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Article

Hermite-Hadamard Fractional Integral Inequalities via Abel-Gontscharoff Green's Function

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Abstract: The Hermite-Hadamard inequalities for κ-Riemann-Liouville fractional integrals (R-LFI) are presented in this study using a relatively novel approach based on Abel-Gontscharoff Green's function. In this new technique, we first established some integral identities. Such identities are used to obtain new results for monotonic functions whose second derivative is convex (concave) in absolute value. Some previously published inequalities are obtained as special cases of our main results. Various applications of our main consequences are also explored to special means and trapezoid-type formulae.

Keywords: Mittag-Leffler; Abel-Gontscharoff Green's function; Hermite-Hadamard inequalities; convex function; κ-Riemann-Liouville fractional integral

MSC: 26D15; 26D10; 26A33; 34B27

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1. Introduction

Fractional calculus is a branch of mathematics that investigates the possibility of using a real or even a complex number as a differential and integral operator order. This theory has gained significant prominence in recent decades due to its wide applications in mathematical sciences. Samraiz et al. [1,2] explored some new fractional operators and their applications in mathematical physics. Tarasov [3] and Mainardi [4] explained, in detail, the history and applications of fractional calculus in mathematical economics and finance. The reader might also explore the literature [5–10] for further information on fractional calculus.

Convexity theory has a long history and has been the subject of significant research for more than a century. Many researchers have been interested in the various speculations, variants, and augmentations of convexity theory see, e.g., the books [11,12] and the articles [13,14]. This idea has aided the advancement of the concept of inequality prominently. Wu et al. [15] presented the Hermite-Hadamard inequalities for R-LFI, and Khan et al. [16] explored the inequalities by involving Green's function. This theory has been widely studied by various researchers [17–20].

Hermite (1822–1901) sent a letter to the journal Mathesis. An extract from that letter was published in Mathesis **3** (1883, p. 82). This inequality asserts that for a convex (concave) function $\digamma$, we can write

$$\digamma\left(\frac{\beta_1+\beta_2}{2}\right) \le (\ge) \frac{1}{\beta_1-\beta_2}\int_{\beta_1}^{\beta_2} \digamma(\tau)d\tau \le (\ge) \frac{\digamma(\beta_1)+\digamma(\beta_2)}{2}$$

Definition 1 ([11]). *A function $\digamma : I \to \mathbb{R}$ is said to be convex (concave) on I convex subset of $\mathbb{R}$ if the inequality*

$$\digamma(\nu\beta_1 + (1-\nu)\beta_2) \leq (\geq)\nu\digamma(\beta_1) + (1-\nu)\digamma(\beta_2)$$

holds for all $0 \leq \nu \leq 1$ and $\beta_1, \beta_2 \in I$.

Definition 2 ([16]). *The left-sided and right-sided RL fractional integrals $I^{\varrho}_{\vartheta_1^+}\digamma$ and $I^{\varrho}_{\vartheta_2^-}\digamma$ of order $\varrho > 0$ on a finite interval $[\vartheta_1, \vartheta_2]$ are defined by*

$$I^{\varrho}_{\vartheta_1^+}\digamma(\varsigma) = \frac{1}{\Gamma(\varrho)}\int_{\vartheta_1}^{\varsigma}(\varsigma-\nu)^{\varrho-1}\digamma(\nu)d\nu, \ \ \varsigma > \vartheta_1,$$

and

$$I^{\varrho}_{\vartheta_2^-}\digamma(\varsigma) = \frac{1}{\Gamma(\varrho)}\int_{\varsigma}^{\vartheta_2}(\nu-\varsigma)^{\varrho-1}\digamma(\nu)d\nu, \ \ \varsigma < \vartheta_2,$$

respectively. The symbol $\Gamma(\varrho)$ represents the usual Euler's gamma function of ϱ defined by

$$\Gamma(\varrho) = \int_{0}^{\infty}\nu^{\varrho-1}e^{-\nu}d\nu. \tag{1}$$

Definition 3 ([15]). *The κ-fractional integrals of order ϱ with $\kappa > 0$ and $\vartheta \geq 0$ are defined as*

$$I^{\varrho,\kappa}_{\vartheta_1^+}\digamma(\varsigma) = \frac{1}{\kappa\Gamma_\kappa(\varrho)}\int_{\vartheta_1}^{\varsigma}(\varsigma-\nu)^{\frac{\varrho}{\kappa}-1}\digamma(\nu)d\nu, \ \ \varsigma > \vartheta_1,$$

and

$$I^{\varrho,\kappa}_{\vartheta_2^-}\digamma(\varsigma) = \frac{1}{\kappa\Gamma_\kappa(\varrho)}\int_{\varsigma}^{\vartheta_2}(\nu-\varsigma)^{\frac{\varrho}{\kappa}-1}\digamma(\nu)d\nu, \ \ \varsigma < \vartheta_2,$$

where $\Gamma_\kappa(\varrho)$ represents the κ-gamma function of ϱ defined by Diaz et al. in [21] with the following integral representation

$$\Gamma_\kappa(\varrho) = \int_{0}^{\infty}\nu^{\varrho-1}e^{\frac{-\nu^\kappa}{\kappa}}d\nu.$$

It is to be noted that $\kappa = 1$ gives the classical gamma function given in (1).

The Abel-Gontscharoff polynomial and theorem for the 'two-point right focal' problem are referenced in [16]. The Abel-Gontscharoff interpolating polynomial for the 'two-point' problem can be stated as

$$\digamma(\varsigma) = \digamma(\vartheta_1) + (\varsigma-\vartheta_1)\digamma'(\vartheta_2) + \int_{\vartheta_1}^{\vartheta_2}G(\varsigma, \upsilon)\digamma''(\upsilon)d\upsilon, \tag{2}$$

where $G(\varsigma, \upsilon)$ is Green's function for the two-point right focal problem.

Mehmood et al. in [22] introduced the following four functions by keeping Abel-Gontscharoff Green's function for the two-point right focal problem

$$G_1(\chi, \upsilon) = \begin{cases} \vartheta_1 - \upsilon, \vartheta_1 \leq \upsilon \leq \chi, \\ \vartheta_1 - \chi, \chi \leq \upsilon \leq \vartheta_2. \end{cases} \tag{3}$$

$$G_2(\chi, \upsilon) = \begin{cases} \chi - \vartheta_2, \vartheta_1 \leq \upsilon \leq \chi, \\ \upsilon - \vartheta_2, \chi \leq \upsilon \leq \vartheta_2. \end{cases} \tag{4}$$

$$G_3(\chi, \upsilon) = \begin{cases} \chi - \vartheta_1, \vartheta_1 \leq \upsilon \leq \chi, \\ \upsilon - \vartheta_1, \chi \leq \upsilon \leq \vartheta_2. \end{cases} \tag{5}$$

$$G_4(\chi, \upsilon) = \begin{cases} \vartheta_2 - \upsilon, \vartheta_1 \leq \upsilon \leq \chi, \\ \vartheta_2 - \chi, \chi \leq \upsilon \leq \vartheta_2. \end{cases} \tag{6}$$

Sarikaya et al. established in [20] the right and left R-LFI of the following Hermite-Hadamard type inequality.

Theorem 1. *Let $F : [\vartheta_1, \vartheta_2] \to R$ be a positive function with $0 \leq \vartheta_1 < \vartheta_2$ and $F \in L[\vartheta_1, \vartheta_2]$. If F is convex function on $[\vartheta_1, \vartheta_2]$, then the following inequalities for fractional integrals hold:*

$$F\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) \leq \frac{\Gamma(\varrho + 1)}{2(\vartheta_2 - \vartheta_1)^{\varrho}} \left(I^{\varrho}_{\vartheta_1^+} F(\vartheta_2) + I^{\varrho}_{\vartheta_2^-} F(\vartheta_1)\right) \leq \frac{F(\vartheta_1) + F(\vartheta_2)}{2} \tag{7}$$

with $\varrho > 0$.

The motivation behind this study is to explore the Hermite-Hadamard inequalities using Green's functions presented above together with Abel-Gontscharoff interpolating polynomial corresponding to the choice $n = 2$.

2. Main Results

In this section, we establish Hermite-Hadamard inequalities for left generalized fractional integral via Abel-Gontscharoff Green's function for the two-point right focal problem (3).

Theorem 2. *Let F be a twice differentiable and convex function on $[\vartheta_1, \vartheta_2]$ that satisfying the relation given in (2). Then, the double inequality*

$$F\left(\frac{\varrho\vartheta_1 + \kappa\vartheta_2}{\varrho + \kappa}\right) \leq \frac{\Gamma_\kappa(\varrho + \kappa)}{(\vartheta_2 - \vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+} F(\vartheta_2) \leq \frac{\varrho F(\vartheta_1) + \kappa F(\vartheta_2)}{\varrho + \kappa} \tag{8}$$

holds, where $\varrho, \kappa > 0$.

Proof. By making a substitution $\varsigma = \frac{\varrho\vartheta_1 + \kappa\vartheta_2}{\varrho + \kappa}$ in an Abel-Gontscharoff polynomial for the two-point right focal problem interpolating the polynomial presented by (2), we obtain

$$F\left(\frac{\varrho\vartheta_1 + \kappa\vartheta_2}{\varrho + \kappa}\right) = F(\vartheta_1) + \left(\frac{\kappa(\vartheta_2 - \vartheta_1)}{\varrho + \kappa}\right) F'(\vartheta_2)$$
$$+ \int_{\vartheta_1}^{\vartheta_2} G\left(\frac{\varrho\vartheta_1 + \kappa\vartheta_2}{\varrho + \kappa}, \upsilon\right) F''(\upsilon) d\upsilon. \tag{9}$$

Multiplying both sides of (2) by $\frac{\varrho(\vartheta_2 - \varsigma)^{\frac{\varrho}{\kappa} - 1}}{\kappa(\vartheta_2 - \vartheta_1)^{\frac{\varrho}{\kappa}}}$ and integrating with respect to ς, we obtain

$$\frac{\varrho}{\kappa(\vartheta_2 - \vartheta_1)^{\frac{\varrho}{\kappa}}} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varsigma)^{\frac{\varrho}{\kappa}} F(\varsigma) d\varsigma = \frac{\varrho}{\kappa(\vartheta_2 - \vartheta_1)^{\frac{\varrho}{\kappa}}} \left(\int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varsigma)^{\frac{\varrho}{\kappa} - 1} F(\vartheta_1) d\varsigma \right.$$
$$\left. + \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varsigma)^{\frac{\varrho}{\kappa} - 1} (\varsigma - \vartheta_1) F'(\vartheta_2) d\varsigma + \int_{\vartheta_1}^{\vartheta_2} \int_{\vartheta_1}^{\vartheta_2} G(\varsigma, \upsilon) (\vartheta_2 - \varsigma)^{\frac{\varrho}{\kappa} - 1} F''(\upsilon) d\upsilon d\varsigma \right)$$

$$= \frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\frac{\kappa}{\varrho}F(\vartheta_1)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}} + \frac{\kappa^2 F'(\vartheta_2)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}}{\varrho(\varrho+\kappa)}\right.$$
$$\left. + \int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}F''(v)dvd\varsigma\right).$$

This can also be written as

$$\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2) = F(\vartheta_1) + \frac{\kappa F'(\vartheta_2)(\vartheta_2-\vartheta_1)}{(\varrho+\kappa)}$$
$$+ \frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2}\int_{\vartheta_1}^{\vartheta_2} G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}F''(v)dvd\varsigma. \quad (10)$$

Subtracting (10) from (9)

$$F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right) - \frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2) = \int_{\vartheta_1}^{\vartheta_2}\left(G\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\right)\right.$$
$$\left. - \frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2} G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma\right)F''(v)dv. \quad (11)$$

Clearly,

$$\int_{\vartheta_1}^{\vartheta_2} G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma = \frac{\kappa^2}{\varrho(\varrho+\kappa)}\left((\vartheta_2-v)^{\frac{\varrho}{\kappa}+1} - (\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right), \quad (12)$$

where

$$G\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\right) = \begin{cases} \vartheta_1 - v, \vartheta_1 \le v \le \frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}, \\ \frac{\kappa(\vartheta_1-\vartheta_2)}{\varrho+\kappa}, \frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa} \le v \le \vartheta_2. \end{cases} \quad (13)$$

If $\vartheta_1 \le v \le \frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}$, then by utilizing (12) and (13), from (11), we obtain

$$G\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\right) - \frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2} G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma$$
$$= \vartheta_1 - v - \frac{\kappa\left((\vartheta_2-v)^{\frac{\varrho}{\kappa}+1} - (\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}.$$

Now, let

$$g(v) = \vartheta_1 - v - \frac{\kappa\left((\vartheta_2-v)^{\frac{\varrho}{\kappa}+1} - (\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)},$$
$$g'(v) = -1 + \frac{(\vartheta_2-v)^{\frac{\varrho}{\kappa}}}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} \le 0.$$

This proved that g is a decreasing function; therefore, we can write

$$G\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\right)-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2}G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma\leq 0. \tag{14}$$

If $\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\leq v\leq\vartheta_2$, then making use of (12) and (13), from (11), we obtain

$$\begin{aligned}
&G\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\right)-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2}G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma\\
&=\frac{\kappa(\vartheta_1-\vartheta_2)}{\varrho+\kappa}-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\frac{\kappa^2\left((\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}-(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{\varrho(\varrho+\kappa)}\right)\\
&=\frac{-\kappa(\vartheta_2-\vartheta_1)}{\varrho+\kappa}-\frac{\kappa\left((\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}-(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\\
&=\frac{\kappa\left(-(\vartheta_2-\vartheta_1)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}-(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}+(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\\
&=\frac{\kappa\left(-(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}-(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}+(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\\
&=\frac{-\kappa(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\leq 0.
\end{aligned} \tag{15}$$

Since $\digamma$ is convex, therefore $\digamma''(v)\geq 0$ and by using (14) and (15) in (11), we obtain

$$\digamma\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\leq\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}\digamma(\vartheta_2), \tag{16}$$

which is the left half inequality of (8).

Next, we prove the right half inequality of (8). For this purpose, we choose $\varsigma=\vartheta_2$ in Equation (2), and we obtain

$$\digamma(\vartheta_2)=\digamma(\vartheta_1)+(\vartheta_2-\vartheta_1)\digamma'(\vartheta_2)+\int_{\vartheta_1}^{\vartheta_2}G(\vartheta_2,v)\digamma''(v)dv.$$

Adding $\frac{\varrho}{\kappa}\digamma(\vartheta_1)$ on both sides and then dividing by $(\frac{\varrho}{\kappa}+1)$, we obtain

$$\frac{\varrho\digamma(\vartheta_1)+\kappa\digamma(\vartheta_2)}{\varrho+\kappa}=\digamma(\vartheta_1)+\frac{\kappa(\vartheta_2-\vartheta_1)\digamma'(\vartheta_2)}{\varrho+\kappa}+\frac{\kappa}{\varrho+\kappa}\int_{\vartheta_1}^{\vartheta_2}G(\vartheta_1,v)\digamma''(v)dv. \tag{17}$$

Subtracting (10) from (17), we have

$$\begin{aligned}
&\frac{\varrho\digamma(\vartheta_1)+\kappa\digamma(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}\digamma(\vartheta_2)=\int_{\vartheta_1}^{\vartheta_2}\left(\frac{\kappa G(\vartheta_2,v)}{\varrho+\kappa}\right.\\
&\left.-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2}G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma\right)\digamma''(v)dv.
\end{aligned} \tag{18}$$

Using the value of Green's function $(\vartheta_1 - \upsilon)$ for $\vartheta_1 \leq \upsilon \leq \vartheta_2$ and Equation (12), we can write

$$\begin{aligned}
&\frac{\kappa G(\vartheta_2, \upsilon)}{\varrho+\kappa} - \frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\frac{\kappa^2\left((\vartheta_2-\upsilon)^{\frac{\varrho}{\kappa}+1}-(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{\varrho(\varrho+\kappa)}\right)\\
&=\frac{\kappa(\vartheta_1-\upsilon)}{\varrho+\kappa}-\frac{\kappa\left((\vartheta_2-\upsilon)^{\frac{\varrho}{\kappa}+1}-(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\\
&=\frac{\kappa\left((\vartheta_1-\upsilon)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}-(\vartheta_2-\upsilon)^{\frac{\varrho}{\kappa}+1}+(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\\
&=\frac{\kappa\left((\vartheta_2-\upsilon)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}-(\vartheta_2-\upsilon)^{\frac{\varrho}{\kappa}+1}\right)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)} \geq 0. \qquad (19)
\end{aligned}$$

Now, using the convexity of F and (18) in (19), we obtain

$$\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa} \geq \frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2). \qquad (20)$$

Finally, by combining (16) and (20), we arrive at required result. □

The following remark proved the generalization of Theorem 2.

Remark 1. *Substituting $\kappa = 1$ in inequality (8), we find the following results presented in ([16], Theorem 2.2).*

$$F\left(\frac{\varrho\vartheta_1+\vartheta_2}{\varrho+1}\right) \leq \frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}} I^{\varrho}_{\vartheta_1^+}F(\vartheta_2) \leq \frac{\varrho F(\vartheta_1)+F(\vartheta_2)}{\varrho+1}.$$

In next result, we consider the absolute value of difference presented in (18) and utilizing (19) along with additional conditions on F.

Theorem 3. *Let F be a twice differentiable function on $[\vartheta_1, \vartheta_2]$ and $\varrho, \kappa > 0$. Then, we have the following inequalities*

(i) *If $|F''|$ is an increasing function, then*

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa} - \frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right| \leq \frac{\varrho\kappa\left|F''(\vartheta_2)\right|\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}. \qquad (21)$$

(ii) *If $|F''|$ is decreasing function, then*

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa} - \frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right| \leq \frac{\varrho\kappa\left|F''(\vartheta_1)\right|\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}. \qquad (22)$$

(iii) *If $|F''|$ is a convex function, then*

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa} - \frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right| \leq \frac{\max\left(\left|F''(\vartheta_1)\right|,\left|F''(\vartheta_2)\right|\right)\varrho\kappa\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}. \qquad (23)$$

Proof. (i) From (18) and (19), we can write

$$\begin{aligned}&\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\\&\leq\frac{\kappa}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\int_{\vartheta_1}^{\vartheta_2}\Big((\vartheta_2-v)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}\\&-(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}\Big)F''(v)dv. \end{aligned}\tag{24}$$

Since $(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\vartheta_2-v)-(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}\geq 0$ and $|F''|$ is an increasing function, this implies

$$\begin{aligned}&\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\\&\leq\frac{\kappa\left|F''(\vartheta_2)\right|}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\int_{\vartheta_1}^{\vartheta_2}\Big((\vartheta_2-v)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}-(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}\Big)dv\\&=\frac{\kappa\left|F''(\vartheta_2)\right|}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\left(\frac{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\vartheta_2-\vartheta_1)^2}{2}-\frac{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+2}}{\varrho+2\kappa}\right)\\&=\frac{\kappa\left|F''(\vartheta_2)\right|\left(\varrho\left(\vartheta_2-\vartheta_1\right)^2\right)}{2(\varrho+\kappa)(\varrho+2\kappa)},\end{aligned}$$

which is inequality (21).
(ii) Again, using (18) and (19) and by following the same procedure as in case (i), we obtain

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\leq\frac{\kappa\left|F''(\vartheta_1)\right|\varrho\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}.$$

(iii) By using (21) and (22) and the fact that F'' is bounded above by $\max\left(\left|F''(\vartheta_1)\right|,\left|F''(\vartheta_2)\right|\right)$ being a convex function on the interval $\left(\vartheta_1,\vartheta_2\right)$, we find

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\leq\frac{\kappa\max\left|F''(\vartheta_1)\right|,\left|F''(\vartheta_2)\right|\varrho\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}.$$

□

The following remark relates the above theorem with the published results in [16].

Remark 2. *With a choice* $\kappa=1$ *in inequalities* (21)–(23)*, we find the following results presented in ([16], Theorem 2.3).*

$$\left|\frac{\varrho F(\vartheta_1)+F(\vartheta_2)}{\varrho+1}-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right|\leq\frac{\left|F''(\vartheta_2)\right|\varrho\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+1)(\varrho+2)},$$

$$\left|\frac{\varrho F(\vartheta_1)+F(\vartheta_2)}{\varrho+1}-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right|\leq\frac{\left|F''(\vartheta_1)\right|\varrho\left(\vartheta_2-\vartheta_1\right)^2}{2(\varrho+1)(\varrho+2)},$$

$$\left|\frac{\varrho F(\vartheta_1)+F(\vartheta_2)}{\varrho+1}-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right|\leq\frac{\max\left(\left|F''(\vartheta_1)\right|,\left|F''(\vartheta_2)\right|\right)\varrho(\vartheta_2-\vartheta_1)^2}{2(\varrho+1)(\varrho+2)}.$$

Using Green's function from (13) and some additional features on F, we obtain the following theorem.

Theorem 4. *Let F be a twice differentiable function on $[\vartheta_1,\vartheta_2]$ and $\varrho,\kappa>0$. Then, the following statements holds.*

(i) *If $|F''|$ is an increasing function, then*

$$\begin{aligned}&\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\\ &\leq\frac{\kappa(\kappa)^{\frac{\varrho}{\kappa}+3}(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+3}(\varrho+2\kappa)}\left(\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|\right.\\ &\left.\left(\frac{\varrho\left((\varrho+\kappa)^{\frac{\varrho}{\kappa}+1}-2(\varrho)^{\frac{\varrho}{\kappa}+1}\right)}{2\kappa(\kappa)^{\frac{\varrho}{\kappa}+1}}\right)+\left|F''(\vartheta_2)\right|\left(\frac{\varrho}{\kappa}\right)^{\frac{\varrho}{\kappa}+2}\right).\end{aligned}$$

(ii) *If $|F''|$ is a decreasing function, then*

$$\begin{aligned}&\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\\ &\leq\frac{\kappa(\kappa)^{\frac{\varrho}{\kappa}+3}(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+3}(\varrho+2\kappa)}\left(\left|F''(\vartheta_1)\right|\right.\\ &\left.\left(\frac{\varrho\left((\varrho+\kappa)^{\frac{\varrho}{\kappa}+1}-2(\varrho)^{\frac{\varrho}{\kappa}+1}\right)}{2\kappa(\kappa)^{\frac{\varrho}{\kappa}+1}}\right)+\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|\left(\frac{\varrho}{\kappa}\right)^{\frac{\varrho}{\kappa}+2}\right).\end{aligned}$$

(iii) *If $|F''|$ is a convex function, then*

$$\begin{aligned}&\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\leq\frac{\kappa(\kappa)^{\frac{\varrho}{\kappa}+3}(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+3}(\varrho+2\kappa)}\\ &\times\left(\max\left(\left|F''(\vartheta_1)\right|,\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|\right)\left(\frac{\varrho\left((\varrho+\kappa)^{\frac{\varrho}{\kappa}+1}-2(\varrho)^{\frac{\varrho}{\kappa}+1}\right)}{2\kappa(\kappa)^{\frac{\varrho}{\kappa}+1}}\right)\right.\\ &\left.+\max\left(\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|,\left|F''(\vartheta_2)\right|\right)\left(\frac{\varrho}{\kappa}\right)^{\frac{\varrho}{\kappa}+2}\right).\end{aligned}$$

Proof. (i) By using (11)–(13), we can write

$$F\Big(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\Big)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)=\int\limits_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\Bigg(G\Big(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\Big)$$
$$-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int\limits_{\vartheta_1}^{\vartheta_2}G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma\Bigg)F''(v)dv+\int\limits_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}\Bigg(G\Big(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa},v\Big)$$
$$-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int\limits_{\vartheta_1}^{\vartheta_2}G(\varsigma,v)(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}d\varsigma\Bigg)F''(v)dv$$
$$=-\frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\Bigg(\int\limits_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\Big((\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}-(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}-\Big(\frac{\varrho+\kappa}{\kappa}\Big)$$
$$\times(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\vartheta_1-v)\Big)F''(v)dv+\int\limits_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}F''(v)dv\Bigg). \tag{25}$$

This can also be written as

$$\Bigg|F\Big(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\Big)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\Bigg|$$
$$\leq\frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\Bigg(\int\limits_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\Big(\vartheta_2-v\Big)^{\frac{\varrho}{\kappa}+1}-\Big(\vartheta_2-\vartheta_1\Big)^{\frac{\varrho}{\kappa}+1}$$
$$-\Big(\frac{\varrho+\kappa}{\kappa}\Big)\Big(\vartheta_2-\vartheta_1\Big)^{\frac{\varrho}{\kappa}}(\vartheta_1-v)\Big|F''(v)\Big|dv$$
$$+\int\limits_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}\Big(\vartheta_2-v\Big)^{\frac{\varrho}{\kappa}+1}\Big|F''(v)\Big|dv\Bigg)$$
$$\leq\frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\Bigg(\Bigg|F''\Big(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\Big)\Bigg|\int\limits_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\Big(\vartheta_2-v\Big)^{\frac{\varrho}{\kappa}+1}$$
$$-\Big(\vartheta_2-\vartheta_1\Big)^{\frac{\varrho}{\kappa}+1}-\Big(\frac{\varrho+\kappa}{\kappa}\Big)\Big(\vartheta_2-\vartheta_1\Big)^{\frac{\varrho}{\kappa}}\Big(\vartheta_1-v\Big)dv$$
$$+\Big|F''(\vartheta_2)\Big|\int\limits_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}\Big(\vartheta_2-v\Big)^{\frac{\varrho}{\kappa}+1}dv\Bigg)$$
$$=\frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\Bigg(\Bigg|F''\Bigg(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\Bigg)\Bigg|\Big(\vartheta_2-\vartheta_1\Big)^{\frac{\varrho}{\kappa}+2}$$
$$\times\Bigg(\frac{-\kappa(\varrho)^{\frac{\varrho}{\kappa}+2}}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+2}(\varrho+2\kappa)}+\frac{\kappa}{\varrho+2\kappa}-\frac{\kappa}{\varrho+\kappa}+\frac{\kappa}{2(\varrho+\kappa)}\Bigg)$$
$$+\Big|F''(\vartheta_2)\Big|\Big(\vartheta_2-\vartheta_1\Big)^{\frac{\varrho}{\kappa}+2}\Bigg(\frac{\kappa(\varrho)^{\frac{\varrho}{\kappa}+2}}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+2}(\varrho+2\kappa)}\Bigg)\Bigg)$$

$$
= \frac{\kappa(\kappa)^{\frac{\varrho}{\kappa}+3}(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+3}(\varrho+2\kappa)}\left(\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|\right.
$$

$$
\left.\times\left(\frac{\varrho\left((\varrho+\kappa)^{\frac{\varrho}{\kappa}+1}-2(\varrho)^{\frac{\varrho}{\kappa}+1}\right)}{2\kappa(\kappa)^{\frac{\varrho}{\kappa}+1}}\right)+\left|F''(\vartheta_2)\right|\left(\frac{\varrho}{\kappa}\right)^{\frac{\varrho}{\kappa}+2}\right).
$$

Part (ii) can be proved by the same procedure as above.
(iii) Since

$$
\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|
$$

$$
\leq \frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\left(\int\limits_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\left(\vartheta_2-v\right)^{\frac{\varrho}{\kappa}+1}\right.\right.
$$

$$
\left.-\left(\vartheta_2-\vartheta_1\right)^{\frac{\varrho}{\kappa}+1}-\left(\frac{\varrho}{\kappa}+1\right)\left(\vartheta_2-\vartheta_1\right)^{\frac{\varrho}{\kappa}}\left(\vartheta_1-v\right)\right)\left|F''(v)\right|dv
$$

$$
\left.+\int\limits_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}\left(\vartheta_2-v\right)^{\frac{\varrho}{\kappa}+1}\left|F''(v)\right|dv\right).
$$

Since every convex function F defined on an interval $[\vartheta_1,\vartheta_2]$ is bounded above by $\max\{F(\vartheta_1),F(\vartheta_2)\}$. Therefore, we have

$$
\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|
$$

$$
\leq \frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\max\left(\left|F''(\vartheta_1)\right|,\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|\right)\right.
$$

$$
\times\int\limits_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\left(\left(\vartheta_2-v\right)^{\frac{\varrho}{\kappa}+1}-\left(\vartheta_2-\vartheta_1\right)^{\frac{\varrho}{\kappa}+1}-\left(\frac{\varrho}{\kappa}+1\right)\right.
$$

$$
\left.\times\left(\vartheta_2-\vartheta_1\right)^{\frac{\varrho}{\kappa}}\left(\vartheta_1-v\right)\right)dv+\max\left(\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|,\left|F''(\vartheta_2)\right|\right)
$$

$$
\left.\times\int\limits_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}dv\right)
$$

$$
=\frac{\kappa(\kappa)^{\frac{\varrho}{\kappa}+3}(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)^{\frac{\varrho}{\kappa}+3}(\varrho+2\kappa)}\left(\max\left(\left|F''(\vartheta_1)\right|,\left|F''\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|\right)\right.
$$

$$
\times\left(\frac{\varrho\left((\varrho+\kappa)^{\frac{\varrho}{\kappa}+1}-2(\varrho)^{\frac{\varrho}{\kappa}+1}\right)}{2\kappa(\kappa)^{\frac{\varrho}{\kappa}+1}}\right)+\max\left(\left|F''\left(\frac{\rho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)\right|,\right.
$$

$$
\left.\left.\left|F''(\vartheta_2)\right|\right)\left(\frac{\varrho}{\kappa}\right)^{\frac{\varrho}{\kappa}+2}\right),
$$

which is the desired inequality. □

Remark 3. *Let $\kappa=1$, and we obtain the following results presented in ([16], Theorem 2.5).*

The following theorem involves the change of variables in Theorem 4.

Theorem 5. *Let F be a twice differentiable and $|F''|$ be a convex function on $[\vartheta_1, \vartheta_2]$. Then, the inequality*

$$\begin{aligned}&\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|\\&\leq\frac{\kappa^3(\vartheta_2-\vartheta_1)^2}{6(\varrho+\kappa)^3(\varrho+3\kappa)}\left(\left|F''(\vartheta_1)\right|\left(9\left(\frac{\varrho^2}{\kappa}\right)+23\varrho+12\kappa\right)\right.\\&\left.+\left|F''(\vartheta_2)\right|\frac{(7(\varrho^2)+17\varrho\kappa+12\kappa^2)}{\varrho+2\kappa}\right).\end{aligned}$$

holds for any $\varrho, \kappa > 0$.

Proof. Equation (25) can be written as

$$\begin{aligned}&F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\\&=\frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\int_{\vartheta_1}^{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}\left(\left(\frac{\varrho+\kappa}{\kappa}\right)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}\right.\right.\\&\left.\times(\vartheta_1-v)-(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}+(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)F''(v)dv\\&\left.-\int_{\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}}^{\vartheta_2}(\vartheta_2-v)^{\frac{\varrho}{\kappa}+1}F''(v)dv\right).\end{aligned}$$

Let $\tau \in [0,1]$ and $v = \tau\vartheta_1 + (1-\tau)\vartheta_2$, then

$$\begin{aligned}&F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\\&=\frac{\kappa}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\int_1^{\frac{\varrho}{\varrho+\kappa}}\left((\frac{\varrho}{\kappa}+1)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}\left(\vartheta_1-\tau\vartheta_1\right.\right.\right.\\&\left.-(1-\tau)\vartheta_2\right)-\left(\vartheta_2-\tau\vartheta_1-(1-\tau)\vartheta_2\right)^{\frac{\varrho}{\kappa}+1}\\&\left.+(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)\left(\vartheta_1-\vartheta_2\right)d\tau\\&-\int_{\frac{\varrho}{\varrho+\kappa}}^{0}\left(\vartheta_2-\tau\vartheta_1-(1-\tau)\vartheta_2\right)^{\frac{\varrho}{\kappa}+1}F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)\\&\left.\times\left(\vartheta_1-\vartheta_2\right)d\tau\right)\\&=-\frac{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+2}}{(\varrho+\kappa)(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\left(\int_1^{\frac{\varrho}{\varrho+\kappa}}\left(-(1-\tau)\left(\frac{\varrho}{\kappa}+1\right)+1\right)\right.\end{aligned} \tag{26}$$

$$\times F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)d\tau - \int_1^{\frac{\varrho}{\varrho+\kappa}} \tau^{\frac{\varrho}{\kappa}+1} F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)d\tau$$

$$+ \int_0^{\frac{\varrho}{\varrho+\kappa}} \tau^{\frac{\varrho}{\kappa}+1} F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)d\tau\Bigg)$$

$$= \frac{\kappa(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)}\Bigg(\int_{\frac{\varrho}{\varrho+\kappa}}^{1} \Big(\Big(\frac{\varrho}{\kappa}\Big)\tau + \tau - \frac{\varrho}{\kappa}\Big) F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)d\tau$$

$$- \int_0^1 \tau^{\frac{\varrho}{\kappa}+1} F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)d\tau\Bigg).$$

Taking the absolute on both sides and using the convexity of $|F''|$, we obtain

$$\left| F\Big(\frac{\varrho\vartheta_1 + \kappa\vartheta_2}{\varrho+\kappa}\Big) - \frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}} I^{\varrho,\kappa}_{\vartheta_1^+} F(\vartheta_2) \right|$$

$$\leq \frac{\kappa(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)}\Bigg(\int_{\frac{\varrho}{\varrho+\kappa}}^{1} \Big(\Big(\frac{\varrho}{\kappa}\Big)\tau + \tau - \frac{\varrho}{\kappa}\Big)$$

$$\times \Big|F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)\Big|d\tau$$

$$+ \int_0^1 \tau^{\frac{\varrho}{\kappa}+1} \Big|F''\Big(\tau\vartheta_1 + (1-\tau)\vartheta_2\Big)\Big|d\tau\Bigg)$$

$$\leq \frac{\kappa(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)}\Bigg(\int_{\frac{\varrho}{\varrho+\kappa}}^{1} \Big(\tau\Big(\frac{\varrho}{\kappa}+1\Big) - \frac{\varrho}{\kappa}\Big)$$

$$\times \Big(\tau\Big|F''(\vartheta_1)\Big| + (1-\tau)\Big|F''(\vartheta_2)\Big|\Big)d\tau$$

$$+ \int_0^1 \tau^{\frac{\varrho}{\kappa}+1}\Big(\tau\Big|F''(\vartheta_1)\Big| + \Big(1-\tau\Big)\Big|F''(\vartheta_2)\Big|\Big)d\tau\Bigg)$$

$$\leq \frac{\kappa(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)}\Bigg(\frac{\kappa\Big|F''(\vartheta_1)\Big|\Big(3\varrho+2\kappa\Big)}{6(\varrho+\kappa)^2} + \frac{\kappa^2\Big|F''(\vartheta_2)\Big|}{6(\varrho+\kappa)^2}$$

$$+ \frac{\kappa\Big|F''(\vartheta_1)\Big|}{\varrho+3\kappa} + \frac{\kappa^2\Big|F''(\vartheta_2)\Big|}{(\varrho+2\kappa)(\varrho+3\kappa)}\Bigg)$$

$$\leq \frac{\kappa^3(\vartheta_2-\vartheta_1)^2}{6(\varrho+\kappa)^3(\varrho+3\kappa)}\Bigg(\Big|F''(\vartheta_1)\Big|\Big(9\Big(\frac{\varrho^2}{\kappa}\Big) + 23\varrho + 12\kappa\Big)$$

$$+ \frac{\Big|F''(\vartheta_2)\Big|\Big(7(\varrho^2) + 17\varrho\kappa + 12\kappa^2\Big)}{\varrho+2\kappa}\Bigg).$$

Hence, the proof is done. □

Remark 4. *By setting $\kappa = 1$, we obtain the following result presented in ([16], Theorem 2.7).*

$$\left|F\left(\frac{\varrho\vartheta_1+\vartheta_2}{\varrho+1}\right)-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)^2}{6(\varrho+1)^3(\varrho+3)}\left(\left|F''(\vartheta_1)\right|(9(\varrho^2)+23\varrho+12)\right.$$
$$\left.+\left|F''(\vartheta_2)\right|\frac{(7(\varrho^2)+17\varrho+12)}{\varrho+2}\right).$$

Theorem 6. *Let F be a twice differentiable and $|F''|$ be a convex function on $[\vartheta_1, \vartheta_2]$. Then, the inequality*

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|$$
$$\leq \frac{\kappa\varrho(\vartheta_2-\vartheta_1)^2}{3(\varrho+\kappa)(\varrho+3\kappa)}\left(\left|F''(\vartheta_1)\right|+\left|F''(\vartheta_2)\right|\frac{(\varrho+5\kappa)}{2(\varrho+2\kappa)}\right)$$

holds for any $\varrho, \kappa > 0$.

Proof. From Equation (18), we can write

$$\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)$$
$$=\frac{\kappa}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\int_{\vartheta_1}^{\vartheta_2}\left(\left(\vartheta_2-v\right)\left(\vartheta_2-\vartheta_1\right)^{\frac{\varrho}{\kappa}}\right.$$
$$\left.-\left(\vartheta_2-v\right)^{\frac{\varrho}{\kappa}+1}\right)F''(v)dv.$$

For $\tau \in [0,1]$, substituting $v = \tau\vartheta_1 + (1-\tau)\vartheta_2$, we obtain

$$\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)$$
$$=\frac{\kappa(\vartheta_2-\vartheta_1)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}(\varrho+\kappa)}\int_0^1\left(\left(\vartheta_2-\vartheta_1\right)^{\frac{\varrho}{\kappa}}\left(\tau(\vartheta_2-\vartheta_1)\right)\right.$$
$$\left.-\left(\tau(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}+1}\right)\right)F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau,$$
$$=\frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau. \quad (27)$$

By using the convexity of $|F''|$, we obtain

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|$$
$$\leq\frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)\left|F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)\right|d\tau,$$

$$\leq \frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)\left(\tau\left|F''(\vartheta_1)\right|+\left(1-\tau\right)\left|F''(\vartheta_2)\right|\right)d\tau,$$
$$=\frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\left(\left|F''(\vartheta_1)\right|\left(\frac{\varrho}{3(\varrho+3\kappa)}\right)\right.$$
$$\left.+\left|F''(\vartheta_2)\right|\left(\frac{(\varrho^2+5\varrho\kappa)}{6(\varrho+2\kappa)(\varrho+3\kappa)}\right)\right)$$
$$=\frac{\varrho\kappa(\vartheta_2-\vartheta_1)^2}{3(\varrho+\kappa)(\varrho+3\kappa)}\left(\left|F''(\vartheta_1)\right|+\left|F''(\vartheta_2)\right|\frac{(\varrho+5\kappa)}{2(\varrho+2\kappa)}\right).$$

This completes the proof. □

Remark 5. *Corresponding to the choice $\kappa = 1$, in Theorem 6, we obtain the following result explored in ([16], Theorem 2.9)*

$$\left|\frac{\varrho F(\vartheta_1)+F(\vartheta_2)}{\varrho+1}-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right|$$
$$\leq\frac{\varrho(\vartheta_2-\vartheta_1)^2}{3(\varrho+1)(\varrho+3)}\left(\left|F''(\vartheta_1)\right|+\left|F''(\vartheta_2)\right|\frac{(\varrho+5)}{2(\varrho+2)}\right).$$

The next theorem is a combination of Equation (26) given in Theorem 5 and the well-known Jensen's inequality.

Theorem 7. *Let F be a twice differentiable and $|F''|$ be a concave function on $[\vartheta_1,\vartheta_2]$. Then, the inequality*

$$\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right|$$
$$\leq\frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\left(\frac{\kappa}{2(\varrho+\kappa)}\left|F''\left(\frac{3\varrho\vartheta_1+2\vartheta_1\kappa+\vartheta_2\kappa}{3(\varrho+\kappa)}\right)\right|\right.$$
$$\left.+\frac{\kappa}{\varrho+2\kappa}\left|F''\left(\frac{\varrho\vartheta_1+2\vartheta_1\kappa+\vartheta_2\kappa}{\varrho+3\kappa}\right)\right|\right)$$

holds for any $\varrho,\kappa>0$.

Proof. Equation (26) can be rewritten in the following way

$$F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)$$
$$=\frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\left(\int_{\frac{\varrho}{\varrho+\kappa}}^1\left(\tau\left(\frac{\varrho}{\kappa}+1\right)-\frac{\varrho}{\kappa}\right)F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau\right.$$
$$\left.-\int_0^1\tau^{\frac{\varrho}{\kappa}+1}F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau\right).$$

By using the condition of absolute value and then Jensen's integral inequality, we find

$$
\begin{aligned}
&\left|F\left(\frac{\varrho\vartheta_1+\kappa\vartheta_2}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right| \\
&\leq \frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\Bigg[\int_{\frac{\varrho}{\varrho+\kappa}}^{1}\left(\tau\left(\frac{\varrho}{\kappa}+1\right)-\frac{\varrho}{\kappa}\right)d\tau \\
&\times\left|F''\left(\frac{\int_{\frac{\varrho}{\varrho+\kappa}}^{1}\left(\tau(\frac{\varrho}{\kappa}+1)-\frac{\varrho}{\kappa}\right)\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau}{\int_{\frac{\varrho}{\varrho+\kappa}}^{1}\left(\tau(\frac{\varrho}{\kappa}+1)-\frac{\varrho}{\kappa}\right)d\tau}\right)\right| \\
&+\int_0^1 t^{\frac{\varrho}{\kappa}+1}\left|F''\left(\frac{\int_0^1\tau^{\frac{\varrho}{\kappa}+1}\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau}{\int_0^1\tau^{\frac{\varrho}{\kappa}+1}d\tau}\right)\right| \\
&=\frac{\kappa(\vartheta_2-\vartheta_1)^2}{\varrho+\kappa}\left(\frac{\kappa}{2(\varrho+\kappa)}\left|F''\left(\frac{3\varrho\vartheta_1+2\vartheta_1\kappa+\vartheta_2\kappa}{3(\varrho+\kappa)}\right)\right|\right. \\
&\left.+\frac{\kappa}{\varrho+2\kappa}\left|F''\left(\frac{\varrho\vartheta_1+2\vartheta_1\kappa+\vartheta_2\kappa}{\varrho+3\kappa}\right)\right|\right).
\end{aligned}
$$

□

Remark 6. *Letting $\kappa=1$ in Theorem 7 gives the following result presented in ([16], Theorem 2.11).*

$$
\begin{aligned}
&\left|F\left(\frac{\varrho\vartheta_1+\vartheta_2}{\varrho+1}\right)-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right| \\
&\leq \frac{(\vartheta_2-\vartheta_1)^2}{\varrho+1}\left(\frac{1}{2(\varrho+1)}\left|F''\left(\frac{3\varrho\vartheta_1+2\vartheta_1+\vartheta_2}{3(\varrho+1)}\right)\right|\right. \\
&\left.+\frac{1}{\varrho+2}\left|F''\left(\frac{\varrho\vartheta_1+2\vartheta_1+\vartheta_2}{\varrho+3}\right)\right|\right).
\end{aligned}
$$

Theorem 8. *Let F be a twice differentiable and $|F''|$ be a concave function on $[\vartheta_1,\vartheta_2]$. Then, for any $\varrho,\kappa>0$, we have the inequality*

$$
\begin{aligned}
&\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}\right)-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}f(\vartheta_2)\right| \\
&\leq \frac{\kappa\varrho(\vartheta_2-\vartheta_1)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}\left|F''\left(\frac{2\vartheta_1\varrho+\varrho\vartheta_2+4\vartheta_1\kappa+5\vartheta_2\kappa}{3(\varrho+3\kappa)}\right)\right|.
\end{aligned}
$$

Proof. Equation (27) can also be expressed by the following relation.

$$
\begin{aligned}
&\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2) \\
&=\frac{\kappa(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)}\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)F''\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau,
\end{aligned}
$$

By using the condition of absolute value and then Jensen's integral inequality, we find

$$\begin{aligned}
&\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\Gamma_\kappa(\varrho+\kappa)}{(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}I^{\varrho,\kappa}_{\vartheta_1^+}F(\vartheta_2)\right| \\
&\leq \frac{\kappa(\vartheta_2-\vartheta_1)^2}{(\varrho+\kappa)}\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)d\tau\left|F''\left(\frac{\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)\left(\tau\vartheta_1+(1-\tau)\vartheta_2\right)d\tau}{\int_0^1\left(\tau-\tau^{\frac{\varrho}{\kappa}+1}\right)d\tau}\right)\right| \\
&= \frac{\kappa\varrho(\vartheta_2-\vartheta_1)^2}{2(\varrho+\kappa)(\varrho+2\kappa)}\left|F''\left(\frac{2\varrho\vartheta_1+\varrho\vartheta_2+4\kappa\vartheta_1+5\kappa\vartheta_2}{3(\varrho+3\kappa)}\right)\right|.
\end{aligned}$$

Hence, the desired result is proven. □

Remark 7. *If we choose $\kappa = 1$ in Theorem 8, we obtain the following result presented in ([16], Theorem 2.13).*

$$\begin{aligned}
&\left|\frac{\varrho F(\vartheta_1)+F(\vartheta_2)}{\varrho+1}-\frac{\Gamma(\varrho+1)}{(\vartheta_2-\vartheta_1)^{\varrho}}I^{\varrho}_{\vartheta_1^+}F(\vartheta_2)\right| \\
&\leq \frac{\varrho(\vartheta_2-\vartheta_1)^2}{2(\varrho+1)(\varrho+2)}\left|F''\left(\frac{2\vartheta_1\varrho+\varrho\vartheta_2+4\vartheta_1+5\vartheta_2}{3(\varrho+3)}\right)\right|.
\end{aligned}$$

3. Some Applications to Special Means

(i) The arithmetic mean:

$$A = A(\vartheta_1,\vartheta_2) = \frac{\vartheta_1+\vartheta_2}{2},\ \vartheta_1,\vartheta_2 > 0. \tag{28}$$

(ii) The logarithmic mean:

$$L(\vartheta_1,\vartheta_2) = \frac{\vartheta_2-\vartheta_1}{\ln\vartheta_2-\ln\vartheta_1},\ \vartheta_1\neq\vartheta_2\ \vartheta_1,\vartheta_2>0.$$

(iii) The generalized logarithmic mean:

$$L_n(\vartheta_1,\vartheta_2) = \left(\frac{\vartheta_2^{n+1}-\vartheta_1^{n+1}}{(n+1)(\vartheta_2-\vartheta_1)}\right)^{\frac{1}{n}},\ n\in\mathbb{Z}\setminus\{-1,0\},\ \vartheta_1\neq\vartheta_2,\ \vartheta_1,\vartheta_2>0. \tag{29}$$

Proposition 1. *Let $\vartheta_1,\vartheta_2\in\Re^+$, $\vartheta_1<\vartheta_2$, then we have the following inequalities.*

$$\left|A(e^{\vartheta_1},e^{\vartheta_2})-L(e^{\vartheta_1},e^{\vartheta_2})\right| \leq \frac{e^{\vartheta_2}(\vartheta_2-\vartheta_1)^2}{12},$$

$$\left|A(e^{\vartheta_1},e^{\vartheta_2})-L(e^{\vartheta_1},e^{\vartheta_2})\right| \leq \frac{e^{\vartheta_1}(\vartheta_2-\vartheta_1)^2}{12},$$

$$\left|A(e^{\vartheta_1},e^{\vartheta_2})-L(e^{\vartheta_1},e^{\vartheta_2})\right| \leq \frac{\max\left(e^{\vartheta_1},e^{\vartheta_2}\right)(\vartheta_2-\vartheta_1)^2}{12}$$

and

$$\left|A(e^{\vartheta_1}, e^{\vartheta_2}) - L(e^{\vartheta_1}, e^{\vartheta_2})\right| \leq \frac{(\vartheta_2 - \vartheta_1)^2 (e^{\vartheta_1} + e^{\vartheta_2})}{24}.$$

Proof. Using Theorem 3 and making some simplification, we can write

$$\left|\frac{\varrho F(\vartheta_1) + \kappa F(\vartheta_2)}{\varrho + \kappa} - \frac{\varrho}{\kappa(\vartheta_2 - \vartheta_1)^{\frac{\varrho}{\kappa}}} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varsigma)^{\frac{\varrho}{\kappa} - 1} F(\varsigma) d\varsigma\right| \leq \frac{\varrho\kappa \left|F''(\vartheta_2)\right| \left(\vartheta_2 - \vartheta_1\right)^2}{2(\varrho + \kappa)(\varrho + 2\kappa)}.$$

By substituting $\varrho = \kappa$, $F(\vartheta) = e^{\vartheta}$ and using simple calculation, we obtain

$$\left|\frac{\kappa(e^{\vartheta_1}) + \kappa(e^{\vartheta_2})}{\kappa + \kappa} - \frac{\kappa}{\kappa(\vartheta_2 - \vartheta_1)^{\frac{\kappa}{\kappa}}} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - \varsigma)^{\frac{\kappa}{\kappa} - 1} (e^{\varsigma}) d\varsigma\right| \frac{e^{\vartheta_2} \left(\vartheta_2 - \vartheta_1\right)^2}{12}.$$

This can also be written as

$$\left|\frac{e^{\vartheta_1} + e^{\vartheta_2}}{2} - \frac{e^{\vartheta_2} - e^{\vartheta_1}}{(\vartheta_2 - \vartheta_1)}\right| \leq \frac{e^{\vartheta_2} \left(\vartheta_2 - \vartheta_1\right)^2}{12}.$$

Now, making use of (28) and (29), we arrive at the result

$$\left|A(e^{\vartheta_1}, e^{\vartheta_2}) - L(e^{\vartheta_1}, e^{\vartheta_2})\right| \leq \frac{|e^{\vartheta_2}|(\vartheta_2 - \vartheta_1)^2}{12}.$$

By using the same procedure in part (ii) and part (iii) of Theorem 3 and Theorem 6, we find the remaining inequalities. □

Proposition 2. *Let* $\vartheta_1, \vartheta_2 \in \Re^+$, $\vartheta_1 < \vartheta_2$, *then the inequalities*

$$\left|A(\vartheta_1^n, \vartheta_2^n) - L_n^n(\vartheta_1, \vartheta_2)\right| \leq \frac{(\vartheta_2 - \vartheta_1)^2}{24} \left(\left|n(n-1)\right| \left(\vartheta_1^{n-2} + \vartheta_2^{n-2}\right)\right),$$

$$\left|A(\vartheta_1^n, \vartheta_2^n) - L_n^n(\vartheta_1, \vartheta_2)\right| \leq \frac{(\vartheta_2 - \vartheta_1)^2 \left|n(n-1)\right| \vartheta_2^{n-2}}{12},$$

$$\left|A(\vartheta_1^n, \vartheta_2^n) - L_n^n(\vartheta_1, \vartheta_2)\right| \leq \frac{(\vartheta_2 - \vartheta_1)^2 \left|n(n-1)\right| \vartheta_1^{n-2}}{12}$$

and

$$\left|A(\vartheta_1^n, \vartheta_2^n) - L_n^n(\vartheta_1, \vartheta_2)\right| \leq \frac{\max\left(\left|n(n-1)\right| \vartheta_1^{n-2}, \left|n(n-1)\right| \vartheta_2^{n-2}\right) (\vartheta_2 - \vartheta_1)^2}{12}.$$

are true for $n \in \mathbb{Z}$ *with* $|n(n-1)| \geq 2$.

Proof. Using Theorem 6 and making some simplification, we obtain

$$\left|\frac{\varrho F(\vartheta_1)+\kappa F(\vartheta_2)}{\varrho+\kappa}-\frac{\varrho}{\kappa(\vartheta_2-\vartheta_1)^{\frac{\varrho}{\kappa}}}\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-\varsigma)^{\frac{\varrho}{\kappa}-1}F(\varsigma)d\varsigma\right|$$
$$\leq\frac{\kappa\varrho(\vartheta_2-\vartheta_1)^2}{3(\varrho+\kappa)(\varrho+3\kappa)}\left(\left|F''(\vartheta_1)\right|+\left|F''(\vartheta_2)\right|\frac{(\varrho+5\kappa)}{2(\varrho+2\kappa)}\right)$$

By substituting $\varrho=\kappa$ and $F(\vartheta)=\vartheta^n$, where $\vartheta>0$ and $\left|n(n-1)\right|\geq 2$, we obtain

$$\left|\frac{\vartheta_1^n+\vartheta_2^n}{2}-\frac{1}{(\vartheta_2-\vartheta_1)}\int_{\vartheta_1}^{\vartheta_2}(\varsigma)^n d\varsigma\right|\leq\frac{(\vartheta_2-\vartheta_1)^2}{24}\left(\left|n(n-1)\right|\vartheta_1^{n-2}+\left|n(n-1)\right|\vartheta_2^{n-2}\right),$$
$$\left|\frac{(\vartheta_1^n)+(\vartheta_2^n)}{2}-\frac{\vartheta_2^{n+1}-\vartheta_1^{n+1}}{(n+1)(\vartheta_2-\vartheta_1)}\right|\leq\frac{(\vartheta_2-\vartheta_1)^2}{24}\left(\left|n(n-1)\right|\left(\vartheta_1^{n-2}+\vartheta_2^{n-2}\right)\right).$$

Now, by using Equations (28) and (29), we obtain the desired result. Similarly by using the same polynomial function in Theorem 3, we obtain the rest of the inequalities. □

4. Conclusions

The bounds of various functions are studied in optimization theory—a branch of mathematics. The innovative fractional Hermite-Hadamard type inequalities established in this research are based on functions whose second order derivatives with absolute values are convex (concave). A new technique is used to explore the main results by involving Green's function and Abel-Gontscharoff interpolating polynomials for two-point problems with a combination of κ-R-LFI. Jensen's inequality is capably utilized with wide applications in optimization theory. Some applications of our main findings are presented to special means. This study motivates the researchers to establish the various Hermite-Hadamard inequalities by using the other Green's functions G_2, G_3, and G_4 with more general fractional operators.

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