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Domains of Quasi Attraction: Why Stable Processes Are Observed in Reality?

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Abstract: From the very start of modelling with power-tail distributions, concerns were expressed about the actual applicability of distributions with infinite expectations to real-world distributions, which usually have bounded ranges. Here, we suggest resolving this issue by shifting the analysis from the true convergence in various CLTs to some kind of quasi convergence, where a stable approximation to, say, normalised sums of n i.i.d. random variables (or more generally, in a functional setting, to the processes of random walks), holds for large n, but not "too large" n. If the range of "large n" includes all imaginable applications, the approximation is practically indistinguishable from the true limit. This approach allows us to justify a stable approximation to random walks with bounded jumps and, moreover, it leads to some kind of cascading (quasi) asymptotics, where for different ranges of a small parameter, one can have different stable or light-tail approximations. The author believes that this development might be relevant to all applications of stable laws (and thus of fractional equations), say, in Earth systems, astrophysics, biological transport and finances.

Keywords: rates of quasi-convergence; domain of quasi attraction; functional limit theorem with stable processes; Wasserstein distances; smooth Wasserstein distances; Kolmogorov's distance

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1. Introduction

1.1. Objectives of the Paper

From the very start of modelling with power-tail distributions, concerns were expressed about the actual applicability of distributions with infinite expectations to real-world distributions, which usually have bounded ranges, see e.g., [1]. We aim to resolve this issue by obtaining explicit rates of approximation in functional limit theorems with stable laws and allowing one to shift the analysis from exact convergence in various CLTs to some kind of quasi convergence. Namely, we look at stable approximations to, say, normalised sums of n i.i.d. random variables (or more generally, in a functional setting, to the processes of random walks), which hold for large n, but not "too large". If the range of "large n" includes any imaginable applications (say, being of the order of the age of the Universe), the approximation is practically indistinguishable from a true limit, and we can say that the corresponding random variables belong to the domain of quasi attraction of a stable law. This idea is supported by supplying explicit rates of approximation in explicitly prescribed ranges of a small parameter. In this way, we justify a stable approximation to normalised sums of i.i.d. random variables having all moments bounded or even having a bounded range. It also leads to some kind of cascading (quasi) asymptotics, where for different ranges of n, one can have different stable or light-tailed approximations. This idea is already well appreciated by physicists, see, e.g., [2], devoted to the analysis of cosmic rays, where it is stated that for fluxes "of actual interest one is relatively far from the Gaussian limit and much closer to the stable law limit" (though Gaussian limit is dictated by the assumption of bounded Universe). In the present paper, we give an exact quantitative and qualitative description of this effect. The author believes that this development might be relevant to all applications of stable laws and processes (and thus of fractional equations), say, in the contexts of Earth systems [3], of astrophysics [2], of biological transport [4], of seismo-dynamics [5] and of finances [1].

A search for the rates of convergence for functional limit theorems with stable laws was initiated in the author's paper [6]. That paper also contained a brief review of the literature on the three related topics: (i) rates of convergence for functional standard central limit theorem, (ii) rates of (nonfunctional) convergence of sequences to stable laws, and (iii) functional central limits with stable laws without the rates. We will not reproduce this review here, but only remind some basic references [7–11] on the rates of convergence of random sequences (not processes of random walks) to stable laws. The second objective of the present paper is to introduce essential improvements to the first result of [6] on the rates of convergence for functional limit theorems with stable laws in finite times. Namely, while in [6] the rates were very rough and were given only for exact power tails, for the one-dimensional case and for stability index $\beta \in (0,1)$; here, we essentially tighten the rates (improve both orders and distances used) and extend to arbitrary β (excluding $\beta = 1$), arbitrary dimensions, and to standard assumptions of asymptotic (not exact) power tails. Apart from theoretical importance, results on convergence rates are crucial for assessing the effectiveness of numeric schemes for solving fractional PDEs by probabilistic methods, see [12–14]. They provide exact rates of convergence for these schemes.

We refer to books [15–17] for a general background on modelling with stable laws.

1.2. Content

In Section 2, we formulate our results and present proofs that are consequences of two types of certain technical estimates for stable laws and their random walk approximations. These two types of estimates are proved in Sections 3 and 4, respectively. In Section 2, we also present corollaries concerning normalised sums of i.i.d. random variables and an example of cascading asymptotics. For the latter, we identify explicitly the regions of different asymptotic regimes and the region of switching between them. Some conclusions and perspectives are drawn in Section 5. In Appendix A we recall the general theorem on the rates of convergence of discrete Markov chains to continuous time Feller processes, which forms the cornerstone for the present derivations.

1.3. Notations for Spaces and Distances

Letters $\mathbf{P}$ and $\mathbf{E}$ will be used to denote probability and expectation. We also use the standard abbreviation i.i.d. for independent identically distributed and r.h.s. (respectively, l.h.s.) for right (respectively, left) hand side.

As usual, let $C(\mathbf{R}^d)$ denote the space of bounded continuous functions on $\mathbf{R}^d$ equipped with the standard sup-norm $\|.\|$. By $C_{Lip}(\mathbf{R}^d)$ we shall denote the space of Lipschitz continuous functions from $C(\mathbf{R}^d)$, the Lipschitz constant being denoted f_{Lip}, with the norm $\|f\|_{Lip} = \max(\|f\|, f_{Lip})$.

For $k \in \mathbf{N}$, let $C^k = C^k(\mathbf{R}^d)$ denote the space of k times continuously differentiable functions on $\mathbf{R}^d$ with bounded derivatives equipped with the standard norm

$$\|f\|_{C^k} = \max\{\|f\|, \max_{m=1}^{k} \|f^{(m)}\|\},$$

where $\|f^{(m)}\|$ denotes the sup-norm of the norms of multi-linear operators $f^{(m)}(x)$. Let $C_\infty(\mathbf{R}^d)$ denote the closed subspace of $C(\mathbf{R}^d)$ consisting of functions vanishing at infinity, $C^k_\infty(\mathbf{R}^d)$ the closed subspace of $C^k(\mathbf{R}^d)$ consisting of functions such that itself and all its derivatives up to order k belong to $C_\infty(\mathbf{R}^d)$.

For $\alpha \in (0,1]$, let $H_\alpha = H_\alpha(\mathbf{R}^d)$ denote the space of bounded α-Hölder continuous functions f having a finite Hölder constant

$$f_\alpha = \sup_{0<|x-y|\le 1} \frac{|f(x)-f(y)|}{|x-y|^\alpha}. \tag{1}$$

This is a Banach space equipped with the norm

$$\|f\|_\alpha = \max\{\|f\|, f_\alpha\}. \tag{2}$$

For instance, $H_1(\mathbf{R}^d) = C_{Lip}(\mathbf{R}^d)$ so that $f_1 = f_{Lip}$.

The same notation is used for the space $H_\alpha = H_\alpha(\mathbf{R}^d, \mathbf{R}^n)$ of bounded α-Hölder continuous functions $f : \mathbf{R}^d \to \mathbf{R}^n$ with Euclidean norm used in (1) instead of magnitude. We shall need this extension mostly for the gradient mapping

$$f'(x) = \nabla f(x) = (\partial f/\partial x_1, \cdots, \partial f/\partial x_d)(x).$$

Similarly we use this notation for the square-matrix-valued functions (notably for the second derivatives of a real-valued function f), where, by the corresponding norm, we mean the usual norm of a matrix.

For $k \in \mathbf{N}$, let $H_{k,\alpha} = H_{k,\alpha}(\mathbf{R}^d)$ denote the subspace of $C^k(\mathbf{R}^d)$ of functions with α-Hölder continuous derivatives of order k equipped with the norm

$$\|f\|_{k,\alpha} = \max\{\|f\|_{C^k}, f_\alpha^{(k)}\}, \tag{3}$$

where $f_\alpha^{(k)}$ is the Hölder constant (of index α) of the mapping $f^{(k)}$ from $\mathbf{R}^d$ to the space of k-linear forms in $\mathbf{R}^d$.

For a subspace B of $C(\mathbf{R}^d)$, which is itself a Banach space equipped with the norm $\|.\|_B$, one can introduce a metric on the set of $\mathbf{R}^d$-valued random variables (more precisely, on the space of distributions of random variables) by the equation

$$d_B(X, Y) = \sup\{|\mathbf{E}f(X) - \mathbf{E}f(Y)| : \|f\|_B \le 1\}. \tag{4}$$

For instance, if $B = C^k$, $k \in \mathbf{N}$, the corresponding metrics d_{C^k} are often referred to as the smooth Wasserstein metrics (see, e.g., [9]). Intermediate metrics can be defined by using the spaces of Hölder functions H_α as the subspace B. For the space H_1 the corresponding metric is referred to as the bounded Lipschitz metric or as the (standard) Wasserstein 1-distance, and it is usually denoted W_1.

The Kolmogorov distance between real random variables X and Y is defined by the formula

$$d_{Kol}(X, Y) = \sup_z |\mathbf{P}(X \le z) - \mathbf{P}(Y \le z)|. \tag{5}$$

When one of the variables, say Y, has a continuous density, $p(y)$, the Kolmogorov distance can be estimated by the smooth Wasserstein distance. In particular, as was shown in [6] (extending the arguments from [9]), for any $\alpha \in (0, 1]$,

$$d_{Kol}(X, Y) \le (M + 1)[d_{H_\alpha}(X, Y)]^{1/(\alpha+1)}, \tag{6}$$

$$d_{Kol}(X, Y) \le (M + 3/2)[d_{H_{1,\alpha}}(X, Y)]^{1/(\alpha+2)}, \tag{7}$$

where $M = \sup_y p(y)$. For a stable process, estimates for the maximum of densities can be easily found. For instance, the maximum of the density $p_t^\beta(x)$ of the stable process generated by (10) below was estimated in [6] as follows:

$$M_t^\beta = \sup_x \sup_{s \ge t} p_s^\beta(x) \le \frac{1}{2} t^{-1/\beta}. \tag{8}$$

2. Main Results

Let τ_i, $i \in \mathbf{N}$, be a sequence of i.i.d. real-valued random variables with a bounded probability density p, and let

$$\Phi_t^h = \sum_{i=1}^{[t/h]} h^{1/\beta}\tau_i, \tag{9}$$

be the corresponding scaled random walk (where we set $\Phi_t^h = 0$ for $t < h$). We shall denote by $V_h^{[t/h]}$ the transition operators of the discrete Markov chain Φ_t^h.

For the probability p, we shall assume the following rather standard condition (P) of an asymptotic power tail, but with the marked difference that this power tail holds for large but finite distances.

Condition (P): The probability density $p(y)$ on $\mathbf{R}$ is bounded and such that

$$p(y) = Ay^{-1-\beta}(1+\epsilon(y)y^{-1}) \text{ for } y \in [B_m, B_M],$$

$$p(y) \le Ay^{-1-\beta}(1+\epsilon_0 y^{-1}) \text{ for } y \ge B_M,$$

and $p(y) = 0$ for $y \le -B$, where $\epsilon(y)$ is a measurable function on $\mathbf{R}_+$ such that $|\epsilon(y)| \le \epsilon_0$, with some constants $\epsilon_0, B \ge 0$, $A, \beta > 0$, $B_M > B_m > 0$ such that $\beta B_m^\beta > A$. No additional assumptions on the behaviour of $p(y)$ on the interval $[-B, B_m]$ are made.

The latter condition is taken for simplicity as being a bit stronger than

$$\beta B_m^\beta \ge A[1-(B_m/B_M)^\beta],$$

which is equivalent to the requirement that $A\int_{B_m}^{B_M} y^{-1-\beta}dy \le 1$.

The generator of a one-sided stable Lévy process of index $\beta \in (0,2)$ (or, in the language of analysis, fractional derivative operator of order β) is defined by the formulas

$$L_\beta f(x) = \int_0^\infty \frac{f(x+y)-f(x)}{y^{1+\beta}}\,dy, \quad \beta \in (0,1), \tag{10}$$

$$L_\beta f(x) = \int_0^\infty \frac{f(x+y)-f(x)-f'(x)y}{y^{1+\beta}}\,dy = \int_0^\infty \int_0^y (f'(w+x)-f'(x))dw\frac{dy}{y^{1+\beta}}, \quad \beta \in [1,2). \tag{11}$$

Remark 1. *In fact, the actual standard generators and derivatives differ from these formulas by a constant multiplier that we omit for simplicity.*

Let T_β^t be the Feller semigroup of the β-stable Lévy process Σ_t^β in $\mathbf{R}$ generated by the operator L_β.

Theorem 1. *Let $\beta \in (0,1)$ and the probability density $p(y)$ satisfy the assumption (P). (i) If $\beta < 1/2$ and $h \in [(\beta B_M^\beta)^{-1/2}, 1]$, then*

$$\sup_{s\le t} \|(V_h^{[s/h]} - T_\beta^{As})f\| \le htC_0\|f\|_{Lip}, \tag{12}$$

where

$$C_0 = 4A(1+\frac{\epsilon_0}{B_M}) + 2\epsilon_0 + B + \frac{B_m}{1-\beta} + \frac{4A^2}{\beta^2(1-\beta)(1-2\beta)},$$

which in terms of the Wasserstein 1-distance rewrites as

$$\sup_{s\le t} W_1(\Phi_s^h, \Sigma_{As}^\beta) \le C_0 ht. \tag{13}$$

(ii) If $\beta \geq 1/2$ and $h \in [(\beta B_M^\beta)^{-\beta}, 1]$, then

$$\sup_{s \leq t} \|(V_h^{[s/h]} - T_\beta^{As})f\| \leq th^{(1-\beta)/(2-\beta)} C_1 \|f\|_{Lip}, \tag{14}$$

where

$$C_1 = 4A(1 + \frac{\epsilon_0}{B_M}) + 2\epsilon_0 + B + \frac{B_m}{1-\beta} + \frac{6A^2}{\beta^2(1-\beta)^2},$$

which in terms of the Wasserstein 1*-distance rewrites as*

$$\sup_{s \leq t} W_1(\Phi_s^h, \Sigma_{As}^\beta) \leq C_1 th^{(1-\beta)/(2-\beta)} \tag{15}$$

Proof. It is a consequence of general estimate (A4), where ϵ_h is given by Theorem 4 (i) and $\varkappa_h$ is given by Proposition 3, proved in the next two sections. □

Remark 2. *1. It may seem strange at first sight that the range of h depends only on B_M and not on B_m, as one can hardly expect any approximation if, say, $B_m = B_M$, which is not banned by our conditions. However, all constants C_i depend linearly on B_m, so that, for large B_m, our estimates become essentially void. 2. In [6] we obtained much weaker estimates for the distances between Φ_s^h and Σ_{As}^β, and, moreover, only in case $\epsilon(y) = 0$ and $B_M = \infty$.*

Applying (6) and (8) yields the following.

Corollary 1. *In case (i) and (ii), we have the following estimates for the Kolmogorov distances for any $t_0 \leq t$:*

$$\sup_{s \in [t_0, t]} d_{Kol}(\Phi_s^h, \Sigma_{As}^\beta) \leq \left(1 + \frac{1}{2}(At_0)^{-1/2\beta}\right)\sqrt{C_0 ht}, \tag{16}$$

$$\sup_{s \in [t_0, t]} d_{Kol}(\Phi_s^h, \Sigma_{As}^\beta) \leq \left(1 + \frac{1}{2}(At_0)^{-1/2\beta}\right)\sqrt{C_1 t} h^{(1-\beta)/2(2-\beta)}, \tag{17}$$

respectively.

As a direct consequence, let us derive the approximating rates for normalised sums, that is, a non-functional (quasi) central limit theorem (CLT) with stable laws. Namely, setting $t = 1$ and $n = 1/h$ in the formulas above yields the following.

Corollary 2. *Let $\beta \in (0, 1)$ and the probability density $p(y)$ satisfy the assumption (P). (i) If $\beta < 1/2$ and $n < \sqrt{\beta B_M^\beta}$, then*

$$W_1\left(\frac{\tau_1 + \cdots + \tau_n}{n^{1/\beta}}, \Sigma_A^\beta\right) \leq C_0/n, \quad d_{Kol}\left(\frac{\tau_1 + \cdots + \tau_n}{n^{1/\beta}}, \Sigma_A^\beta\right) \leq \left(1 + \frac{1}{2}A^{-1/2\beta}\right)\sqrt{C_0/n}, \tag{18}$$

(ii) If $\beta \geq 1/2$ and $n < (\beta B_M^\beta)^\beta$, then

$$W_1\left(\frac{\tau_1 + \cdots + \tau_n}{n^{1/\beta}}, \Sigma_A^\beta\right) \leq C_1 n^{-(1-\beta)/(2-\beta)},$$

$$d_{Kol}\left(\frac{\tau_1 + \cdots + \tau_n}{n^{1/\beta}}, \Sigma_A^\beta\right) \leq \left(1 + \frac{1}{2}A^{-1/2\beta}\right)\sqrt{C_1} n^{-(1-\beta)/2(2-\beta)}. \tag{19}$$

When the upper bound for n is large, we may say that the distribution $p(y)$ belongs to the *domain of quasi attraction* of the β-stable law, in the sense that the normalised sums of the corresponding i.i.d. random variables behave in the same way as for distributions from the actual domain of attraction, for all practical purposes. The parameter B_M is the main parameter measuring the level of deviation from the actual domain of attraction.

Remark 3. *Rates obtained for non-functional approximations (18), (19) are surely far from being optimal. The proofs (given below) show the essential flexibility of our approach. In this paper, our main stress was on functional approximations, and moreover, we planned to develop and demonstrate some methodology and did not fight for the best estimates. Nevertheless, for non-functional results and for the exact convergence (when $B_M = \infty$) it can be instructive to compare our rates with those in the literature, which are in abundance. It seems that, even in this case, our results are not consequences of any known results but complement them. The nearest to us seem to be the estimates from [9] that also operate with smooth Wasserstein distances and makes the same standard assumptions on the densities of τ. However, using Hölder spaces, we managed to obtain estimates of weak convergence in terms of just once differentiable functions for $\beta \in (0,1)$ and of twice differentiable functions for $\beta \in (1,2)$ (unlike twice and thrice differentiable, respectively, in [9]). Additionally, our approach allows one to further weaken these regularity assumptions (that is, decrease the order of smooth Wasserstein distances). In other papers, most notably [8], the assumptions on τ are made in terms of characteristic functions, which makes a direct comparison with our rates not straightforward.*

Let us turn to the case $\beta \in (1,2)$. In this paper, we decided to avoid dealing with several technical complications arising in the case $\beta = 1$.

Theorem 2. *Let $\beta \in (1,2)$, the probability density $p(y)$ satisfy assumption (P), and $h \geq \tilde{h}_0 = [(\beta-1)\tilde{B}_M^{\beta-1}]^{-\beta}$, where coefficients with tilde are defined in (33).*

(i) If $\beta \in (1,3/2)$, then

$$\sup_{s \leq t} \|(V_h^{[s/h]} - T_\beta^{As})f\| \leq C_2 t h^{(2-\beta)/(3-\beta)} \|f'\|_{Lip}, \tag{20}$$

where

$$C_2 = 4A(1 + \frac{\tilde{\epsilon}_0}{\tilde{B}_M}) + \frac{\beta \tilde{B}_m \tilde{\epsilon}_0}{(\beta-1)} + \frac{\tilde{B}^2}{2} + \frac{\tilde{B}_m^2}{(2-\beta)} + \frac{12A^2}{(3-2\beta)(\beta-1)^2}.$$

(ii) If $\beta \in [3/2,2)$, then

$$\sup_{s \leq t} \|(V_h^{[s/h]} - T_\beta^{As})f\| \leq C_3 t h^{(2-\beta)/\beta} \|f''\|_{Lip}, \tag{21}$$

where

$$C_3 = 4A(1 + \frac{\tilde{\epsilon}_0}{\tilde{B}_M}) + \frac{\beta \tilde{B}_m \tilde{\epsilon}_0}{(\beta-1)} + \frac{\tilde{B}^2}{2} + \frac{\tilde{B}_m^2}{(2-\beta)} + \frac{9A^2}{2(2-\beta)^2(\beta-1)^2}.$$

Proof. It is a consequence of estimate (A4), where ϵ_h is given by Theorem 4 case (iii) and $\varkappa_h$ is given by Proposition 6. □

Inequalities of Theorem 2 estimate the smooth Wasserstein distances $d_{C^2}(\Phi_s^h, \Sigma_{As}^\beta)$. Analogously to the case $\beta \in (0,1)$ above one can obtain an estimate for the corresponding Kolmogorov distances using (7), and for the distances of normalised sums.

Next, let τ_i, $i \in \mathbf{N}$, be a sequence of i.i.d. $\mathbf{R}^d$-valued random variables with a bounded probability density p, $d > 1$. The random walk Φ_t^h and its transition operator $V_h^{[t/h]}$ are defined as above in case $d = 1$.

For p, we shall assume the condition (Pd), which is a natural extension of the one-dimensional case above.

Condition (Pd): With given constants $\epsilon_0 \geq 0$, $\beta > 0$, $B_M > B_m > 0$, the probability density $p(y)$ on $\mathbf{R}^d$ is bounded and such that

$$p(y) = A(\bar{y})|y|^{-d-\beta}(1 + \epsilon(y)|y|^{-1}) \text{ for } |y| \in [B_m, B_M],$$

$$p(y) \leq A(\bar{y})|y|^{-d-\beta}(1+\epsilon_0|y|^{-1}) \text{ for } |y| \geq B_M,$$

where $\bar{y} = y/|y|$, $\epsilon(y)$ is a measurable function on $\mathbf{R}^d$ such that $|\epsilon(y)| \leq \epsilon_0$, and $A(\bar{y})$ is a continuous non-negative function on the sphere S^{d-1} such that

$$A = \int_{S^{d-1}} A(\bar{y})d\bar{y} < \beta B_m^{\beta}.$$

The generator of a d-dimensional stable Lévy process of index $\beta \in (0,2)$ with a spectral measure specified by the density function $A(\bar{y})$ is defined by the formulas

$$L_\beta f(x) = \int_{\mathbf{R}^d} \frac{f(x+y)-f(x)}{|y|^{d+\beta}} A(\bar{y})\, dy, \quad \beta \in (0,1), \tag{22}$$

$$L_\beta f(x) = \int_{\mathbf{R}^d} \frac{f(x+y)-f(x)-f'(x)y}{|y|^{d+\beta}} A(\bar{y})\, dy$$

$$= \int_{\mathbf{R}^d} \int_0^1 (y, f'(x+wy)-f'(x))dw \frac{A(\bar{y})dy}{|y|^{d+\beta}}, \quad \beta \in [1,2). \tag{23}$$

One-dimensional results above are presented in a way that they extend straightforwardly to the present d-dimensional case. For instance, for $\beta \in (1,2)$, we find the following.

Theorem 3. *Estimates (12) and (14) of Theorem 1 and estimates (20) and (21) of Theorem 2 still hold in d-dimensional case, for densities p satisfying condition (Pd), where one has to plug in the semigroup-generated by (23) in place of* T_β^{At}.

Let us now provide an example of cascading asymptotics showing different regimes for large and for "very large" number of terms.

Let $\beta = 3/4$, B_M a positive constant, $A = \beta/(1-B_M^{-\beta})$, and

$$p(y) = \begin{cases} Ay^{-1-\beta}, & y \in [1, B_M] \\ 0, & y \notin [1, B_M]. \end{cases} \tag{24}$$

Aiming at dealing with large B_M, let us assume for definiteness that $B_M \geq 27$, so that $1 - B_M^{-2/3} \in [8/9, 1)$ and thus $A \in (2/3, 3/4]$.

A random variable τ with distribution $p(y)$ satisfies the requirement of Theorem 1 (i) with $B_m = 1$ and

$$C_1 = \frac{4\beta}{1-B_M^{-\beta}} + \frac{1}{1-\beta} + \frac{6}{(1-\beta)^2(1-B_M^{-\beta})^2} \leq 67$$

On the other hand, the distribution $p(y)$ has finite moments

$$\mu = \mathbf{E}\tau = 3A(B_M^{1/3}-1), \quad \sigma^2 = Var(\tau) = \frac{3}{4}A(B_M^{4/3}-1) - [3A(B_M^{1/3}-1)]^2 \geq B_M^{4/3}/6,$$

$$\rho = \mathbf{E}|\tau - \mathbf{E}\tau|^3 \leq A\int_1^{B_M} [y^{4/3} + (3A)^3 B_M y^{-5/3}]dy \leq 12B_M^{7/3}.$$

Hence, we can apply the Berry–Essen theorem for the distance of normalised sums of τ to the standard law. Combining this theorem with Theorem 1 yields the following result.

Proposition 1. *For a sequence of i.i.d. random variables* τ_i *distributed like* τ *with the distribution* $p(y)$ *given by (24), with* $B_M \geq 27$*, it follows that*

$$d_{Kol}\left(\frac{\tau_1 + \cdots + \tau_n}{n^{3/2}}, \Sigma_A^{2/3}\right) \leq 2\sqrt{C_1}n^{-1/8}, \tag{25}$$

for $n \le B_M^{4/9}/2$. *On the other hand, for all* n,

$$d_{Kol}\left(\frac{(\tau_1-\mu)+\cdots+(\tau_n-\mu)}{\sigma n^{1/2}}, N(0,1)\right) \le C\frac{\rho}{\sigma^3}n^{-1/2} \le 174CB_M^{1/3}n^{-1/2}, \tag{26}$$

where C *is the Berry–Essen constant and* $N(0,1)$ *is a standard normal random variable.*

Remark 4. *The Berry–Essen constant* C *belongs to the interval* $(0.4, 0.5)$. *We refer to [18,19] for the best-known results on its approximation.*

We see that roughly speaking, in order for estimate (26) to make sense, we must have $n \gg B_M^{2/3}$. Thus the interval $n \in [B_M^{4/9}, B_M^{2/3}]$ is the switching region, where the (quasi) $2/3$-stable asymptotics is transferred to the normal CLT. Clearly, if B_M is sufficiently large so that observations for n beyond the level of $B_M^{4/9}$ are not available or feasible, the random variable τ looks like it belongs to the domain of attraction of the $3/2$-stable law and its true asymptotics cannot be revealed. However, the example shows exactly where this quasi attraction actually breaks down and when the true limit becomes visible.

3. Technical Estimates I: Random Walk Approximation for Stable Generators

In this section, we supply the first group of inequalities needed for the application of Proposition A1 in our setting, namely estimates of type (A1).

Theorem 4. *Let* $\epsilon(y)$ *be a measurable bounded function on* $\mathbf{R}_+$ *satisfying assumption (P).*

(i) Let $\beta \in (0,1)$, $\delta \in (1, 1/\beta]$ *and*

$$h_\delta = (\beta B_M^\beta)^{-1/\delta}. \tag{27}$$

Then

$$\left|h^{-1}\int_{-\infty}^{\infty} f(h^{1/\beta}y)p(y)dy - A\int_0^\infty \frac{f(y)dy}{y^{1+\beta}}\right|$$
$$\le \left(2A(1+\frac{\epsilon_0}{B_M}) + \epsilon_0 + B + \frac{B_m}{1-\beta}\right)h^{\delta-1}\|f\|_{Lip}, \tag{28}$$

for any $f \in C_{Lip}(\mathbf{R})$ *vanishing at zero and any* $h \in [h_\delta, 1]$.

In particular, for $g \in C_{Lip}(\mathbf{R})$,

$$\left|h^{-1}\int_0^\infty [g(x \pm h^{1/\beta}y) - g(x)]p(y)dy - A\int_0^\infty \frac{[g(x\pm y)-g(x)]dy}{y^{1+\beta}}\right|$$
$$\le \left(4A(1+\frac{\epsilon_0}{B_M}) + 2\epsilon_0 + B + \frac{B_m}{1-\beta}\right)\|g\|_{Lip}h^{\delta-1}. \tag{29}$$

(ii) Let $\beta \in (1,2)$. *Then*

$$\left|h^{-1}\int_{-\infty}^{\infty} f(h^{1/\beta}y)p(y)dy - A\int_0^\infty \frac{f(y)dy}{y^{1+\beta}}\right|$$
$$\le \left(2A(1+\frac{\epsilon_0}{B_M}) + \frac{\beta B_m\epsilon_0}{2(\beta-1)} + \frac{B^2}{2} + \frac{B_m^2}{(2-\beta)}\right)h^{1+2/\beta}\|f'\|_{Lip}, \tag{30}$$

for

$$h \ge h_0 = [(\beta-1)B_M^{\beta-1}]^{-\beta}.$$

and any differentiable f *vanishing at zero together with its first derivative and such that* $f' \in C_{Lip}(\mathbf{R})$.

In particular,

$$\Big| h^{-1}\int_{-\infty}^{\infty}[g(x\pm h^{1/\beta}y)-g(x)\mp g'(x)h^{1/\beta}y]p(y)dy - A\int_0^\infty \frac{[g(x\pm y)-g(x)\mp g'(x)y]dy}{y^{1+\beta}}\Big|$$

$$\leq \left(4A(1+\frac{\epsilon_0}{B_M})+\frac{\beta B_m\epsilon_0}{(\beta-1)}+\frac{B^2}{2}+\frac{B_m^2}{(2-\beta)}\right)\|g'\|_{Lip}h^{-1+2/\beta}, \tag{31}$$

for $g\in H_{1,1}(\mathbf{R})$ *and* $h\geq h_0$.

(iii) Let again $\beta\in(1,2)$ *and set* $m=\int yp(y)\,dy$ *the first moment of* p *(which is well-defined due to the assumptions of the theorem). Let us assume (for definiteness) that* $m\geq 0$ *and*

$$\max(m,B_m)+m<B_M.$$

Then

$$\Big| h^{-1}\int_{-\infty}^{\infty}[g(x+h^{1/\beta}(y-m))-g(x)]p(y)dy - A\int_0^\infty \frac{[g(x+y)-g(x)-g'(x)y]dy}{y^{1+\beta}}\Big|$$

$$\leq \left(4A(1+\frac{\tilde\epsilon_0}{\tilde B_M})+\frac{\beta\tilde B_m\tilde\epsilon_0}{(\beta-1)}+\frac{\tilde B^2}{2}+\frac{\tilde B_m^2}{(2-\beta)}\right)\|g'\|_{Lip}h^{-1+2/\beta}, \tag{32}$$

for $g\in H_{1,1}(\mathbf{R})$ *and* $h\geq\tilde h_0$, *where*

$$\tilde B=B+m,\tilde B_m=\max(m,B_m),\tilde B_M=B_M-m,\quad \tilde\epsilon_0=\epsilon_0+(m+\epsilon_0)(1+\beta). \tag{33}$$

(iv) Finally, let $\tilde p(y)=[p(y)+p(-y)]/2$ *be the symmetrized version of the probability density* p. *Then*

$$\Big| h^{-1}\int_{-\infty}^{\infty}[g(x+h^{1/\beta}y)-g(x)]\tilde p(y)dy - \frac{A}{2}\int_{-\infty}^\infty \frac{[g(x+y)-g(x)]dy}{y^{1+\beta}}\Big|$$

$$\leq \left(4A(1+\frac{\epsilon_0}{B_M})+\frac{\beta B_m\epsilon_0}{(\beta-1)}+\frac{B^2}{2}+\frac{B_m^2}{(2-\beta)}\right)\|g'\|_{Lip}h^{-1+2/\beta}, \tag{34}$$

for $\beta\in(1,2)$, $g\in H_{1,1}(\mathbf{R})$ *and* $h\geq h_0$. *Integrals in this formula are understood in the sense of the main value.*

Remark 5. *(i) If* $B_M=\infty$, *it is natural to choose* $\delta=1/\beta$ *in Statement (i). We have taken here arbitrary* δ, *because for small* β, *the interval* $[h_{1/\beta},1]$ *can become void even for sufficiently large* B_M. *In addition, notice that the bound* $h\leq 1$ *was used only for* $\delta<1/\beta$ *and is not required whenever* $\delta=1/\beta$. *(ii) Statements (iii) and (iv) are particular cases of a more general situation with different power asymptotics for* p *on positive and negative half-lines. This general case is dealt with in the next result concerning stable limits in arbitrary dimensions. (iii) As seen from the proofs below, most of the explicit constants on the r.h.s. of the estimates above can be essentially tightened. We tried to give the simplest versions that, at the same time, clearly indicate the role of all parameters. (iv) The case with* $\beta=1$ *requires certain modifications that we are not touching here.*

Proof. (i) We shall compare the integrals separately in the domains $[B_m,B_M]$, $[-B,B_m]$ and $[B_M,\infty)$.

Firstly,

$$h^{-1}\int_{B_m}^{B_M}f(h^{1/\beta}y)p(y)\,dy = Ah^{-1}\int_{B_m}^{B_M}f(h^{1/\beta}y)\frac{(1+\epsilon(y)y^{-1})\,dy}{y^{1+\beta}}$$

$$=A\int_{h^{1/\beta}B_m}^{h^{1/\beta}B_M}\frac{f(y)\,dy}{y^{1+\beta}}+Ah^{1/\beta}\int_{h^{1/\beta}B_m}^{h^{1/\beta}B_M}\frac{f(y)\epsilon(h^{-1/\beta}y)\,dy}{y^{2+\beta}}. \tag{35}$$

For the second term, we have the following estimate:

$$\left|Ah^{1/\beta}\int_{h^{1/\beta}B_m}^{h^{1/\beta}B_M}\frac{f(y)\epsilon(h^{-1/\beta}y)\,dy}{y^{2+\beta}}\right|\le A\epsilon_0 f_{Lip}h^{1/\beta}\int_{h^{1/\beta}B_m}^{h^{1/\beta}B_M}\frac{dy}{y^{1+\beta}}$$

$$=\frac{A\epsilon_0}{\beta}f_{Lip}h^{-1+1/\beta}B_m^{-\beta}\le\epsilon_0 f_{Lip}h^{-1+1/\beta},$$

where we used the inequality $\beta B_m^\beta > A$.

Secondly,

$$\left|h^{-1}\int_{-B}^{B_m}f(h^{1/\beta}y)p(y)dy\right|\le h^{-1+1/\beta}f_{Lip}\int_{-B}^{B_m}|y|p(y)dy\le h^{-1+1/\beta}f_{Lip}(B_m+B),$$

and

$$\left|\int_0^{h^{1/\beta}B_m}\frac{f(y)dy}{y^{1+\beta}}\right|\le f_{Lip}\int_0^{B_m h^{1/\beta}}\frac{dy}{y^\beta}=\frac{B_m^{1-\beta}}{1-\beta}f_{Lip}h^{-1+1/\beta},$$

so that

$$\left|h^{-1}\int_{-B}^{B_m}f(h^{1/\beta}y)p(y)dy-A\int_0^{h^{1/\beta}B_m}\frac{f(y)dy}{y^{1+\beta}}\right|$$

$$\le\left(B+B_m+\frac{AB_m^{1-\beta}}{1-\beta}\right)h^{-1+1/\beta}f_{Lip}\le\left(B+\frac{B_m}{1-\beta}\right)h^{-1+1/\beta}f_{Lip}.$$

The latter estimate follows from the inequality $\beta B_m^\beta > A$.

Thirdly,

$$\left|h^{-1}\int_{B_M}^{\infty}f(h^{1/\beta}y)p(y)\,dy\right|\le h^{-1}A\int_{B_M}^{\infty}\frac{|f(h^{1/\beta}y)|(1+\epsilon_0 y^{-1})\,dy}{y^{1+\beta}}$$

$$=A\int_{h^{1/\beta}B_M}^{\infty}\frac{|f(z)|(1+\epsilon_0 h^{1/\beta}z^{-1})\,dz}{z^{1+\beta}}\le h^{-1}A\|f\|\left[\frac{1}{\beta B_M^\beta}+\epsilon_0\frac{1}{(1+\beta)B_M^{1+\beta}}\right]$$

$$\le h^{-1+\delta}A\left(1+\frac{\epsilon_0}{B_M}\right)\|f\|.$$

In the last inequality we used the definition of h_δ and the inequality $\beta B_m^\beta > A$.

Finally, combining the three estimates above yields (28).

(ii) Proof of (30) is analogous. Firstly, for the second term of (35) we obtain the upper bound

$$\frac{1}{2}A\epsilon_0 f'_{Lip}h^{1/\beta}\int_{h^{1/\beta}B_m}^{h^{1/\beta}B_M}\frac{dy}{y^\beta}$$

$$=\frac{A\epsilon_0}{2(\beta-1)}f'_{Lip}h^{-1+2/\beta}B_m^{-(\beta-1)}\le\frac{\beta}{2(\beta-1)}B_m\epsilon_0 f'_{Lip}h^{-1+2/\beta}.$$

Secondly,

$$\left|h^{-1}\int_{-B}^{B_m}f(h^{1/\beta}y)p(y)dy-A\int_0^{h^{1/\beta}B_m}\frac{f(y)dy}{y^{1+\beta}}\right|$$

$$\le\frac{1}{2}\left(B^2+B_m^2+\frac{AB_m^{2-\beta}}{2-\beta}\right)h^{-1+2/\beta}f'_{Lip}\le\left(\frac{B^2}{2}+\frac{B_m^2}{2-\beta}\right)h^{-1+1/\beta}f'_{Lip}.$$

Thirdly,

$$\left|h^{-1}\int_{B_M}^{\infty} f(h^{1/\beta}y)p(y)\,dy\right| \le A\int_{h^{1/\beta}B_M}^{\infty}\frac{|f(z)|(1+\epsilon_0 h^{1/\beta}z^{-1})\,dz}{z^{1+\beta}}$$

$$\le A\int_{h^{1/\beta}B_M}^{\infty}\frac{(1+\epsilon_0 h^{1/\beta}z^{-1})dz}{z^{\beta}}\|f'\|$$

$$\le h^{-1+1/\beta}A\|f'\|\left[\frac{1}{(\beta-1)B_M^{\beta-1}}+\epsilon_0\frac{1}{\beta B_M^{\beta}}\right] \le h^{-1+2/\beta}A\left(1+\frac{\epsilon_0}{B_M}\right)\|f'\|.$$

(iii) We have

$$\int_{-\infty}^{\infty}[g(x+h^{1/\beta}(y-m))-g(x)]p(y)\,dy = \int_{-\infty}^{\infty}[g(x+h^{1/\beta}z)-g(x)]p(m+z)\,dz$$

$$= \int_{-\infty}^{\infty}[g(x+h^{1/\beta}z)-g(x)-g'(x)h^{1/\beta}z]p(m+z)\,dz. \tag{36}$$

We are going to apply Statement (ii) to the probability density $p_m(z) = p(m+z)$ with parameters (33).

If $z \in [\tilde{B}_m, \tilde{B}_M]$, then $z+m \in [B_m, B_M]$ and therefore

$$p_m(z) = p(m+z) = \frac{A(1+\epsilon(m+z)(m+z)^{-1})}{(m+z)^{1+\beta}}$$

$$= Az^{-1-\beta}[1+\epsilon(m+z)z^{-1}(1+q)^{-1}](1+q)^{-1-\beta} = Az^{-1-\beta}[1+\tilde{\epsilon}(z)z^{-1}],$$

where $q = m/z \le 1$ and

$$|\tilde{\epsilon}(z)| \le \epsilon_0 + m(1+\beta) + \epsilon_0(1+\beta)q \le \tilde{\epsilon}_0,$$

because $|(1+q)^{-1-\beta}-1| \le q(1+\beta)$.

Now, by Statement (ii), we can conclude that

$$\int_{-\infty}^{\infty}[g(x+h^{1/\beta}z)-g(x)-g'(x)h^{1/\beta}z]p(m+z)\,dy$$

differs from

$$A\int_{-\infty}^{\infty}[g(x+z)-g(x)-g'(x)z]\frac{dy}{y^{1+\beta}}$$

by the r.h.s. of (31) with all constants with tilde. Consequently, by (36), this implies (32).

(iv) Changing the integration variable y to $-y$ in the second inequality of (31) and summing up with the first one yields

$$|h^{-1}\int_{-\infty}^{\infty}[g(x+h^{1/\beta}y)-g(x)-g'(x)h^{1/\beta}y]\tilde{p}(y)dy - \frac{A}{2}\int_{-\infty}^{\infty}\frac{[g(x+y)-g(x)-g'(x)y]dy}{y^{1+\beta}}|$$

$$\le \left(4A(1+\epsilon_0)+\frac{\beta B_m\epsilon_0}{(\beta-1)}+\frac{B^2}{2}+\frac{B_m^2}{(2-\beta)}\right)\|g'\|_{Lip}h^{-1+2/\beta}.$$

The integrals containing $g'(x)$ vanish yielding (34). □

Let us now obtain a multidimensional extension of these estimates, reducing attention to $\beta \ge 1$.

Recall that for a differentiable function f on $\mathbf{R}^d$ we shall denote by $\|f'\|$ the sup-norm of the Euclidean length of the gradient vector $f' = \nabla f$, and by $\|f''\|$ the sup-norm of the standard matrix norm of the matrix of the second derivatives of f and $\|f'\|_{C^1(\mathbf{R}^d)} = \max(\|f'\|, \|f''\|)$.

Theorem 5. *Let a density p on $\mathbf{R}^d$ satisfy condition (Pd) and $\beta \in (1,2)$.*

Let $m = \int y p(y) dy \in \mathbf{R}^d$ denote the first moment of p (which is well-defined due to the assumptions of the theorem). All estimates below are supposed to hold for

$$h \geq h_0 = [(\beta - 1) B_M^{\beta-1}]^{-\beta}$$

(with the corresponding $\tilde{B}_M$ and $\tilde{h}_0$ in case (iii)) and twice continuously differentiable functions f and g.

(i) For a differentiable f vanishing at zero together with its first derivative, it follows that

$$\left| h^{-1} \int_{\mathbf{R}^d} f(h^{1/\beta} y) p(y) dy - \int_{\mathbf{R}^d} A(\bar{y}) \frac{f(y) dy}{|y|^{d+\beta}} \right|$$
$$\leq \left(2A(1 + \frac{\epsilon_0}{B_M}) + \frac{\beta B_m \epsilon_0}{2(\beta - 1)} + \frac{B_m^2}{(2 - \beta)} \right) h^{-1+2/\beta} \|f'\|_{C^1(\mathbf{R}^d)}. \tag{37}$$

(ii) If $m = 0$, then

$$\left| h^{-1} \int_{\mathbf{R}^d} [g(x + h^{1/\beta} y) - g(x)] p(y)\, dy - \int_{\mathbf{R}^d} A(\bar{y}) \frac{[g(x+y) - g(x)] dy}{y^{|d|+\beta}} \right|$$
$$\leq \left(4A(1 + \frac{\epsilon_0}{B_M}) + \frac{\beta B_m \epsilon_0}{\beta - 1} + \frac{B_m^2}{(2 - \beta)} \right) h^{-1+2/\beta} \|g'\|_{C^1(\mathbf{R}^d)}, \tag{38}$$

(iii) If $m \neq 0$ and $3\|m\| < B_M - B_m$, then

$$|h^{-1} \int_{\mathbf{R}^d} [g(x + h^{1/\beta}(y - m)) - g(x)] p(y) dy - \int_{\mathbf{R}^d} A(\bar{y}) \frac{[g(x+y) - g(x) - g'(x) y] dy}{|y|^{d+\beta}} |$$
$$\leq \left(4A(1 + \frac{\tilde{\epsilon}_0}{\tilde{B}_M}) + \frac{\beta \tilde{B}_m \tilde{\epsilon}_0}{(\beta - 1)} + \frac{\tilde{B}_m^2}{(2 - \beta)} \right) \|g'\|_{C^1(\mathbf{R})} h^{-1+2/\beta}, \tag{39}$$

where

$$\tilde{B}_m = 2\|m\| + B_m, \tilde{B}_M = B_M - m, \quad \tilde{\epsilon}_0 = 2\epsilon_0 + 2^{2+\beta}(|m| + \epsilon_0)(1 + \beta). \tag{40}$$

Proof. Statement (i) is a straightforward extension of the proof of part (ii) of Theorem 4. Statement (ii) is obtained by applying (i) to the function $g(x+y) - g(x) - g'(x)y$ and noting that the integrals containing $g'(x)$ vanish. To prove (iii), we follow the line of arguments of part (iii) of Theorem 4 and start by writing

$$\int_{\mathbf{R}^d} [g(x + h^{1/\beta}(y - m)) - g(x)] p(y)\, dy = \int_{\mathbf{R}^d} [g(x + h^{1/\beta} z) - g(x)] p(m + z)\, dz.$$

Then we apply Statement (ii) to this integral with respect to the probability density $p_m(z) = p(m+z)$. Notice that if $|z| \in [\tilde{B}_m, \tilde{B}_M]$, then $|z + m| \in [B_m, B_M]$. At the same time, if $|z| \in [\tilde{B}_m, \tilde{B}_M]$, then $|z| > 2|m|$, and therefore

$$\frac{|z|}{|z+m|} < 2, \quad \left| \left(\frac{|z|}{|z+m|} \right)^{1+\beta} - 1 \right| \leq 2^{\beta+2}(1+\beta) \frac{|m|}{|z|},$$

implying the required estimate for $\tilde{\epsilon}_0$. □

4. Technical Estimates II: Stable Generators from Stable Semigroups

In this section, we supply the second group of inequalities needed for the application of Proposition A1 in our setting, namely estimates (A2). Thus the main results are given by Propositions 3, 5 and 7. Preliminary Lemmas 1 and 2 must be essentially known to specialists, but explicit constants for the corresponding estimates are not easy to find in the literature, and we sketch proofs for the convenience of readers.

Lemma 1. *Let $\beta \in (0,1)$. Then*

$$\|L_\beta f\| \le \frac{2\alpha - \beta}{\beta(\alpha - \beta)} \|f\|_\alpha, \tag{41}$$

for any $\alpha \in (\beta, 1]$.

Furthermore, if $\alpha \in (0, 1-\beta]$ and $f \in H_{\alpha+\beta}$, then $L_\beta f \in H_\alpha$ and

$$(L_\beta f)_\alpha \le \frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\alpha}\right) f_{\alpha+\beta}. \tag{42}$$

In particular,

$$\|L_\beta f\|_{1-\beta} \le \frac{1}{\beta(1-\beta)} \max(1/2, 2-\beta) \|f\|_1. \tag{43}$$

Proof. Estimate (41) is straightforward from dividing the integral in (10) in two parts, over the interval $[0,1]$ and over the rest of $\mathbf{R}_+$ (more details are given below for the analogous case of $\beta \in (1,2)$).

To prove (42), let us write

$$|L_\beta f(x) - L_\beta f(y)| \le \int_\epsilon^\infty \left| \frac{[f(x+z) - f(y+z)] - [f(x) - f(y)]}{z^{1+\beta}} \right| dz$$

$$+ \int_0^\epsilon \left| \frac{[f(x+z) - f(x)] - [f(y+z) - f(y)]}{z^{1+\beta}} \right| dz.$$

We can estimate the first term in magnitude by

$$2 f_{\alpha+\beta} |x-y|^{\alpha+\beta} \int_\epsilon^\infty \frac{dz}{z^{1+\beta}} = 2 f_{\alpha+\beta} |x-y|^{\alpha+\beta} \frac{1}{\beta \epsilon^\beta},$$

and the second term by

$$2 f_{\alpha+\beta} \int_0^\epsilon z^{\alpha-1}\, dz = 2 f_{\alpha+\beta} \frac{\epsilon^\alpha}{\alpha}.$$

Choosing $\epsilon = |x-y|$ yields the result required.

Finally, to obtain (43) we use (42) with $\alpha = 1-\beta$ and (41) with $\alpha = 1$. □

Let T_β^t be the Feller semigroup of the β-stable Lévy process generated by operator L_β.

Proposition 2. *(i) If $\alpha \in (\beta, 1]$, then*

$$\|T_\beta^h f - f\| \le \frac{2\alpha - \beta}{\beta(\alpha - \beta)} h \|f\|_\alpha. \tag{44}$$

(ii) For any $\alpha \in (0,1)$ and $\gamma \in (\max\{\alpha, \beta\}, 1]$,

$$\|T_\beta^h f - f\| \le \frac{4}{\beta(\gamma - \beta)} h^{\alpha/(\gamma+\alpha)} \|f\|_\alpha. \tag{45}$$

Proof. (i) Since

$$(T_\beta^h - 1) f = \int_0^h T_\beta^t L_\beta f\, dt,$$

estimate (44) is a direct consequence of (41).

(ii) Let ϕ be an even nonnegative smooth function on $\mathbf{R}$ with support in $[-1,1]$ such that $\phi(0) = 1$, ϕ is increasing on $[-1,0]$ and $\int \phi(x)\,dx = 1$. For $\delta \in (0,1]$, let $\phi_\delta(x) = \delta^{-1}\phi(x/\delta)$. For an $f \in C(\mathbf{R})$, let

$$(f \star \phi_\delta)(x) = \int f(y)\phi_\delta(x-y)\,dy = \int f(x-y)\phi_\delta(y)\,dy.$$

If $f \in H_\alpha$, then

$$\|f - f \star \phi_\delta\| \le \sup_x \int |f(x) - f(y)|\phi((x-y)/\delta)\,\frac{dy}{\delta} \le f_\alpha \sup_x \int |x-y|^\alpha \phi((x-y)/\delta)\,\frac{dy}{\delta}$$

$$= f_\alpha \delta^\alpha \int |z|^\alpha \phi(z)\,dz \le f_\alpha \delta^\alpha.$$

On the other hand, $\|(f \star \phi_\delta)'\| \le 2\|f\|/\delta$ and therefore

$$\|(f \star \phi_\delta)\|_{Lip} \le 2\|f\|/\delta, \quad \|f \star \phi_\delta\|_\alpha \le 2\|f\|/\delta^\alpha,$$

for any $\alpha \in (0,1)$.

Writing

$$\|(T_\beta^h - 1)f\| \le \|(T_\beta^h - 1)(f - f \star \phi_\delta)\| + \|(T_\beta^h - 1)(f \star \phi_\delta)\| \le 2\|f - f \star \phi_\delta\| + h\|L_\beta(f \star \phi_\delta)\|$$

and estimating

$$\|L_\beta(f \star \phi_\delta)\| \le \frac{2\gamma - \beta}{\beta(\gamma - \beta)}\|f \star \phi_\delta\|_\gamma \le \frac{2\gamma - \beta}{\beta(\gamma - \beta)}\frac{2}{\delta^\gamma}\|f\|$$

yields

$$\|(T_\beta^h - 1)f\| \le 2\|f\|_\alpha \left(\delta^\alpha + \frac{2\gamma - \beta}{\beta(\gamma - \beta)}\frac{h}{\delta^\gamma}\right).$$

Choosing $\delta = h^{1/(\gamma+\alpha)}$ yields

$$\|(T_\beta^h - 1)f\| \le 2\|f\|_\alpha h^{\alpha/(\alpha+\gamma)}(1 + \frac{2\gamma - \beta}{\beta(\gamma - \beta)}) \le \frac{4}{\beta(\gamma - \beta)}\|f\|_\alpha h^{\alpha/(\alpha+\gamma)}.$$

□

Since

$$\left(\frac{T_\beta^t - 1}{t} - L_\beta\right)f = \frac{1}{t}\int_0^t (T_\beta^s - 1)L_\beta f\,ds,$$

it follows that

$$\|\left(\frac{T_\beta^t - 1}{t} - L_\beta\right)f\| \le \sup_{s \in [0,t]} \|(T_\beta^s - 1)L_\beta f\|.$$

Varying α in (42) and Proposition 2, we can obtain, as direct corollaries, various estimates for the l.h.s. of this inequality. A particular choice of $\alpha = 1 - \beta$ and $\gamma = 1$ leads to the following result.

Proposition 3. *Under assumptions of Proposition 2,*

$$\|\left(\frac{T_\beta^h - 1}{h} - L_\beta\right)f\| \le \begin{cases} \frac{4h}{\beta^2(1-\beta)(1-2\beta)}\|f\|_1, & \beta < 1/2, \\ \frac{6}{\beta^2(1-\beta)^2}h^{(1-\beta)/(2-\beta)}\|f\|_1, & \beta \ge 1/2. \end{cases} \tag{46}$$

Proof. If $\beta < 1/2$, then $\alpha = 1-\beta > \beta$ and we can use (43) and (44) to obtain

$$\|\left(\frac{T_\beta^h - 1}{h} - L_\beta\right) f\| \le \frac{2-3\beta}{\beta(1-2\beta)} h \frac{2-\beta}{\beta(1-\beta)} \|f\|_1 \le \frac{4h}{\beta^2(1-\beta)(1-2\beta)} \|f\|_1.$$

If $\beta \ge 1/2$, then $\alpha = 1-\beta \le \beta$ and we use (43) and (45) to obtain the required estimate. □

Remark 6. *Using arbitrary γ from(45), the second line of the r.h.s. of (46) can be substituted by a more general expression*

$$\frac{6\gamma}{\beta^2(1-\beta)(\gamma-\beta)} h^{(1-\beta)/(1+\gamma-\beta)} \|f\|_1, \quad \beta \ge 1/2, \gamma > \beta.$$

Thus the power of h can be made arbitrary close to $1-\beta$, which is bigger than $(1-\beta)/(2-\beta)$ used in (46).

Let us turn to the case $\beta \in (1,2)$.

Lemma 2. *Let $\beta \in (1,2)$. Then*

$$\|L_\beta f\| \le \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \|f'\|_\alpha, \tag{47}$$

for $\alpha \in (\beta-1, 1]$.

Furthermore, if $\alpha \in (0, 2-\beta]$ and $f' \in H_{\alpha+\beta-1}$, then

$$(L_\beta f)_\alpha \le \frac{4}{\alpha(\alpha+\beta)(\beta-1)} f'_{\alpha+\beta-1} \tag{48}$$

and

$$\|L_\beta f\|_\alpha \le \frac{4}{\alpha(\alpha+\beta)(\beta-1)} \|f'\|_{\alpha+\beta-1}. \tag{49}$$

Proof. Let $\alpha \in (\beta-1, 1]$. Then

$$L_\beta f(x) = \int_0^1 \int_0^y (f'(w+x) - f'(x))\, dw \frac{dy}{y^{1+\beta}} + \int_1^\infty \int_0^y (f'(w+x) - f'(x))\, dw \frac{dy}{y^{1+\beta}},$$

and

$$|L_\beta f(x)| \le f'_\alpha \int_0^1 \int_0^y w^\alpha\, dw \frac{dy}{y^{1+\beta}} + 2\|f'\| \int_1^\infty \frac{y\, dy}{y^{1+\beta}}$$

$$= \|f'\|_\alpha \left[\frac{1}{(\alpha+1)(\alpha-\beta+1)} + \frac{2}{\beta-1}\right] \le \|f'\|_\alpha \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)},$$

proving (47).

Proof of (48) is analogous to the proof of (42). Namely, we decompose the integral in (11) in two parts, over the interval $[0, \epsilon]$ and the rest of $\mathbf{R}$ leading to the estimate

$$|L_\beta f(x) - L_\beta f(y)| \le 2 f'_{\alpha+\beta-1} \left(\frac{\epsilon^\alpha}{\alpha(\alpha+\beta)} + \frac{|x-y|^{\alpha+\beta-1}}{(\beta-1)\epsilon^{\beta-1}}\right).$$

Choosing $\epsilon = |x-y|$ yields

$$(L_\beta f)_\alpha \le 2 f'_{\alpha+\beta-1} \left(\frac{1}{\alpha(\alpha+\beta)} + \frac{1}{\beta-1}\right) = 2 f'_{\alpha+\beta-1} \frac{\beta-1+\alpha(\alpha+\beta)}{\alpha(\alpha+\beta)(\beta-1)}.$$

Since $\alpha \le 2-\beta$ the numerator in the fraction is bounded by $3-\beta \le 2$ yielding (48).

To obtain (49) we apply (48) and then (47) with $\alpha+\beta-1$ instead of α. □

Proposition 4. *Let $\beta \in (1,2)$ and $\alpha \in (\beta-1,1]$. Then*

$$\|T_\beta^h f - f\| \le \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} h\, \|f'\|_\alpha, \tag{50}$$

$$\|T_\beta^h f - f\| \le \frac{9}{2(\alpha+1)(\alpha-\beta+1)(\beta-1)} h^{1/(1+\alpha)}\, \|f'\|. \tag{51}$$

and

$$\|T_\beta^h f - f\| \le \frac{9}{2(\alpha+1)(\alpha-\beta+1)(\beta-1)} h^{\alpha/(1+\alpha)}\, \|f\|_\alpha. \tag{52}$$

Remark 7. *One sees from Proposition 4 that for an effective estimate of $\|T_\beta^h f - f\|$ one either uses higher regularity of f with better behaviour in small h, or less regularity in f resulting in worse estimate in h. Different versions can be used depending on the regularity requirement.*

Proof. Estimate (50) is a direct consequence of (47). To prove the second inequality, we work as if in the proof of (45) exploiting the approximation $f \star \phi_\delta$ to an arbitrary f.

Writing

$$\|(T_\beta^h - 1)f\| \le \|(T_\beta^h - 1)(f - f\star\phi_\delta)\| + \|(T_\beta^h - 1)(f\star\phi_\delta)\| \le 2\|f - f\star\phi_\delta\| + h\|L_\beta(f\star\phi_\delta)\|$$

and estimating

$$\|f - f\star\phi_\delta\| \le \|f'\|\delta,$$

and, for $\alpha \in (\beta-1,1]$,

$$\|L_\beta(f\star\phi_\delta)\| \le \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \|(f\star\phi_\delta)'\|_\alpha \le \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \|f'\| \frac{2}{\delta^\alpha},$$

yields

$$\|(T_\beta^h - 1)f\| \le 2\|f'\| \left(\delta + \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \frac{h}{\delta^\alpha}\right).$$

Choosing $\delta = h^{1/(\alpha+1)}$ yields

$$\|(T_\beta^h - 1)f\| \le 2\|f'\| h^{1/(1+\alpha)} \left(1 + \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)}\right),$$

implying (51) by a rough estimate of the term in the bracket.

Alternatively, we can estimate

$$\|f - f\star\phi_\delta\| \le f_\alpha \delta^\alpha,$$

and, for $\alpha \in (\beta-1,1]$,

$$\|L_\beta(f\star\phi_\delta)\| \le \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \|(f\star\phi_\delta)'\|_\alpha \le \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \|f\|_\alpha \frac{2}{\delta},$$

yields

$$\|(T_\beta^h - 1)f\| \le 2\|f\|_\alpha \left(\delta^\alpha + \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)} \frac{h}{\delta}\right).$$

Choosing again $\delta = h^{1/(\alpha+1)}$ yields

$$\|(T_\beta^h - 1)f\| \le 2\|f\|_\alpha h^{\alpha/(1+\alpha)} \left(1 + \frac{4}{(\alpha+1)(\alpha-\beta+1)(\beta-1)}\right),$$

implying (51). □

As above, for the case of $\beta \in (0,1)$, we obtain the following as a direct corollary.

Proposition 5. *Let $\beta \in (1,2)$ and $\alpha \in (\beta-1,1]$. Then*

$$\| \left(\frac{T_\beta^h - 1}{h} - L_\beta \right) f \| \le \frac{18}{(\alpha+1)^2(\alpha-\beta+1)^2(\beta-1)^2} h^{1/(1+\alpha)} \|f''\|_\alpha. \tag{53}$$

If $\beta \in (1,3/2)$ and $\alpha \in (\beta-1, 2-\beta]$, then also

$$\| \left(\frac{T_\beta^h - 1}{h} - L_\beta \right) f \| \le \begin{cases} \dfrac{18}{\alpha(\alpha+1)(\alpha+\beta)(\alpha-\beta+1)(\beta-1)^2} h^{\alpha/(1+\alpha)} \|f'\|_{\alpha+\beta-1}, \\ \dfrac{16}{\alpha(\alpha+1)(\alpha+\beta)(\alpha-\beta+1)(\beta-1)^2} h \|f''\|_{\alpha+\beta-1}. \end{cases} \tag{54}$$

Proof. Estimate (53) is obtained by combining estimates (51) and (47). Estimate (54) is obtained by combining estimates (50), (52) and (49). □

Choosing $\alpha = 1$ in the first case and $\alpha = 2-\beta$ in the second (also estimating $2-\beta \ge 1/2, 3-\beta \ge 3/2$ in the second case) we obtain the following consequence.

Proposition 6. *Let $\beta \in (1,2)$. Then*

$$\| \left(\frac{T_\beta^h - 1}{h} - L_\beta \right) f \| \le \frac{9}{2(2-\beta)^2(\beta-1)^2} \sqrt{h} \|f''\|_{Lip}.$$

If $\beta \in (1,3/2)$, then also

$$\| \left(\frac{T_\beta^h - 1}{h} - L_\beta \right) f \| \le \begin{cases} \dfrac{12}{(3-2\beta)(\beta-1)^2} h^{(2-\beta)/(3-\beta)} \|f'\|_{Lip}, \\ \dfrac{11}{(3-2\beta)(\beta-1)^2} h \|f''\|_{Lip}. \end{cases} \tag{55}$$

All these estimates and their proofs extend automatically to the d-dimensional case leading to the following result.

Proposition 7. *Let $A(\bar{y})$ be a continuous nonnegative function on S^{d-1} and $A = \int A(\bar{y}) d\bar{y}$. Then operators (22) and (23) generate Feller semigroups on $C_\infty(\mathbf{R}^d)$ satisfying estimates of Lemmas 1 and 2, Proposition 2 with an additional multiplier A on the r.h.s. and Proposition 3 with an additional multiplier A^2 on the r.h.s.*

5. Conclusions

In this paper, we proved various rates of convergence for functional limit theorems with stable laws. In particular, we paid attention to some kind of quasi convergence, where stable approximation holds for large, but not too large n, and in fact, it can vary in different regions of these large n. The method of proof was based essentially on the theory of semigroups.

Let us draw some further perspective.

First of all, our results have more or less straightforward extensions for the convergence of position-dependent random walks to stable-like processes. Unlike the method of Fourier transform, which is tailored to the analysis of constant-coefficient equations, our approach is more robust. To extend our main theorems to variable coefficients, one just has to use general estimate (A3), rather than its simplified version (A4).

Next, we excluded the case $\beta = 1$ that requires certain additional efforts. Bringing this case to the theory is also connected to working out the best rates available for various β and various distances (Kolmogorov, Wasserstein, etc.). As seen from our proofs, several possibilities arise in choosing various intermediate parameters, and our choice here was motivated by simplicity and not by proper consideration of optimality. One can also weaken the assumption (P) on an asymptotic similarity of $p(y)$ with an exact power.

Essential improvement of the results of [6] on functional CLT with stable laws (as performed here) would naturally imply improvements in the results of [6] for the convergence of continuous time random walks (CTRW), which we did not touch here at all.

Finally, the author believes that the methods developed here can be successfully applied to many other related models, as described, for instance, in [20].

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Appendix A. Rates of Convergence for Scaled Markov Chains

Our proofs are all derived from a general estimate for the difference between a Feller semigroup and its discrete (random walk) approximation. The following result was proved essentially in Theorem 8.1.1 of [21] (see also [22]), though here we modify it by stating that all estimates hold only for $h \in [h_0, 1]$ (rather than all positive h in [21]), which does not affect the proof.

Proposition A1. *Let $F_t = e^{tL}$ be a Feller semigroup in the Banach space $B = C_\infty(\mathbf{R}^d)$, generated by an operator L, having a core D, which is itself a Banach space with a norm $\|.\|_D \geq \|.\|_B$. Let F_t be also a bounded semigroup in D such that*

$$\max_{s\in[0,t]} \|F_t\|_{D\to D} \leq e^{mt},$$

with a constant $m \geq 0$ (the growth rate of the semigroup).

Let U_h be a family of contractions in B, and let

$$\|\left(\frac{U_h - 1}{h} - L\right) f\|_B \leq \epsilon_h \|f\|_D, \tag{A1}$$

$$\|\left(\frac{F_h - 1}{h} - L\right) f\|_B \leq \varkappa_h \|f\|_D, \tag{A2}$$

for $h \in [h_0, 1]$ with some constant $h_0 \in (0,1)$ and with some positive continuous functions ϵ_h and $\varkappa_h$ on $[h_0, 1]$. Then the scaled discrete semigroups $(U_h)^{[t/h]}$ are close to the semigroup F_t in the sense that

$$\sup_{s\leq t} \|(U_h)^{[s/h]} f - F_s f\|_B \leq (\varkappa_h + \epsilon_h)\|f\|_D \int_0^t e^{ms}\, ds \tag{A3}$$

for $h \in [h_0, 1]$.

In all our examples, we deal with spatially homogeneous Feller processes and with D being spaces of differentiable or Hölder continuous functions. In these cases, functions and all their derivatives satisfy the same evolution equations and therefore F_t are contractions in all these spaces. Hence (A3) reduces to the simpler relation

$$\sup_{s\leq t} \|(U_h)^{[s/h]} f - F_s f\|_B \leq t(\varkappa_h + \epsilon_h)\|f\|_D \tag{A4}$$

for $h \in [h_0, 1]$.

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