to investigate initial-boundary value problems for some nonlinear systems of partial differential equations modeling viscoelastic media thermoconvection.

However, the used conditions for the operator B in [17] do not allow general results to be applied for partial differential equations with spatial derivatives in the nonlinear part. In the operator semigroup theory [18–20], the consideration of integer order equations with such nonlinearities in the framework of first order equations in Banach spaces is possible due to using fractional powers A^γ, $\gamma \in (0,1)$, of a continuously invertible generator $-A$ of an analytic resolving semigroup of operators and spaces $\mathcal{Z}_\gamma$ as the domains of A^γ with the corresponding graph norms. If an operator B is locally Lipschitz continuous with respect to the norm in $\mathcal{Z}_\gamma$, the local existence of a unique solution of the Cauchy problem for a semilinear first order equation with the operator A in the linear part is proved. In [21], these results were extended to the case of the Cauchy type problem for Equation (1) with Riemann–Liouville fractional derivatives. To this end, fractional powers of an operator A, such that $-A \in \mathcal{A}_\alpha(\theta_0, a_0)$, were constructed and their properties were investigated. In this work, we use the results on fractional powers for the study of the Cauchy problem for semilinear Equation (1) with Gerasimov–Caputo derivatives and with a Lipschitz continuous with respect to the norm in $\mathcal{Z}_\gamma$ operator B, $\gamma \in (0,1)$. We use the abstract results to prove the existence of a unique solution of an initial-boundary value problem for a partial differential equation with a nonlinear part, which contains partial derivatives with respect to spatial variables.

Let us note the works [22–25], in which other approaches are used in the study of initial problems for nonlinear equations with fractional derivatives in Banach spaces.

The structure of this work is as follows. Section 2 contains preliminaries on sectorial operators and complex powers A^γ for such operators. Note that the auxiliary results obtained in [21] and listed here, including some estimates on the operators of resolving families and fractional powers of the operator generating them, are similar to the corresponding results of the theory of semigroups of operators but much more complicated in technical terms. In Section 3, the proof of the local unique solvability of the Cauchy problem to Equation (1) with a nonlinear operator B, which is locally Lipschitz continuous with respect to the norm in $\mathcal{Z}_\gamma$, is obtained. Section 4 contains an analogous result on the nonlocal existence of a unique solution for the Cauchy problem to Equation (1) with a Lipschitzian with respect to the norm in the $\mathcal{Z}_\gamma$ nonlinear operator. Abstract results are applied for the consideration of an initial boundary value problem for a time-fractional order nonlinear diffusion equation.

2. Complex Powers of a Fractional Sectorial Operator

Let $\mathcal{Z}$ be a Banach space. For $t_0 \in \mathbb{R}$, $h : (t_0, \infty) \to \mathbb{Z}$, the Riemann–Liouville integral of order $\beta > 0$ is

$$D^{-\beta}h(t) := J^\beta h := \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-s)^{\beta-1} h(s)ds, \quad t > t_0,$$

$J^0 h(t) := h(t)$. For $m \in \mathbb{N}$, $\beta \in (m-1, m]$ the Gerasimov–Caputo derivative of the order β has the form

$$D^\beta h(t) := D^m J^{m-\beta} \left(h(t) - \sum_{k=0}^{m-1} D^k h(t_0) \frac{(t-t_0)^k}{k!} \right), \quad t > t_0.$$

Let $h : \mathbb{R}_+ \to \mathcal{Z}$, $\omega \in \mathbb{R}$, $H : \{\mu \in \mathbb{C} : \mathrm{Re}\mu > \omega\} \to \mathcal{Z}$. We denote the Laplace transform by $\mathfrak{L}[h]$ and the inverse Laplace transform by $\mathfrak{L}^{-1}[H]$. For $\beta > 0$ it is known that (see, e.g., [1])

$$\mathfrak{L}[J^\beta h](\mu) = \mu^{-\beta} \mathfrak{L}[h](\mu), \quad \mathfrak{L}[D^\beta h](\mu) = \mu^\beta \mathfrak{L}[h](\mu) - \sum_{k=0}^{m-1} \mu^{\beta-1-k} D^k h(0).$$

Denote by $\mathcal{Cl}(\mathcal{Z})$ the set of all linear closed operators in a Banach space $\mathcal{Z}$, which are densely defined in $\mathcal{Z}$. Let $A \in \mathcal{Cl}(\mathcal{Z})$, denote by D_A the domain of A, which is endowed by the graph norm $\|\cdot\|_{D_A} = \|\cdot\|_{\mathcal{Z}} + \|A\cdot\|_{\mathcal{Z}}$; $\rho(A) := \{\mu \in \mathbb{C} : R_\mu(A) := (\mu I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$, $\sigma(A) := \mathbb{C} \setminus \rho(A)$, $S_{\theta_0,a_0} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a_0)| < \theta_0,\ \lambda \neq a_0\}$, $\Sigma_\varphi := \{\tau \in \mathbb{C} : |\arg\tau| < \varphi,\ \tau \neq 0\}$.

For some $\alpha > 0$, $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ denote by $\mathcal{A}_\alpha(\theta_0, a_0)$ a set of all operators $A \in \mathcal{Cl}(\mathcal{Z})$, such that the following hold:

(i) For all $\lambda \in S_{\theta_0,a_0}$ we have $\lambda^\alpha \in \rho(A)$;

(ii) For every $\theta \in (\pi/2, \theta_0)$, $a \geq a_0$ there exists $K = K(\theta, a) > 0$, such that

$$\forall \lambda \in S_{\theta,a} \quad \|R_{\lambda^\alpha}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta, a)}{|\lambda^{\alpha-1}(\lambda - a)|}.$$

Hereafter, for a power function, its main branch is taken.

Ii is known [12] that operators from $\mathcal{A}_\alpha(\theta_0, a_0)$ with $\alpha > 2$ are bounded. If $\alpha \in (0, 2)$, an operator $A \in \mathcal{A}_\alpha(\theta_0, a_0)$ is often called sectorial, and it generates an analytic in a sector $\Sigma_{\theta_0 - \pi/2}$ resolving the family of operators for the equation $D^\alpha z(t) = Az(t)$ [12].

Let $\alpha \in (0, 2)$, $-A \in \mathcal{A}_\alpha(\theta_0, 0)$ and $0 \in \rho(A)$; then, $\rho(-A)$ contains a neighborhood of zero cut along the negative semi-axis in which $R_{\lambda^\alpha}(-A)$ is bounded. Hence, for a small $a > 0$ and $\theta \in (\pi/2, \theta_0)$,

$$\exists r_1 > 0 \quad \exists K_1 > 0 \quad \forall \lambda \in S_{\theta,a} \cup \{\mu \in \mathbb{C} : |\mu| < r_1\} \quad \|R_{\lambda^\alpha}(-A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K_1}{1 + |\lambda|^\alpha}.$$

For a sufficiently small $\varepsilon > 0$ and for $\omega \in (\frac{\pi - \theta_0}{\alpha}, \frac{\pi}{2\alpha})$ denote a contour $\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_0 \cup \mathcal{C}_-$, which goes from top to bottom, where $\mathcal{C}_\pm := \{z = re^{\pm i\alpha\omega} : r \in (\alpha\varepsilon, \infty)\}$, $\mathcal{C}_0 := \{z = \alpha\varepsilon e^{i\varphi} : \varphi \in [-\alpha\omega, \alpha\omega]\}$, and operators

$$A^{-\gamma} := \frac{1}{2\pi i} \int_{\mathcal{C}} z^{-\gamma}(zI - A)^{-1} dz, \quad \mathrm{Re}\gamma > 0. \tag{2}$$

Since at some $K_2 > 0$ for all $z \in \mathcal{C}$

$$\|z^{-\gamma}(zI - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K_2}{|z|^{1+\mathrm{Re}\gamma}},$$

the integral (2) converges in the operator norm.

Lemma 1 ([21]). *Let $\alpha > 0$, $-A \in \mathcal{A}_\alpha(\theta_0, 0)$, $0 \in \rho(A)$. Then, for* Re$\gamma > 0$, *the operator $A^{-\gamma}$ is bounded and injective.*

For Re$\gamma > 0$, define the operator $A^\gamma := (A^{-\gamma})^{-1}$ with the domain $D_{A^\gamma} = \mathrm{im}A^{-\gamma} := \{y = A^{-\gamma}x : x \in \mathcal{Z}\}$. We also define the operator $A^0 := I$.

For Re$\gamma = 0$, define $A^\gamma := A^{\gamma-\beta}A^\beta$ with $D_{A^\gamma} := D_{A^\beta}$ for some $\beta > 0$. The independence of $\beta > 0$ of this definition can be proved easily (see [21]).

Theorem 1 ([21]). *Let $\alpha > 0$, $-A \in \mathcal{A}_\alpha(\theta_0, 0)$, $0 \in \rho(A)$. Then, the following hold:*

(i) *The family $\{A^{-\gamma} : \mathrm{Re}\gamma > 0\}$ forms an analytic semigroup, while for any $\theta \in (0, \pi/2)$, $z \in \mathcal{Z}$, we have the equality $\lim\limits_{\substack{\gamma \to 0 \\ |\arg\gamma| \leq \theta}} A^{-\gamma}z = z$;*

(ii) *For $\gamma \in \mathbb{C}$ A^γ is a closed operator;*

(iii) *If* Re$\gamma >$ Re$\beta \geq 0$, *then $D_{A^\gamma} \subset D_{A^\beta}$;*

(iv) *$\overline{D}_{A^\gamma} = \mathcal{Z}$ for every* Re$\gamma \geq 0$;

(v) *If $\gamma, \beta \in \mathbb{C}$, then $A^{\gamma+\beta}z = A^\gamma A^\beta z$ for every $z \in D_{A^\gamma} \cap D_{A^\beta} \cap D_{A^{\gamma+\beta}}$;*

(vi) *If* $0 < \text{Re}\gamma < 1$, $z \in D_A$, *then*

$$A^{\gamma}z = \frac{\sin \pi\gamma}{\pi} \int\limits_0^{\infty} t^{\gamma-1} A(tI + A)^{-1} z dt.$$

If $\alpha > 0$, $-A \in \mathcal{A}_{\alpha}(\theta_0, a_0)$, $\Gamma := \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_{\pm} := \{\mu \in \mathbb{C} : \mu = a + re^{\pm i\theta}, r \in [\delta, \infty)\}$, $\Gamma_0 := \{\mu \in \mathbb{C} : \mu = a + \delta e^{i\varphi}, \varphi \in (-\theta, \theta)\}$ for $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$, then the operators

$$Z_{\beta}(t) := \frac{1}{2\pi i} \int\limits_{\Gamma} \mu^{\alpha-1+\beta} R_{\mu^{\alpha}}(-A) e^{\mu t} d\mu, \quad t \in \mathbb{R}_+, \quad \beta \in \mathbb{R},$$

are defined [26]. These satisfy the inequalities for every $a > a_0$ (see [26]):

$$\|Z_{\beta}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_{\beta}(\theta, a) e^{at} (t^{-1} + a)^{\beta}, \quad t > 0, \quad \beta \geq 0, \tag{3}$$

$$\|Z_{\beta}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_{\beta}(\theta, a) e^{at} t^{-\beta}, \quad t > 0, \quad \beta < 0. \tag{4}$$

Theorem 2 ([21])**.** *Let* $\alpha > 0$, $-A \in \mathcal{A}_{\alpha}(\theta_0, a_0)$. *Then, for all* $\beta < 1$, $\delta < 1$, $s, t > 0$

$$Z_{\beta}(s) Z_{\delta}(t) = -\frac{1}{\alpha} Z_{\beta+\delta}(s+t) + \frac{t^{-\delta}}{2\pi i} \int\limits_{\Gamma} \mu^{\alpha-1+\beta} R_{\mu^{\alpha}}(-A) E_{\alpha,1-\delta}(\mu^{\alpha} t^{\alpha}) e^{\mu s} d\mu +$$

$$+ \frac{s^{-\beta}}{2\pi i} \int\limits_{\Gamma} \mu^{\alpha-1+\delta} R_{\mu^{\alpha}}(-A) E_{\alpha,1-\beta}(\mu^{\alpha} s^{\alpha}) e^{\mu t} d\mu.$$

It is known that for $\alpha = 1$, $\{Z_0(t) \in \mathcal{L}(\mathcal{Z}) : t \in \mathbb{R}_+\}$ is an analytic semigroup of operators [18–20,27,28]. Consider Theorem 2 $\alpha = 1$, $\beta = \delta = 0$ and obtain the semigroup property $Z_0(t) Z_0(s) = Z_0(t+s)$, $t, s > 0$. Thus, Theorem 2 gives some generalization of the semigroup property for resolving families of operators, which are generated by an operator from the class $\mathcal{A}_{\alpha}(\theta_0, a_0)$.

Theorem 3 ([21])**.** *Let* $\alpha > 0$, $-A \in \mathcal{A}_{\alpha}(\theta_0, 0)$, $0 \in \rho(A)$. *Then, the following hold:*

(i) $Z_{\beta}(t) : \mathcal{Z} \to D(A^{\gamma})$ *for all* $\beta \in \mathbb{R}$, $\text{Re}\gamma \in [0, 1)$, $t > 0$;

(ii) $Z_{\beta}(t) A^{\gamma} z = A^{\gamma} Z_{\beta}(t) z$ *for* $\beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $z \in D(A^{\gamma})$;

(iii) *For* $\beta \in \mathbb{R}$, $\text{Re}\gamma < 1$, $t > 0$ *the operator* $A^{\gamma} Z_{\beta}(t)$ *is bounded;*

(iv) *For* $\beta < 1$, $\text{Re}\gamma \in (0, 1)$

$$A^{-\gamma} = \frac{\alpha \sin \pi\gamma}{\sin(\pi(\alpha + \gamma\beta)) \Gamma(\alpha\gamma + \beta)} \int\limits_0^{\infty} t^{\alpha\gamma+\beta-1} Z_{\beta}(t) dt;$$

(v) *For* $\beta \in \mathbb{R}$, $t > 0$ $\|A Z_{\beta}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C t^{-\alpha-\beta}$;

(vi) *For* $\beta \in (-\alpha\text{Re}\gamma, 1)$, $\text{Re}\gamma \in (0, 1)$, $t > 0$ $\|A^{\gamma} Z_{\beta}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_{\gamma} t^{-\alpha\text{Re}\gamma-\beta}$;

(vii) *For* $\beta < 1$, $\text{Re}\gamma \in (0, 1)$, $z \in D(A^{\gamma})$

$$\|D^{-\beta} Z_{\beta}(t) z - z\|_{\mathcal{Z}} \leq C_{\gamma} t^{\alpha\text{Re}\gamma} \|A^{\gamma} z\|_{\mathcal{Z}}.$$

3. Local Solvability of Quasilinear Equation

Consider the Cauchy problem

$$D^k z(t_0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{5}$$

for a quasilinear equation

$$D^{\alpha}z(t) + Az(t) = B(t, D^{\alpha_1}z(t), D^{\alpha_2}z(t), \dots, D^{\alpha_n}z(t)), \tag{6}$$

where $m-1 < \alpha \leq m \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{Z}$, $l = 1, 2, \dots, n$. Some of α_l may be negative.

Let $\gamma \in (0,1)$, $\mathcal{Z}_\gamma := D_{A^\gamma}$ is a normed space with the norm $\|\cdot\|_\gamma := \|A^\gamma \cdot\|_{\mathcal{Z}}$. It is a Banach space, since A^γ is a continuously invertible closed operator. Let U be an open subset of $\mathbb{R} \times \mathcal{Z}_\gamma^n$, a mapping $B : U \to \mathcal{Z}$ is given; for every point $(t, x_1, x_2, \dots, x_n) \in U$, there exists its neighborhood $V \subset U$ and constants $C > 0$, $\delta \in (0,1]$ such that for all $(s, y_1, y_2, \dots, y_n), (t, v_1, v_2, \dots, v_n) \in V$

$$\|B(s, y_1, y_2, \dots, y_n) - B(t, v_1, v_2, \dots, v_n)\|_{\mathcal{Z}} \leq C\left(|s-t|^{\delta} + \sum_{l=1}^{n} \|y_l - v_l\|_\gamma\right). \tag{7}$$

A function $z \in C((t_0, t_1]; D_A)$, such that $z \in C^{m-1}([t_0, t_1]; \mathcal{Z})$, $D^\alpha z \in C((t_0, t_1]; \mathcal{Z})$, $D^{\alpha_1}z, D^{\alpha_2}z, \dots, D^{\alpha_n}z \in C([t_0, t_1]; \mathcal{Z})$, is called a solution of the Cauchy problem (5), (6) on a segment $[t_0, t_1]$, if it satisfies conditions (5) for all $t \in [t_0, t_1]$ $(D^{\alpha_1}z(t), D^{\alpha_2}z(t), \dots, D^{\alpha_n}z(t)) \in U$ and for all $t \in (t_0, t_1]$ equality (6) holds.

The next theorem on the unique solvability of the Cauchy problem for an inhomogeneous linear equation was proved in [13] for a Hölderian function $f \in C^\nu([t_0, T]; \mathcal{Z})$, $\nu \in (0,1]$, and for the case $f \in C([t_0, T]; \mathcal{Z}_1)$ in [14].

Theorem 4 ([13,14])**.** *Let $\alpha > 0$, $-A \in \mathcal{A}_\alpha(\theta_0, a_0)$, $f \in C([t_0, T]; \mathcal{Z}_1) \cup C^\nu([t_0, T]; \mathcal{Z})$, $\nu \in (0,1]$. Then for all $z_0, z_1, \dots, z_{m-1} \in D_A$ the function*

$$z(t) = \sum_{k=0}^{m-1} Z_{-k}(t-t_0)z_k + \int_{t_0}^{t} Z_{1-\alpha}(t-s)f(s)ds, \quad t > t_0,$$

is a unique solution of Cauchy problem (5) for the equation $D^\alpha z(t) + Az(t) = f(t)$.

Lemma 2 ([29])**.** *Let $p - 1 < \beta \leq p \in \mathbb{N}$. Then*

$$\exists C > 0 \quad \forall h \in C^p([t_0, t_1]; \mathcal{Z}) \quad \|D^\beta h\|_{C([t_0,t_1];\mathcal{Z})} \leq C\|h\|_{C^p([t_0,t_1];\mathcal{Z})}.$$

For $t_1 > t_0$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, define the space

$$C^{m-1,\{\alpha_l\}}([t_0, t_1]; \mathcal{Z}) := \{z \in C^{m-1}([t_0, t_1]; \mathcal{Z}) : D^{\alpha_l}z \in C([t_0, t_1]; \mathcal{Z}),\ l = 1, 2, \dots, n\}$$

and endow it by the norm

$$\|z\|_{C^{m-1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})} = \|z\|_{C^{m-1}([t_0,t_1];\mathcal{Z})} + \sum_{l=1}^{n} \|D^{\alpha_l}z\|_{C([t_0,t_1];\mathcal{Z})}.$$

Denote $l_{m-1} := \min\{l \in \{1, 2, \dots, n\} : \alpha_l > m-1\}$, if the set $\{l \in \{1, 2, \dots, n\} : \alpha_l > m-1\}$ is not empty, otherwise, $l_{m-1} := n+1$. Due to Lemma 2 the norm $\|z\|_{C^{m-1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})}$ is equivalent to

$$\|z\|_{C^{m-1}([t_0,t_1];\mathcal{Z})} + \sum_{l=l_{m-1}}^{n} \|D^{\alpha_l}z\|_{C([t_0,t_1];\mathcal{Z})}.$$

Hence, $C^{m-1,\{\alpha_l\}}([t_0, t_1]; \mathcal{Z}) = C^{m-1}([t_0, t_1]; \mathcal{Z})$, if and only if $\alpha_n \leq m-1$.

Lemma 3. *The normed space $C^{m-1,\{\alpha_l\}}([t_0, t_1]; \mathcal{Z})$ is complete.*

Proof. Take a fundamental sequence $\{x_p\}$ from the space $C^{m-1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$; then, there exist limits $x \in C^{m-1}([t_0,t_1];\mathcal{Z})$ for $\{x_p\}$ in the space $C^{m-1}([t_0,t_1];\mathcal{Z})$, y_l for the sequences $\{D^{\alpha_l}x_p\}$ in $C([t_0,t_1];\mathcal{Z})$, $l = 1,2,\ldots,n$. Hence, for $t \in [t_0,t_1]$ we have

$$J^{\alpha_l}y_l(t) = \lim_{p\to\infty} J^{\alpha_l}D^{\alpha_l}x_p(t) = \lim_{p\to\infty}\left(x_p(t) - \sum_{j=0}^{m_l-1} D^j x_p(t_0)\frac{(t-t_0)^j}{j!}\right) =$$

$$= x(t) - \sum_{j=0}^{m_l-1} D^j x(t_0)\frac{(t-t_0)^j}{j!}, \quad y_l = D^{\alpha_l}x \in C([t_0,t_1];\mathcal{Z}),\ l = 1,2,\ldots,n.$$

Thus, $C^{m-1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$ is a Banach space. □

Lemma 4. *Let $\beta \in (0,1)$, $h, D^\beta h \in C([t_0,t_1];\mathcal{Z})$. Then $h \in C^\beta([t_0,t_1]\mathcal{Z})$, moreover, there exists $C > 0$, such that for all $t,\tau \in [t_0,t_1]$*

$$\|h(t) - h(\tau)\| \le \frac{\|D^\beta h\|_{C([t_0,t_1];\mathcal{Z})}}{\Gamma(\beta+1)}|t-\tau|^\beta.$$

Proof. For $t_0 \le \tau < t \le t_1$,

$$\|h(t) - h(\tau)\|_{\mathcal{Z}} = \|J^\beta D^\beta h(t) - J^\beta D^\beta h(\tau)\|_{\mathcal{Z}} \le$$

$$\le \frac{(t-t_0)^\beta - (\tau-t_0)^\beta}{\Gamma(\beta+1)}\|D^\beta h\|_{C([t_0,t_1];\mathcal{Z})} \le \frac{(t-\tau)^\beta}{\Gamma(\beta+1)}\|D^\beta h\|_{C([t_0,t_1];\mathcal{Z})}.$$

Here, we use the decreasing function

$$\frac{(t-t_0)^\beta - (\tau-t_0)^\beta}{(t-\tau)^\beta}$$

of $\tau \in [t_0,t)$ for $\beta \in (0,1)$. □

Denote

$$\tilde{z}(t) := z_0 + (t-t_0)z_1 + \cdots + \frac{(t-t_0)^{m-1}}{(m-1)!}z_{m-1}, \quad \tilde{z}_l := D^{\alpha_l}\tilde{z}(t_0), \quad l = 1,2,\ldots,n.$$

If $\alpha_l = m_l = k \in \{0,1,\ldots,m-1\}$, then $\tilde{z}_l = z_k$, otherwise, $\tilde{z}_l = 0$, $l = 1,2,\ldots,n$.

Theorem 5. *Let $\alpha \in (1,2]$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $-A \in \mathcal{A}_\alpha(\theta_0,0)$, $0 \in \rho(A)$, a mapping $B : U \to \mathcal{Z}$ satisfies condition (7) with $\gamma \in (0,1)$, $z_0, z_1 \in \mathcal{Z}_{1+\gamma}$, $(t_0,\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_n) \in U$. Then, for some $t_1 > t_0$, there exists a unique solution of problems (5) and (6) on $[t_0,t_1]$.*

Proof. For $(t_0,\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_n) \in U$ choose $t_1 > t_0$ and $\varepsilon > 0$, such that on the set

$$V := \{(t,x_1,x_2,\ldots,x_n) \in \mathbb{R}\times\mathcal{Z}_\gamma^n : t \in [t_0,t_1],\ \|x_l - \tilde{z}_l\|_\gamma \le \varepsilon,\ l = 1,2,\ldots,n\}$$

inequality (7) holds with some $C > 0$, $\delta > 0$.

By the construction of $C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$, we have $\|D^{\alpha_l}x\|_{C([t_0,t_1];\mathcal{Z})} \le C\|x\|_{C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})}$, $l = 1,2,\ldots,n$. Therefore, a subset

$$S_{t_1} := \{x \in C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z}) : D^{\alpha_l}x(t_0) = A^\gamma\tilde{z}_l,\ \|D^{\alpha_l}x(t) - A^\gamma\tilde{z}_l\|_{\mathcal{Z}} \le \varepsilon,\ l = 1,2,\ldots,n\}$$

of the Banach space $C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$ is closed. Hence, S_{t_1} is a complete metric space with the metric $d(x,y) = \|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})}$. For $x \in S_{t_1}$, define a mapping

$$Fx(t) := \sum_{k=0}^{1} Z_{-k}(t-t_0)A^{\gamma}z_k + \int\limits_{t_0}^{t} A^{\gamma}Z_{1-\alpha}(t-s)B^x(s)ds,$$

where $B^x(s) := B(s, A^{-\gamma}D^{\alpha_1}x(s), A^{-\gamma}D^{\alpha_2}x(s), \ldots, A^{-\gamma}D^{\alpha_n}x(s))$.

In the proof of Theorem 4, it was shown that for $k = 0,1$ $D^kZ_{1-\alpha}(0) = 0$; hence, for $x \in \mathcal{S}_{t_1}$, $D^kFx(t_0) = A^{\gamma}z_k$, for $l = 1,2,\ldots,n$,

$$D^{\alpha_l}Fx(t) = \sum_{k=0}^{m_l-1} Z_{\alpha_l-k-\alpha}(t-t_0)A^{\gamma+1}z_k + \sum_{k=m_l}^{1} Z_{\alpha_l-k}(t-t_0)A^{\gamma}z_k +$$

$$+ \int\limits_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^x(s)ds.$$

Therefore, for $\alpha_l \geq 0$

$$D^{\alpha_l}Fx(t_0) = Z_{\alpha_l-m_l}(0)A^{\gamma}z_k + \int\limits_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^x(s)ds|_{t=t_0} = A^{\gamma}\tilde{z}_l,$$

since $\alpha_l - k - \alpha < 0$, $\alpha_l - k \leq 0$ for $k = m_l, 1$, $Z_{\alpha_l-m_l}(0) = 0$ for $\alpha_l < m_l$, $Z_{\alpha_l-m_l}(0) = I$ for $\alpha_l = m_l$. If $\alpha_l < 0$, then $D^{\alpha_l}Fx(t_0) = 0 = A^{\gamma}\tilde{z}_l$. Theorem 3 (vi) implies that

$$\left\|\int\limits_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^x(s)ds\right\|_{\mathcal{Z}} = \left\|\int\limits_{t_0}^{t} A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\alpha+\alpha_l}(t-s)(B^x(s)-\tilde{B}(s))ds\right\|_{\mathcal{Z}} +$$

$$+\left\|\int\limits_{t_0}^{t} A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\alpha+\alpha_l}(t-s)\tilde{B}(s)ds\right\|_{\mathcal{Z}} \leq$$

$$\leq C_1(C\varepsilon n + C_2)\int\limits_{t_0}^{t}(t-s)^{\alpha(1-\gamma-\delta_l)-\alpha_l-1}ds = C_3(t_1-t_0)^{\alpha(1-\gamma-\delta_l)-\alpha_l},$$

where $\delta_l \in (1-\gamma-\frac{\alpha_l+1}{\alpha}, 1-\gamma-\frac{\alpha_l}{\alpha})$ is chosen, since the mapping B is continuous, $\tilde{B}(s) := B(s,\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_n)$, $\|\tilde{B}(s)\|_{\mathcal{Z}} \leq C_2$ for $s \in [t_0,t_1]$. Thus, for every $x \in S_{t_1}$, we have $Fx \in S_{t_1}$, if t_1 is close enough to t_0. Note that we can choose t_1 regardless of x.

For $x,y \in S_{t_1}$, $t \in (t_0,t_1]$, $\delta_{2n+1} \in (1-\gamma-\frac{1}{\alpha}, 1-\gamma)$, $\delta_{2n+2} \in (1-\gamma-\frac{2}{\alpha}, 1-\gamma-\frac{1}{\alpha})$

$$\|Fx(t)-Fy(t)\|_{\mathcal{Z}} \leq \int\limits_{t_0}^{t} \|A^{-\delta_{2n+1}}A^{\gamma+\delta_{2n+1}}Z_{1-\alpha}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^x(s)-B^y(s)\|_{\mathcal{Z}}ds \leq$$

$$\leq C_1(t_1-t_0)^{\alpha(1-\gamma-\delta_{2n+1})}d(x,y) \leq \frac{1}{6}d(x,y),$$

$$\|D^1Fx(t)-D^1Fy(t)\|_{\mathcal{Z}} \leq \int\limits_{t_0}^{t} \|A^{-\delta_{2n+2}}A^{\gamma+\delta_{2n+2}}Z_{2-\alpha}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^x(s)-B^y(s)\|_{\mathcal{Z}}ds \leq$$

$$\leq C_1(t_1-t_0)^{\alpha(1-\gamma-\delta_{2n+2})-1}d(x,y) \leq \frac{1}{6}d(x,y),$$

$$\|D^{\alpha_l}Fx(t)-D^{\alpha_l}Fy(t)\|_{\mathcal{Z}}\leq\int\limits_{t_0}^{t}\|A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\alpha+\alpha_l}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^x(s)-B^y(s)\|_{\mathcal{Z}}ds\leq$$

$$\leq C_1(t_1-t_0)^{\alpha(1-\gamma-\delta_l)-\alpha_l}d(x,y)\leq\frac{1}{6n}d(x,y),\quad l=1,2,\dots,n.$$

Hence, $d(Fx,Fy)\leq\frac{1}{2}d(x,y)$ and by the Banach theorem, there exists a unique $y\in S$, such that $y(t)=Fy(t)$, $t\in[t_0,t_1]$.

Besides, for $t_0\leq\tau<t\leq t_1$, $\alpha_l\geq 0$

$$\|D^{\alpha_l}y(t)-D^{\alpha_l}y(\tau)\|_{\mathcal{Z}}=\|D^{\alpha_l}Fy(t)-D^{\alpha_l}Fy(\tau)\|_{\mathcal{Z}}\leq$$

$$\leq\sum_{k=0}^{m_l-1}\|Z_{\alpha_l-k-\alpha}(t-t_0)A^{\gamma+1}z_k-Z_{\alpha_l-k-\alpha}(\tau-t_0)A^{\gamma+1}z_k\|_{\mathcal{Z}}+$$

$$+\sum_{k=m_l}^{1}\|J^{\alpha}D^{\alpha}Z_{\alpha_l-k}(t-t_0)A^{\gamma}z_k-J^{\alpha}D^{\alpha}Z_{\alpha_l-k}(\tau-t_0)A^{\gamma}z_k\|_{\mathcal{Z}}+$$

$$+\left\|\int\limits_{t_0}^{t}A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^y(s)ds-\int\limits_{0}^{\tau}A^{\gamma}Z_{1-\alpha+\alpha_l}(\tau-s)B^y(s)ds\right\|_{\mathcal{Z}}\leq$$

$$\leq\sum_{k=0}^{1}\max_{t\in[t_0,t_1]}\|D^{\frac{\alpha-\alpha_l}{2}}Z_{\alpha_l-k-\alpha}(t-t_0)A^{\gamma+1}z_k\|_{\mathcal{Z}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}+$$

$$+\max_{t\in[t_0,t_1]}\left\|D^{\frac{\alpha-\alpha_l}{2}}\int\limits_{t_0}^{t}A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^y(s)ds\right\|_{\mathcal{Z}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}\leq$$

$$\leq\sum_{k=0}^{1}\max_{t\in[t_0,t_1]}\|Z_{-k-\frac{\alpha-\alpha_l}{2}}(t-t_0)A^{\gamma+1}z_k\|_{\mathcal{Z}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}+$$

$$+\max_{t\in[t_0,t_1]}\left\|\int\limits_{t_0}^{t}A^{-\delta_{n+l}}A^{\gamma+\delta_{n+l}}Z_{1-\frac{\alpha-\alpha_l}{2}}(t-s)B^y(s)ds\right\|_{\mathcal{Z}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}\leq$$

$$\leq C_1|t-\tau|^{\frac{\alpha-\alpha_l}{2}}+C_2(t_1-t_0)^{\frac{\alpha(1-2\gamma-2\delta_{n+l})-\alpha_l}{2}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}\leq C_3|t-\tau|^{\frac{\alpha-\alpha_l}{2}},$$

if we take $\delta_{n+l}\in\left(\frac{1-\frac{\alpha_l}{\alpha}}{2}-\frac{1}{\alpha}-\gamma,\frac{1-\frac{\alpha_l}{\alpha}}{2}-\gamma\right)$. Here, we used the Lagrange formula, inequalities (4) and Lemma 4. Partially,

$$\|Z_{\alpha_l-\alpha}(t-t_0)-Z_{\alpha_l-\alpha}(\tau-t_0)\|_{\mathcal{L}(\mathcal{Z})}\leq\max_{t\in[t_0,t_1]}\|D^{\frac{\alpha-\alpha_l}{2}}Z_{\alpha_l-\alpha}(t-t_0)\|_{\mathcal{L}(\mathcal{Z})}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}=$$

$$=\max_{t\in[t_0,t_1]}\|Z_{\frac{\alpha_l-\alpha}{2}}(t-t_0)\|_{\mathcal{L}(\mathcal{Z})}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}\leq C_1(t_1-t_0)^{\frac{\alpha-\alpha_l}{2}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}}\leq C_2|t-\tau|^{\frac{\alpha-\alpha_l}{2}}.$$

If $\alpha_l<0$, take $\beta_l=\min\{1,-\alpha_l\}$ and $\delta_{n+l}\in\left(1-\frac{1+\alpha_l+\beta_l}{\alpha}-\gamma,1-\frac{\alpha_l+\beta_l}{\alpha}-\gamma\right)$, then

$$\|D^{\alpha_l}y(t)-D^{\alpha_l}y(\tau)\|_{\mathcal{Z}}=\|D^{\alpha_l}Fy(t)-D^{\alpha_l}Fy(\tau)\|_{\mathcal{Z}}\leq$$

$$\leq\sum_{k=0}^{1}\|Z_{\alpha_l-k}(t-t_0)A^{\gamma}z_k-Z_{\alpha_l-k}(\tau-t_0)A^{\gamma}z_k\|_{\mathcal{Z}}+$$

$$+\left\|\int\limits_{t_0}^{t}A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^y(s)ds-\int\limits_{0}^{\tau}A^{\gamma}Z_{1-\alpha+\alpha_l}(\tau-s)B^y(s)ds\right\|_{\mathcal{Z}}\leq$$

$$\leq \sum_{k=0}^{1} \max_{t\in[t_0,t_1]} \|D^{\beta_l} Z_{\alpha_l-k}(t-t_0)A^{\gamma}z_k\|_{\mathcal{Z}}|t-\tau|^{\beta_l}+$$

$$+\max_{t\in[t_0,t_1]}\left\|D^{\beta_l}\int_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^{y}(s)ds\right\|_{\mathcal{Z}}|t-\tau|^{\beta_l}\leq$$

$$\leq \sum_{k=0}^{1} \max_{t\in[t_0,t_1]} \|Z_{\alpha_l+\beta_l-k}(t-t_0)A^{\gamma+1}z_k\|_{\mathcal{Z}}|t-\tau|^{\beta_l}+$$

$$+\max_{t\in[t_0,t_1]}\left\|\int_{t_0}^{t} A^{-\delta_{n+l}}A^{\gamma+\delta_{n+l}}Z_{1-\alpha+\alpha_l+\beta_l}(t-s)B^{y}(s)ds\right\|_{\mathcal{Z}}|t-\tau|^{\beta_l}\leq$$

$$\leq C_1|t-\tau|^{\beta_l}+C_2(t_1-t_0)^{\alpha(1-\gamma-\delta_{n+l})-\alpha_l-\beta_l}|t-\tau|^{\beta_l}\leq C_3|t-\tau|^{\beta_l}.$$

Therefore, for the fixed point y of the mapping F, we have $B^y \in C^{\nu}([t_0,t_1];\mathcal{Z})$ for some $\nu \in (0,\delta)$ due to condition (7).

Theorem 4 implies that a solution of (5), (6) is a function $z \in C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$, such that

$$z(t)=\sum_{k=0}^{m-1} Z_{-k}(t-t_0)z_k+\int_{t_0}^{t} Z_{1-\alpha}(t-s)B^{z}(s)ds=A^{-\gamma}FA^{\gamma}z(t)=A^{-\gamma}y(t), \tag{8}$$

where y is a fixed point of F. Inversely, if a function $z \in C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$ satisfies Equation (8), then $B^{A^{\gamma}z}(s)=B^{y}(s)=B^{Fy}(s)$ satisfies the Hölder condition, and due to Theorem 4. z is a solution of (5) and (6). Thus, a function $z \in C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$ is a solution of (5) and (6), if and only if $y=A^{\gamma}z$ is a fixed point of F, the existence and uniqueness of which is proved above. □

Theorem 6. *Let $\alpha \in (0,1]$, $\alpha_1<\alpha_2<\cdots<\alpha_n<\alpha$, $-A \in \mathcal{A}_{\alpha}(\theta_0,0)$, $0 \in \rho(A)$, a mapping $B : U \to \mathcal{Z}$ satisfy condition (7) with $\gamma \in (0,1)$, $z_0 \in \mathcal{Z}_{1+\gamma}$, $(t_0,\tilde{z}_1,\tilde{z}_2,\ldots,\tilde{z}_n) \in U$. Then, for some $t_1>t_0$, there exists a unique solution of problems (5) and (6) on $[t_0,t_1]$.*

Proof. Take $t_1>t_0$ and $\varepsilon>0$, such that on

$$V:=\{(t,x_1,x_2,\ldots,x_n)\in\mathbb{R}\times\mathcal{Z}_{\gamma}^{n} : t\in[t_0,t_1], \|x_l-\tilde{z}_l\|_{\gamma}\leq\varepsilon,\ l=1,2,\ldots,n\}$$

inequality (7) with some $C>0$, $\delta>0$ is satisfied. The set

$$S_{t_1}:=\{x\in C^{0,\{\alpha_l\}}([t_0,t_1];\mathcal{Z}) : D^{\alpha_l}x(t_0)=A^{\gamma}\tilde{z}_l,\ \|D^{\alpha_l}x(t)-A^{\gamma}\tilde{z}_l\|_{\mathcal{Z}}\leq\varepsilon,\ l=1,2,\ldots,n\}$$

is a complete metric space with the metric $d(x,y)=\|x-y\|_{C^{0,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})}$. For $x\in S_{t_1}$, define a mapping

$$Fx(t):=Z_0(t-t_0)A^{\gamma}z_0+\int_{t_0}^{t} A^{\gamma}Z_{1-\alpha}(t-s)B^{x}(s)ds$$

with $B^{x}(s):=B(s,A^{-\gamma}D^{\alpha_1}x(s),A^{-\gamma}D^{\alpha_2}x(s),\ldots,A^{-\gamma}D^{\alpha_n}x(s))$. It is obvious that for $x\in\mathcal{S}_{t_1}$ $Fx(t_0)=A^{\gamma}z_0$. For $l=1,2,\ldots,n$, in the case of $\alpha_l>0$ we have

$$D^{\alpha_l}Fx(t)=Z_{\alpha_l-\alpha}(t-t_0)A^{\gamma+1}z_0+\int_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^{x}(s)ds,$$

$D^{\alpha_l}Fx(t_0) = Z_{\alpha_l-\alpha}(0)A^{\gamma}z_0 = 0 = A^{\gamma}\tilde{z}_l$, since $\alpha_l - \alpha < 0$. Otherwise,

$$D^{\alpha_l}Fx(t) = Z_{\alpha_l}(t-t_0)A^{\gamma}z_0 + \int_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^{x}(s)ds,$$

$\alpha_l = 0$ and $D^{\alpha_l}Fx(t_0) = A^{\gamma}z_0 = A^{\gamma}\tilde{z}_l$, or $\alpha_l < 0$ and $D^{\alpha_l}Fx(t_0) = 0 = A^{\gamma}\tilde{z}_l$. By Theorem 3 (vi) for $t \in (t_0, t_1]$, $x \in S_{t_1}$ we have

$$\left\| \int_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^{x}(s)ds \right\|_{\mathcal{Z}} = \left\| \int_{t_0}^{t} A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\alpha+\alpha_l}(t-s)B^{x}(s)ds \right\|_{\mathcal{Z}} \le$$

$$\le C_1(t_1-t_0)^{\alpha(1-\gamma-\delta_l)-\alpha_l},$$

where $\delta_l \in (1-\gamma-\frac{\alpha_l+1}{\alpha}, 1-\gamma-\frac{\alpha_l}{\alpha})$. Thus, $Fx \in S_{t_1}$ for every $x \in S_{t_1}$, if t_1 is sufficiently close to t_0.

If $x, y \in S_{t_1}$, $t \in (t_0, t_1]$, $\delta_{2n+1} \in (1-\gamma-\frac{1}{\alpha}, 1-\gamma)$, then

$$\|Fx(t) - Fy(t)\|_{\mathcal{Z}} \le \int_{t_0}^{t} \|A^{\gamma}Z_{1-\alpha}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^{x}(s) - B^{y}(s)\|_{\mathcal{Z}}ds \le$$

$$\le C_1(t_1-t_0)^{\alpha(1-\gamma-\delta_{2n+1})}d(x,y) \le \frac{1}{4}d(x,y),$$

$$\|D^{\alpha_l}Fx(t) - D^{\alpha_l}Fy(t)\|_{\mathcal{Z}} \le \int_{t_0}^{t} \|A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^{x}(s) - B^{y}(s)\|_{\mathcal{Z}}ds \le$$

$$\le C_1(t_1-t_0)^{\alpha(1-\gamma-\delta_l)-\alpha_l}d(x,y) \le \frac{1}{4n}d(x,y), \quad l = 1,2,\ldots,n.$$

Therefore, $d(Fx, Fy) \le \frac{1}{2}d(x,y)$, and there exists a unique $y \in S$ such that $y(t) = Fy(t)$ for all $t \in [t_0, t_1]$.

Further, for $t_0 \le \tau < t \le t_1$, $\alpha_l \ge 0$

$$\|D^{\alpha_l}y(t) - D^{\alpha_l}y(\tau)\|_{\mathcal{Z}} = \|D^{\alpha_l}Fy(t) - D^{\alpha_l}Fy(\tau)\|_{\mathcal{Z}} \le$$

$$\le \|Z_{\alpha_l-\alpha}(t-t_0)A^{\gamma+1}z_0 - Z_{\alpha_l-\alpha}(\tau-t_0)A^{\gamma+1}z_0\|_{\mathcal{Z}} +$$

$$+ \left\| \int_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^{y}(s)ds - \int_{0}^{\tau} A^{\gamma}Z_{1-\alpha+\alpha_l}(\tau-s)B^{y}(s)ds \right\|_{\mathcal{Z}} \le$$

$$\le \max_{t\in[t_0,t_1]} \|D^{\frac{\alpha-\alpha_l}{2}}Z_{\alpha_l-\alpha}(t-t_0)A^{\gamma+1}z_0\|_{\mathcal{Z}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}} +$$

$$+ \max_{t\in[t_0,t_1]} \left\| D^{\frac{\alpha-\alpha_l}{2}} \int_{t_0}^{t} A^{\gamma}Z_{1-\alpha+\alpha_l}(t-s)B^{y}(s)ds \right\|_{\mathcal{Z}} |t-\tau|^{\frac{\alpha-\alpha_l}{2}} \le$$

$$\le \max_{t\in[t_0,t_1]} \|Z_{-\frac{\alpha-\alpha_l}{2}}(t-t_0)A^{\gamma+1}z_0\|_{\mathcal{Z}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}} +$$

$$+ \max_{t\in[t_0,t_1]} \left\| \int_{t_0}^{t} A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\frac{\alpha-\alpha_l}{2}}(t-s)B^{y}(s)ds \right\|_{\mathcal{Z}} |t-\tau|^{\frac{\alpha-\alpha_l}{2}} \le$$

$$\le C_1|t-\tau|^{\frac{\alpha-\alpha_l}{2}} + C_2(t_1-t_0)^{\frac{\alpha(1-2\gamma-2\delta_{n+l})-\alpha_l}{2}}|t-\tau|^{\frac{\alpha-\alpha_l}{2}} \le C_3|t-\tau|^{\frac{\alpha-\alpha_l}{2}}$$

with $\delta_{n+l} \in \left(\frac{1-\frac{\alpha_l}{\alpha}}{2} - \frac{1}{\alpha} - \gamma, \frac{1-\frac{\alpha_l}{\alpha}}{2} - \gamma\right)$. If $\alpha_l < 0$, then for $\beta_l = \min\{1, -\alpha_l\}$ and $\delta_{n+l} \in \left(1 - \frac{1+\alpha_l+\beta_l}{\alpha} - \gamma, 1 - \frac{\alpha_l+\beta_l}{\alpha} - \gamma\right)$

$$\|D^{\alpha_l}y(t) - D^{\alpha_l}y(\tau)\|_{\mathcal{Z}} = \|D^{\alpha_l}Fy(t) - D^{\alpha_l}Fy(\tau)\|_{\mathcal{Z}} \leq$$
$$\leq \|Z_{\alpha_l}(t-t_0)A^\gamma z_0 - Z_{\alpha_l}(\tau-t_0)A^\gamma z_0\|_{\mathcal{Z}} +$$
$$+\left\|\int_{t_0}^{t} A^\gamma Z_{1-\alpha+\alpha_l}(t-s)B^y(s)ds - \int_0^{\tau} A^\gamma Z_{1-\alpha+\alpha_l}(\tau-s)B^y(s)ds\right\|_{\mathcal{Z}} \leq$$
$$\leq \max_{t\in[t_0,t_1]} \|D^{\beta_l}Z_{\alpha_l}(t-t_0)A^\gamma z_0\|_{\mathcal{Z}}|t-\tau|^{\beta_l} +$$
$$+ \max_{t\in[t_0,t_1]}\left\|D^{\beta_l}\int_{t_0}^{t} A^\gamma Z_{1-\alpha+\alpha_l}(t-s)B^y(s)ds\right\|_{\mathcal{Z}} |t-\tau|^{\beta_l} \leq$$
$$\leq \max_{t\in[t_0,t_1]} \|Z_{\alpha_l+\beta_l}(t-t_0)A^\gamma z_0\|_{\mathcal{Z}}|t-\tau|^{\beta_l} +$$
$$+ \max_{t\in[t_0,t_1]}\left\|\int_{t_0}^{t} A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\alpha+\alpha_l+\beta_l}(t-s)B^y(s)ds\right\|_{\mathcal{Z}} |t-\tau|^{\beta_l} \leq$$
$$\leq C_1|t-\tau|^{\beta_l} + C_2(t_1-t_0)^{\alpha(1-\gamma-\delta_{n+l})-\alpha_l-\beta_l}|t-\tau|^{\beta_l} \leq C_3|t-\tau|^{\beta_l}$$

Hence, $D^{\alpha_l}y \in C^\nu([t_0,t_1];\mathcal{Z})$ for all $l = 1,2,\ldots,n$ and due to (7) $B^y \in C^\nu([t_0,t_1];\mathcal{Z})$ for some $\nu \in (0,\delta]$.

Arguing as in the end of the proof of Theorem 5, we can obtain that $z \in C^{0,\{\alpha_l\}}(t_0,t_1;\mathcal{Z})$ is a solution of (5) and (6), if and only if $y = A^\gamma z$ is a fixed point of F. □

4. Nonlocal Solvability of Quasilinear Equation

Now, consider the Cauchy problem

$$D^k z(t_0) = z_k, \quad k = 0,1,\ldots,m-1, \tag{9}$$

for the quasilinear equation

$$D^\alpha z(t) + Az(t) = B(t, D^{\alpha_1}z(t), D^{\alpha_2}z(t), \ldots, D^{\alpha_n}z(t)) \tag{10}$$

on a given segment $[t_0,T]$. Here, as before, $m-1 < \alpha \leq m \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{Z}$, $l = 1,2,\ldots,n$. Some of α_l may be negative.

Let the mapping $B : [t_0,T] \times \mathcal{Z}_\gamma^n \to \mathcal{Z}$ be given; thus, there exist constants $C > 0$, $\delta \in (0,1]$, such that for all $(s,x_1,x_2,\ldots,x_n), (t,y_1,y_2,\ldots,y_n) \in [t_0,T] \times \mathcal{Z}_\gamma^n$

$$\|B(s,x_1,x_2,\ldots,x_n) - B(t,y_1,y_2,\ldots,y_n)\|_{\mathcal{Z}} \leq C\left(|s-t|^\delta + \sum_{l=1}^{n}\|x_l - y_l\|_\gamma\right). \tag{11}$$

Theorem 7. *Let $\alpha \in (1,2]$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $-A \in \mathcal{A}_\alpha(\theta_0,0)$, $0 \in \rho(A)$, a mapping $B : [t_0,T] \times \mathcal{Z}_\gamma^n \to \mathcal{Z}$ satisfies condition (11) with $\gamma \in (0,1)$, $z_0, z_1 \in \mathcal{Z}_{1+\gamma}$. Then, there exists a unique solution of problems (9) and (10) on $[t_0,T]$.*

Proof. Consider a mapping

$$Fx(t) := \sum_{k=0}^{1} Z_{-k}(t-t_0)A^\gamma z_k + \int_{t_0}^{t} A^\gamma Z_{1-\alpha}(t-s)B^x(s)ds,$$

with $B^x(s) := B(s, A^{-\gamma}D^{\alpha_1}x(s), A^{-\gamma}D^{\alpha_2}x(s), \ldots, A^{-\gamma}D^{\alpha_n}x(s))$. As in Theorem 5, it is not difficult to show that $Fx \in C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$ for every $x \in C^{1,\{\alpha_l\}}([t_0,t_1];\mathcal{Z})$.

For $x, y \in C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})$, $\delta_{2n+1} \in (1-\gamma-\frac{1}{\alpha}, 1-\gamma)$, $\delta_{2n+2} \in (1-\gamma-\frac{2}{\alpha}, 1-\gamma-\frac{1}{\alpha})$ and for all $t \in (t_0, T]$,

$$\|Fx(t) - Fy(t)\|_{\mathcal{Z}} \leq \int_{t_0}^{t} \|A^{-\delta_{2n+1}}A^{\gamma+\delta_{2n+1}}Z_{1-\alpha}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^x(s) - B^y(s)\|_{\mathcal{Z}}ds \leq$$

$$\leq C_1(t-t_0)^{\alpha(1-\gamma-\delta_{2n+1})}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})},$$

$$\|D^1Fx(t) - D^1Fy(t)\|_{\mathcal{Z}} \leq \int_{t_0}^{t} \|A^{-\delta_{2n+2}}A^{\gamma+\delta_{2n+2}}Z_{2-\alpha}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^x(s) - B^y(s)\|_{\mathcal{Z}}ds \leq$$

$$\leq C_1(t-t_0)^{\alpha(1-\gamma-\delta_{2n+2})-1}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})},$$

$$\|D^{\alpha_l}Fx(t) - D^{\alpha_l}Fy(t)\|_{\mathcal{Z}} \leq \int_{t_0}^{t} \|A^{-\delta_l}A^{\gamma+\delta_l}Z_{1-\alpha+\alpha_l}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^x(s) - B^y(s)\|_{\mathcal{Z}}ds \leq$$

$$\leq C_1(t-t_0)^{\alpha(1-\gamma-\delta_l)-\alpha_l}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})}, \quad l = 1,2,\ldots,n.$$

Here, $\delta_l \in (1-\gamma-\frac{\alpha_l+1}{\alpha}, 1-\gamma-\frac{\alpha_l}{\alpha})$, as before. Take $\chi = \min\{\alpha(1-\gamma-\delta_{2n+1}), \alpha(1-\gamma-\delta_{2n+2})-1, \alpha(1-\gamma-\delta_1)-\alpha_1, \ldots, \alpha(1-\gamma-\delta_n)-\alpha_n\}$, then $\|Fx-Fy\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})} \leq C_2(t-t_0)^{\chi}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})}$ and

$$\|F^2x(t) - F^2y(t)\|_{\mathcal{Z}} \leq \int_{t_0}^{t} \|A^{-\delta_1}A^{\gamma+\delta_1}Z_{1-\alpha}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|B^{Fx}(s) - B^{Fy}(s)\|_{\mathcal{Z}}ds \leq$$

$$\leq C_1(t-t_0)^{\alpha(1-\gamma-\delta_1)}\|Fx-Fy\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})} \leq C_1C_2\frac{(t-t_0)^{2\chi}}{\chi+1}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})},$$

$$\|D^1Fx(t) - D^1Fy(t)\|_{\mathcal{Z}} \leq C_1C_2\frac{(t-t_0)^{2\chi}}{\chi+1}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})},$$

$$\|D^{\alpha_l}Fx(t) - D^{\alpha_l}Fy(t)\|_{\mathcal{Z}} \leq C_1C_2\frac{(t-t_0)^{2\chi}}{\chi+1}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})},$$

$\|F^2x - F^2y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})} \leq C_2^2\frac{(t-t_0)^{2\chi}}{\chi+1}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})}$. Analogously we can get the inequalities

$$\|F^3x - F^3y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})} \leq C_2^3\frac{(t-t_0)^{3\chi}}{(\chi+1)(2\chi+1)}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})} \leq$$

$$\leq C_2^3\frac{(t-t_0)^{3\chi}}{\chi^2 2!}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})}, \ldots,$$

$$\|F^px - F^py\|_{C^{1,\{\alpha_l\}}([t_0,T];\mathcal{Z})} \leq C_2^p\frac{(T-t_0)^{p\chi}}{\chi^{p-1}(p-1)!}\|x-y\|_{C^{1,\{\alpha_l\}}([t_0,T];\mathcal{Z})}, \quad p \in \mathbb{N}.$$

Therefore, for a large enough $p \in \mathbb{N}$, the operator F^p is a contraction, and there exists a unique $y \in C^{1,\{\alpha_l\}}([t_0,t];\mathcal{Z})$ such that $y(t) = Fy(t)$, $t \in [t_0,T]$. As in the previous section, we can prove that $B^y \in C^{\nu}([t_0,t_1];\mathcal{Z})$, $\nu \in (0,\delta]$, due to condition (11), and that $z \in C^{1,\{\alpha_l\}}(t_0,t_1;\mathcal{Z})$ is a solution of (9) and (10), if and only if $y = A^{\gamma}z$ is a fixed point of F. $\square$

Theorem 8. *Let $\alpha \in (0,1]$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $-A \in \mathcal{A}_\alpha(\theta_0, 0)$, $0 \in \rho(A)$, a mapping $B : [t_0, T] \times \mathcal{Z}_\gamma^n \to \mathcal{Z}$ satisfies condition (11) with $\gamma \in (0,1)$, $z_0 \in \mathcal{Z}_{1+\gamma}$. Then, there exists a unique solution of problems (9) and (10) on $[t_0, T]$.*

Proof. As in Theorem 6 and Theorem 7, we can prove that the mapping

$$Fx(t) := Z_0(t-t_0)A^\gamma z_0 + \int_{t_0}^{t} A^\gamma Z_{1-\alpha}(t-s)B\big(s, A^{-\gamma}D^{\alpha_1}x(s), \dots, A^{-\gamma}D^{\alpha_n}x(s)\big)ds$$

has a unique fixed point y in the space $C^{0,\{\alpha_l\}}([t_0, T]; \mathcal{Z})$ and a function z is a solution of problems (9) and (10), if and only if $z = A^{-\gamma}y$. $\square$

5. Application

In a bounded region $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$, consider a problem with initial conditions

$$v(\xi, t_0) = v_0(\xi), \quad D_t^1 v(\xi, t_0) = v_1(\xi), \quad \xi \in \Omega, \tag{12}$$

for $\alpha \in (1,2]$, or with a unique initial condition

$$v(\xi, t_0) = v_0(\xi), \quad \xi \in \Omega, \tag{13}$$

in the case $\alpha \in (0,1]$, and with a boundary condition

$$v(\xi, t) = 0, \quad \xi \in \partial\Omega,\ t > t_0, \tag{14}$$

for an equation

$$D_t^\alpha v(\xi, t) = \Delta v(\xi, t) + \sum_{l=1}^{n} D_t^{\alpha_l} v(\xi, t) \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} D_t^{\alpha_l} v(\xi, t), \quad \xi \in \Omega,\ t > t_0, \tag{15}$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $D_t^{\alpha_l} v$ are partial Gerasimov–Caputo fractional derivatives for $\alpha_l > 0$, or Riemann–Liouville fractional integrals for $\alpha_l \leq 0$ with respect to t. Take $\mathcal{Z} = L_2(\Omega)$, $A = -\Delta$, $D_A = H^2(\Omega) \cap H_0^1(\Omega)$, then $-A \in \mathcal{A}_\alpha(\theta_0, 0)$ at $\alpha \in (0,2)$, $\theta_0 \in (\pi/2, \pi)$ (see Theorem 4 in [30] for $n = 0$, $P_0 \equiv 1$, $p = 1$, $Q_1(\lambda) = \lambda$). Reasoning as in Theorem 8.3.5 ([19]), we can obtain that the nonlinear operator of the form

$$f(v_1, v_2, \dots, v_n) = \sum_{l=1}^{n} v_l \sum_{i=1}^{3} \frac{\partial}{\partial \xi_i} v_l$$

satisfies the conditions of Theorem 6 and Theorem 5 at $\gamma > 3/4$. Therefore, for all $v_0 \in D_{A^{1+\gamma}}$, or $v_0, v_1 \in D_{A^{1+\gamma}}$, there exists a unique solution of problems (13)–(15) in the case of $\alpha \in (0,1]$, or problems (12), (14) and (15), if $\alpha \in (1,2)$, in $\Omega \times [t_0, t_1]$ with some $t_1 > t_0$.

Author Contributions: Conceptualization, V.E.F.; methodology, V.E.F. and M.K.; software, T.A.Z.; validation, M.K. and T.A.Z.; formal analysis, T.A.Z.; investigation, V.E.F. and M.K.; resources, T.A.Z.; data curation, T.A.Z.; writing—original draft preparation, V.E.F.; writing—review and editing, V.E.F. and M.K.; visualization, T.A.Z.; supervision, V.E.F. and M.K.; project administration, M.K.; funding acquisition, V.E.F. All authors have read and agreed to the published version of the manuscript.

Funding: Marko Kostić is partially supported by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia. The work of Vladimir E. Fedorov and Tatyana A. Zakharova was funded by the grant of President of the Russian Federation to support leading scientific schools, project number NSh-2708.2022.1.1.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analysis, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Publishing: Amsterdam, The Netherlands ; Boston, MA, USA; Heidelberg, Germany, 2006.
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science: Philadelphia, PA, USA, 1993.
3. Prüss, J. *Evolutionary Integral Equations and Applications*; Birkhäuser: Basel, Switzerland, 1993.
4. Podlubny, I. *Fractional Differential Equations*; Academic: Boston, MA, USA, 1999.
5. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka: Moscow, Russia, 2005. (In Russian)
6. Kostić, M. *Abstract Volterra Integro-Differential Equations*; CRC Press: Boca Raton, FL, USA, 2015.
7. Gerasimov, A.N. Generalization of linear laws of deformation and their application to problems of internal friction. *Prikl. Mat. I Mekhanika* **1948**, *12*, 251–260. (In Russian)
8. Caputo, M. Linear model of dissipation whose Q is almost frequancy independent. II. *Geophys. J. R. Soc.* **1967**, *13*, 529–539. [CrossRef]
9. Nakhushev, A.M. *Fractional Calculus and Its Applications*; Fizmatlit: Moscow, Russia, 2003. (In Russian)
10. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles*; Springer: New York, NY, USA, 2011.
11. Uchaykin, V.V. *Fractional Derivatives for Physicists and Engineers*; Higher Education Press: Beijing, China, 2012.
12. Bajlekova, E.G. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
13. Fedorov, V.E. A class of fractional order semilinear evolutions in Banach spaces. In *Integral Equations and Their Applications, Proceeding of the University Network Seminar on the Occasion of the Third Mongolia–Russia–Vietnam Workshop on NSIDE 2018, 27–28 October 2018*; Hanoi Mathematical Society, Hung Yen University of Technology and Education: Hung Yen, Vietnam, 2018; pp. 11–20.
14. Fedorov, V.E.; Romanova, E.A. Inhomogeneous fractional evolutionary equation in the sectorial case. *J. Math. Sci.* **2020**, *250*, 819–829. [CrossRef]
15. Fedorov, V.E.; Boyko, K.V.; Phuong, T.D. Initial value problems for some classes of linear evolution equations with several fractional derivatives. *Math. Notes NEFU* **2021**, *28*, 85–104.
16. Boyko, K.V.; Fedorov, V.E. The Cauchy Problem for a class of multi-term equations with Gerasimov–Caputo derivatives. *Lobachevskii J. Math.* **2022**, *43*, 1293–1302. [CrossRef]
17. Fedorov, V.E.; Zakharova, T.A. Nonlocal solvability of quasilinear degenerate equations with Gerasimov–Caputo derivatives. *Lobachevskii J. Math.* **2023**, *44*, 593–605.
18. Henry, D. *Geometric Theory of Semilinear Parabolic Equations*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1981.
19. Pazy, A. *Semigroups and Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.
20. Arendt, W.; Batty, C.J.K.; Hieber, M.; Neubrander, F. *Vector-Valued Laplace Transforms and Cauchy Problems*; Springer Basel AG: Berlin, Germany, 2011.
21. Fedorov, V.E.; Avilovich, A.S.; Zakharova, T.A. Complex powers of fractional sectorial operators and quasilinear equations with Riemann–Liouville derivatives. *Lobachevskii J. Math.* **2023**, *44*, 579–592.
22. Baybulatova, G.D.; Plekhanova, M.V. An initial problem for a class of weakly degenerate semilinear equations with lower order fractional derivatives. *Bull. Irkutsk. State Univ. Ser. Math.* **2021**, *35*, 34–48. [CrossRef]
23. Plekhanova, M.V.; Shuklina, A.F.; Baybulatova, G.D. Mixed control for degenerate nonlinear equations with fractional derivatives. *Chelyabinsk Phys. Math. J.* **2022**, *7*, 287–300. [CrossRef]
24. Plekhanova, M.V.; Izhberdeeva, E.M. Local unique solvability of a quasilinear equation with the Dzhrbashyan–Nersesyan derivatives. *Lobachevskii J. Math.* **2022**, *43*, 1379–1388. [CrossRef]
25. Plekhanova, M.V.; Izhberdeeva, E.M. Degenerate quasilinear equations with the Dzhrbashyan–Nersesyan derivative. *J. Math. Sci.* **2023**, *269*, 217–228. [CrossRef]
26. Fedorov, V.E.; Avilovich, A.S. A Cauchy type problem for a degenerate equation with the Riemann–Liouville derivative in the sectorial case. *Sib. Math. J.* **2019**, *60* , 359–372. [CrossRef]
27. Krein, S.G. *Linear Differential Equations in Banach Space*; American Mathematical Society: Providence, RI, USA, 1972.
28. Clement, P.; Heijmans, H.J.A.M.; Angenent, S.; van Duijn, C.J.; de Pagter, B. *One-Parameter Semigroups*; North-Holland: Amsterdam, The Netherlands; New York, NY, USA; Oxford, UK; Tokyo, Japan, 1987.

29. Plekhanova, M.V.; Baybulatova, G.D. Semilinear equations in Banach spaces with lower fractional derivatives. *Springer Proc. Math. Stat.* **2019**, *292*, 81–93.
30. Fedorov, V.E.; Nagumanova, A.V.; Avilovich, A.S. A class of inverse problems for evolution equations with the Riemann–Liouville derivative in the sectorial case. *Math. Methods Appl. Sci.* **2021**, *44*, 11961–11969. [CrossRef]