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Operations with Nested Named Sets as a Tool for Artificial Intelligence

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Abstract: Knowledge and data representations are important for artificial intelligence (AI), as well as for intelligence in general. Intelligent functioning presupposes efficient operation with knowledge and data representations in particular. At the same time, it has been demonstrated that named sets, which are also called fundamental triads, instantiate the most fundamental structure in general and for knowledge and data representations in particular. In this context, named sets allow for effective mathematical portrayal of the key phenomenon, called nesting. Nesting plays a weighty role in a variety of fields, such as mathematics and computer science. Computing tools of AI include nested levels of parentheses in arithmetical expressions; different types of recursion; nesting of several levels of subroutines; nesting in recursive calls; multilevel nesting in information hiding; a variety of nested data structures, such as records, objects, and classes; and nested blocks of imperative source code, such as nested repeat-until clauses, while clauses, if clauses, etc. In this paper, different operations with nested named sets are constructed and their properties obtained, reflecting different attributes of nesting. An AI system receives information in the form of data and knowledge and processing information, performs operations with these data and knowledge. Thus, such a system needs various operations for these processes. Operations constructed in this paper perform processing of data and knowledge in the form of nested named sets. Knowing properties of these operations can help to optimize the processing of data and knowledge in AI systems.

Keywords: artificial intelligence; knowledge; data; nesting; named set; operation; function; name; information; structure

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1. Introduction

Knowledge and data representations are important for artificial intelligence (AI), as well as for intelligence in general. With respect to knowledge and data representation, it has been demonstrated that named sets, which are also called fundamental triads, instantiate the most fundamental structure [1].

On the one hand, it has been proved that basic mathematical structures, such as functions, relations, morphisms in categories, functors, operators, graphs, multigraphs, and sets, are named sets. As a result, the theory of named sets forms the unified foundation for mathematics, comprising set theory, logic, category theory, algorithms, univalent foundations, and intuitionism [2].

On the other hand, named sets have demonstrated their efficacy in a diversity of computer and network applications. For instance, as demonstrated in [3], natural language processing applications and downstream tasks often employ named entity recognition (NER). At the same time, contemporary NER systems work with flat annotation from widespread datasets, avoiding the semantic information contained in nested entities. To exclude these deficiencies, a more powerful technique—nested named entity recognition (NNER)—was elaborated [4].

Named sets also form the structure of dynamics at different levels. In particular, it is found that events, actions, and processes have the structure of a named set, or what is the same, a fundamental triad [5,6]. An efficient model of concurrent computations to express branching time is based on the special case of named sets called a Chu space [7,8].

Named sets play an important role in programming languages because utilization of identifiers must be completely namespace-qualified, whereas namespaces are special named sets. Programming languages with direct support for namespaces regularly provide tools for programmers to declare up front what identifiers from a specific namespace they are going to use, and then they can use them without references to the namespace for the remainder of the block [9]. Named sets were useful for constructing a mathematical model of professional organizations and processing XML structures [10].

Named sets form the most fundamental and useful data structure in databases and knowledge bases. Because an important special case of named sets is set-theoretical relations, all relational databases stockpile named sets as data structures and use them to provide information to their users [11,12]. It has also been demonstrated that in temporal databases, named set chains are basic structures [13,14]. Utilization of named sets for data visualization and information retrieval in databases was developed in [15–18], while their usage for database management was elaborated in [19,20].

Additionally, named sets are used as a unified data metamodel, which allows for adaptation of data models on all levels of data representations: from conceptual or high-level to implementation or representational to physical models on the low level. In turn, named sets and their chains serve as resourceful high-level metadata for a variety of applications (cf., [21]).

One of the highest achievements of modern technology is the Internet. At the same time, any network—especially the World Wide Web—extensively uses names and, consequently, name sets because names are related to what they name, which produces a named set [22]. For instance, the essential component of the Internet is the Domain Name System (DNS). It converts user-level domain names into IP addresses. This means that DNS builds in named sets for web operation. A more advanced approach to the Internet employs the Intentional Naming System (INS) instead of DNS [23]. This is a new naming system intended for naming and discovering a variety of resources in future networks of devices and services by building named sets. In INS, names describe applications and other resources rather than network locations, as is the case in DNS. This means that INS also builds in named sets for web operation. The Internet uses naming schemata and, in particular, intentional naming schemata, which provide dynamic construction of named sets used in the functioning of the Internet.

Although DNS has one level of name resolution in building a dynamic named set, researchers argue that there should be three levels of name resolution: from user-level descriptors to service identifiers; from service identifiers to endpoint identifiers; and from endpoint identifiers to IP addresses [24]. This implies the construction of a more complex mathematical system called a named set chain [1].

Moreover, utilization of names on the Internet has become even more important with anew network technology, Named Data Networking (NDN), a future internet architecture inspired by years of empirical research into network usage and a growing awareness of unsolved problems in contemporary internet architectures [25].

Studying and utilizing data and knowledge, researchers found that complex structures often include nested substructures. Informally, nesting means that one system is a subsystem of another system. Nesting exists in many areas as an essential technique in computer science and technology. Examples of nesting include: nested levels of parentheses in arithmetical expressions; different types of recursion; nesting of several levels of subroutines; nesting in recursive calls; multilevel nesting in information hiding; a variety of nested data structures, such as records, objects, and classes; and nested blocks of imperative source code, such as nested repeat-until clauses, while clauses, if clauses, etc. [3,4,26–29].

Taking into account the fundamental nature of named sets, it is natural to utilize named sets for building a mathematical model of nesting. The goal of this model is to develop a system of efficient operations with nested entities for various applications in general intelligence and artificial intelligence, forming a base for algorithmic structures of artificial intelligence. With this in mind, the remainder of this paper is structured as follows.

In Section 2, basic definitions and constructions from named set theory [1] are presented. In Section 3, basic definitions and constructions from nested named set theory [30] are presented. In Section 4, we develop a theory of operations with nested named sets aimed at creating an operational foundation for the development of artificial intelligence. Section 5 contains conclusions and direction for further research.

2. General Named Sets: Basic Definitions and Constructions

We start with the definition of the basic structure for this work: a *named set* or a *fundamental triad*. Calling the same object by different names reflects dissimilar features of the object. The name *named set* directs our attention to the inner structure of the object. The name *fundamental triad* reveals the unity of the object. To achieve higher generality and flexibility, herein, we utilize relative named sets (fundamental triads).

Let us consider two arbitrary classes of objects $\mathfrak{V}$ and $\mathfrak{B}$ as well as a third class $\mathfrak{R}$ of relations, correspondences, ties or connections between objects from $\mathfrak{V}$ and objects from $\mathfrak{B}$. They form the triad $(\mathfrak{V}, \mathfrak{R}, \mathfrak{B})$, which is also a fundamental triad or named set.

Definition 1. *A named set (fundamental triad) in $(\mathfrak{V}, \mathfrak{R}, \mathfrak{B})$ is a triad $\mathbf{X} = (X, f, N)$ where the component X of $\mathbf{X}$ is an object from $\mathfrak{V}$, which is called the support of $\mathbf{X}$ and denoted as $S(\mathbf{X})$; the component N of $\mathbf{X}$ is an object from $\mathfrak{B}$, which is called the component of names (reflector) or set of names of $\mathbf{X}$ and denoted as $N(\mathbf{X})$; and f is a relation (correspondence, tie, or connection) from $\mathfrak{R}$ between X and N, which is called the naming correspondence (reflection) of $\mathbf{X}$ and denoted as $r(\mathbf{X})$.*

This means that $\mathbf{X} = (\mathrm{S}(\mathbf{X}), \mathrm{r}(\mathbf{X}), \mathrm{N}(\mathbf{X}))$. Note that in $\mathbf{X}$, components X and N are not automatically sets, whereas f is not necessarily a mapping or a function, even if X and N are sets. For instance, X and N are sets of words, and f is an algorithm.

When a triad $(\mathfrak{V}, \mathfrak{R}, \mathfrak{B})$ of classes is not specified, we simply call $\mathbf{X} = (X, f, N)$ a named set or fundamental triad.

If we take a set of people P, the set of their names L, and the connection c between people and their names, we obtain a basic named set (fundamental triad) of the form (P, c, L).

Taking a set X of things, the set N of their names and the connection f between things and their names, we have a basic named set (fundamental triad) of the form (X, f, N) [31].

Sometimes people do not understand the difference between triples and triads. Whereas a *triple* is any group with three elements, a *triad* forms a system that has three connected elements or parts. This informal notion of a triple was formalized in mathematics in the form of a structure in an abstract category [32]. This structure is a triad formed from three fundamental triads. Consequently, it is a triad of the second order [1]. When mathematicians understood the composite organization of the *categorical triple*, they changed its name and called it a *monad* [33]. This explicated the relationship between Leibniz's monads and fundamental triads. Besides, a categorical monad is formed from triads and not the other way around. This is similar to the situation when named sets comprise sets as their special case of and not the other way around [1].

Definition 2.

(a) *A named set (fundamental triad) $\mathbf{A} = (A, r, B)$ in $(\mathfrak{V}, \mathfrak{R}, \mathfrak{B})$ is called basic if the relation (correspondence, tie, or connection) goes from A to B (cf. Diagram (1)).*

(b) *A named set (fundamental triad) $\mathbf{A} = (A, r, B)$ in $(\mathfrak{V}, \mathfrak{R}, \mathfrak{B})$ is called bidirectional if the relation (correspondence, tie, or connection) goes from A to B and from B to A (cf. Diagram (2)).*

(c) *A named set (fundamental triad) $\mathbf{A} = (A, r, B)$ in $(\mathfrak{V}, \mathfrak{R}, \mathfrak{B})$ is called cyclic if A = B (cf. Diagram (3)).*

$$A \xrightarrow{\quad r \quad} B \tag{1}$$

$$A \xleftrightarrow{\quad r \quad} B \tag{2}$$

$$A \xrightarrow{\quad R \quad} A \tag{3}$$

Basic and bidirectional named sets are essentially different, even in the case of set-theoretical named sets where the naming correspondence is some binary relation [1]. In this case, a *directed binary relation* between sets A and B consists of pairs with the first element from A and the second element from B. At the same time, an *undirected binary relation* between sets A and B consists of pairs where either the first element is from set A and the second element is from set B or the first element is from set B and the second element is from set A. As a result, the naming correspondence of a basic named set is a directed binary relation. Contrary to this, the naming correspondence of a bidirectional named set is an undirected binary relation.

In many books, binary relations are defined as arbitrary subsets of the Cartesian product of two sets. In contrast to this, Bourbaki constructed a binary relation in the form of a set-theoretical named set in their highly formalized monograph on mathematics. Specifically, they describe a binary relation, R, in the form of an ordered triad (A, G, B), taking A and B as arbitrary sets (or classes) and the naming correspondence, G, as a subset of the Cartesian product, $A \times B$ [34]. This shows that a conventional binary relation has the form of a basic set-theoretical named set.

In many cases, it is possible to represent bidirectional set-theoretical named set with a naming correspondence, which is formed of two directed binary relations. In the case when one of these binary relations is empty, an arbitrary basic set-theoretical named set becomes a special form of a bidirectional set-theoretical named set. In addition, it is possible to represent any bidirectional set-theoretical named set as a composition of some basic set-theoretical named sets.

If we consider two people who are exchanging messages, such as e-mails, or are talking, we obtain a set-theoretical named set (X, f, Y), where the sets X and Y consist of people, and the naming correspondence, f, comprises messages that go between these individuals. A computer network is an example of a cyclic named set (X, f, X) in which the set X consists of computers, and the naming correspondence f consists of all connections between these computers.

Named sets have diverse types and categories [1]. In particular, by varying the classes $\mathfrak{V}$, $\mathfrak{R}$, and $\mathfrak{B}$, we obtain different classes of named sets in the triad ($\mathfrak{V}$, $\mathfrak{R}$, $\mathfrak{B}$).

Definition 3. *Named sets in* ($\mathfrak{V}$, $\mathfrak{R}$, $\mathfrak{B}$) *are called set-based if* $\mathfrak{V}$ *and* $\mathfrak{B}$ *are classes of sets.*

As a host of mathematical constructions are set-based named sets, they are imperative for mathematics. In addition, set-based named sets play a considerable role in networking and computation due to the fact that data have the form of set-theoretical constructions, such as stacks, records, lists, or arrays.

Let us consider some principal classes of set-based named sets in ($\mathfrak{V}$, $\mathfrak{R}$, $\mathfrak{B}$).

Definition 4. *Named sets in* ($\mathfrak{V}$, $\mathfrak{R}$, $\mathfrak{B}$) *are called:*

- *Set-theoretical when* $\mathfrak{R}$ *consists of binary relations between sets;*
- *Algorithmic when* $\mathfrak{R}$ *consists of algorithms;*
- *Automaton named sets when* $\mathfrak{R}$ *consists of automata;*
- *Elementary when* $\mathfrak{V}$ *and* $\mathfrak{B}$ *consist of sets with a single element and* $\mathfrak{R}$ *consists of one tie (relation, connection, or correspondence) between two elements;*
- *Labeled when* $\mathfrak{R}$ *consists of labeled ties (relations, connections, or correspondences);*
- *Labeled set-theoretical when* $\mathfrak{R}$ *consists of labeled binary relations between sets;*
- *Normalized when* $\mathfrak{R}$ *consists of projective ties (relations, connections, or correspondences);*
- *Conormalized when* $\mathfrak{R}$ *consists of totally defined ties (relations, connections, or correspondences);*

- *Binormalized when they are both normalized and conormalized;*
- *Partially functional when $\Re$ consists of partial functions;*
- *Functional when $\Re$ consists of total functions;*
- *Partially cofunctional when $\Re$ is a set of binary relations, the inverses of which are partial functions;*
- *Cofunctional when $\Re$ is a set of binary relations, the inverses of which are total functions;*
- *Categorical when each element in $\mathfrak{V}$ and $\mathfrak{B}$ is a single object that belongs to some category C and each correspondence from $\Re$ consists of morphisms between an object, A, from $\mathfrak{V}$ and between an object B from $\mathfrak{B}$;*
- *Individually named when $\Re$ consists of bijections;*
- *Dynamic when $\Re$ consists of processes;*
- *Topological when $\mathfrak{V}$ and $\mathfrak{B}$ consist of topological spaces and $\Re$ consists of continuous mappings;*
- *Algebraic when $\mathfrak{V}$ and $\mathfrak{B}$ consist of algebraic systems and $\Re$ consists of homomorphisms.*

Remark 1. *Named sets from some of these classes of named sets are individually defined in* [1,18,30].

Examples of named sets from the defined classes:

1. *Cyclic set-theoretical named sets are graphs.*
2. *Cyclic labeled set-theoretical named sets are labeled graphs and hypergraphs.*
3. *A set-theoretical named set* $\mathbf{A} = (A, f, B)$ *where f is a function, is a functional set-theoretical named set.*
4. *An arrow (morphism) in a category is a categorical named set, as well as an elementary named set.*
5. *A triad (A, H(A,B), B), where H(A,B) is the set of all morphisms between objects A and B in a category K is a categorical named set.*
6. *All namespaces are set-theoretical or algorithmic named sets.*
7. *Fiber spaces are topological named sets.*
8. *Sheaves are topological named sets.*

Remark 2. *By introducing set-based named sets, we implicitly assume the existence of two kinds of object sets and named sets. The theory of named sets* [1] *allows for different formalizations of this situation. The simplest approach is to take set as the basic concept and construct named sets using sets, as was done early in the development of named set theory when only set-theoretical named sets were introduced and studied. A more advanced methodology takes named set as the basic concept of the theory, introducing sets as singlenamed sets* [1]. *The synthesized technique is based on two basic types of objects—sets and named sets. The most detailed system starts with three basic categories of objects: named sets; sets; and urelements, which are neither sets nor named sets. However, sets are simply singlenamed sets, whereas any object, including urelements, is a named set because it has a name. In particular, any urelement has the name "urelement".*

In what follows, we consider only set-based named sets.

3. Nested Named Sets: Basic Definitions and Constructions

Set-based named sets are nested when named sets appear as the elements of their supports or/and sets of names. Here is a formal definition.

Definition 5 ([18]). *A set-based named set,* $\mathbf{A} = (A, r, B) = (S(\mathbf{A}), r(\mathbf{A}), N(\mathbf{A}))$*, is called:*

- *Nested from above when some elements from the support,* $N(\mathbf{A})$*,are also named sets;*
- *Nested from below when some elements from the set of names,* $S(\mathbf{A})$*,are also named sets;*
- *Amply nested when some elements from both sets,* $S(\mathbf{A})$*and* $N(\mathbf{A})$*,are also named sets;*
- *Completely nested from above when all elements from the set,* $N(\mathbf{A})$*, are also named sets;*
- *Completely nested from below when all elements from the set,* $S(\mathbf{A})$*, are also named sets;*
- *Completely amply nested when all elements from both sets,* $S(\mathbf{A})$ *and* $N(\mathbf{A})$*, are also named sets.*

Example 1. *An important dynamic structure of computations is recursion* [35]. *A computational process that works with some input and gives some output is a fundamental triad (named set), with its input as the support, its output as the set of names, and the process itself as the correspondence between input and output. In this context, recursion is a nested named set (or more exactly, dynamic self-nested named set) because performance of recursion demands using the same recursion only with another input. This named set is nested from below because inputs on some levels are formed by recursion outputs on the previous level. This means that such inputs, in essence, are processes.*

Example 2. *When a basis B in the two-dimensional vector space* $\boldsymbol{V}$ *over the field of complex numbers* $\boldsymbol{C}$ *is defined, any vector v from* $\boldsymbol{V}$ *is represented by a pair of complex numbers* (a_1, a_2) *from the space* $\boldsymbol{C}^2$*. As a result, we obtain the named set.*

$$\mathbf{Repr} = (\boldsymbol{V}, r_B, \boldsymbol{C}^2)$$

In it, the naming correspondence r_B connects each vector from the space $\boldsymbol{V}$ with its numerical representation in the space $\boldsymbol{C}^2$ with respect to the basis B.

This is an amply nested named set because the complex vector space $\boldsymbol{V}$ has a complex construction with operations, relations, and their properties. For instance, identities in the vector space $\boldsymbol{V}$ define its properties. Consequently, the space $\boldsymbol{V}$ has the structure of a named set, namely:

$$\boldsymbol{V} = (V, p, \{O, R, P\})$$

In this named set, the support V consists of vectors from the space $\boldsymbol{V}$; the naming correspondence, p, connects vectors with their properties, operations, and relations; the set O consists of operations in $\boldsymbol{V}$; the set R consists of relations in $\boldsymbol{V}$; and the set P consists of properties of $\boldsymbol{V}$.

As a result, $\boldsymbol{V}$ is an amply nested named set because any binary relation, operation, or property is a named set [1].

At the same time, $\boldsymbol{C}^2$ is also a named set because each pair (two-dimensional vector) of complex numbers (a_1, a_2) is a named set of the following form:

$$(X, f, \{1, 2, 3\})$$

In this named set, the set X consists of two complex numbers, and the naming correspondence f connects each of these numbers to the natural number indicating its position in the vector (a_1, a_2). Specifically, a_1 is connected to 1, and a_2 is connected to 2. If we take the named set of the vector (2, 3), then the number 2 is connected to 1, whereas the number 3 is connected to 2.

The named set $(X, f, \{1, 2, 3\})$ is also amply nested because each number is (represented by) a named set, and each digit is (represented by) a named set as a symbol [1].

Example 3. *Let us consider* 3×3 *matrices of real numbers. The same reasoning as before shows that each such a matrix is a nested named set because any real number is a named set.*

Example 4. *Any algorithm or program is a named set in its complete representation. The full description of an algorithm/program must include a description of possible inputs I; a description of possible outputs O; and a constructive compressed description of the process DP. As a result, we obtain a named set (fundamental triad) (I, DP, O). Note that usually, only the constructive compressed description of the process is called an algorithm or a program.*
A subroutine is a nested algorithm/program because it is an algorithm or a program in its own right.

Example 5. *Let us consider chess. Chess is an abstract strategy game with no hidden information, in which each position emerges by conditionally emergent extraction when a player chooses the next move.*

Any position in a chess party is a named set, **CH**. Its support P consists of two groups: white and black pieces. At the beginning of the game, each group has 16 pieces. In the process of the game, some pieces are eliminated from the support. The set of names is the chess board B, which is formally represented by a two-dimensional 8 × 8 array or matrix containing 64 squares. The name of each piece from the support is the coordinate of a square on the chess board. The naming correspondence, c, connects each piece with its position (place) on the board B. As a result, for each position, we have the named set (fundamental triad).

$$\mathbf{CH} = (\mathrm{P}, c, \mathrm{B})$$

At the same time, each piece has a name, e.g., pawn or king, and this name is connected to the rules of the piece movements, as well as to the physical image representing this piece. Traditionally, the image was a material thing. When people started printing chess positions and displaying them on the screen, images of pieces became pictures on paper or on the screen.

This means that each piece is also a named set. Consequently, a position is a nested named set.

Because squares of the board have their coordinates and there are relations between them, each square on the board, B, is also a named set.

In the process of the game, the set of names (the board) remains the same, whereas the naming correspondence changes with each move, and the support (the set of pieces) changes from time to time. Each move is an operation with this named set, which is performed according to the definite chess rules. This operation is called a mapping of named sets in the theory of named sets [1].

Many other abstract strategy games, such as GO or checkers, as well as many real-life games, can be represented as processes of sequential application of operations with named sets.

As nested named sets form a special class of named sets, we need to distinguish nested named sets from other named sets.

Definition 6. *Named sets that are not nested named sets are called plain named sets.*

Remark 3. *It is possible to treat any nested named set as a plain named set, ignoring its nested structure.*

Named sets that belong to a nested named set comprise different levels [30].

Definition 7.

(a) *If* **X** *is a nested named set, then it is called the top named set, and it has level 0 with respect to its nesting.*

(b) *The first-level named sets of* **X** *are named sets that are either elements of the set of names, N(***X***), or elements of the support S(***X***).*

(c) *The nth-level named sets of* **X** *are named sets that are either elements of the set of names, N(***Z***) or elements of the support S(***Z***) for some (n -1)th-level named set,* **Z** *of* **X***.*

Remark 4. *It is necessary to make a distinction between named sets nested in some named set* **X** *and named subsets of* **X** [1]. *The difference is similar to the difference between subsets and elements of a set.*

Example 6. *Let us consider the named set* **CH** *= (P, c, B), representing a position in a chess party. Each piece is a first-level named set of* **CH***, whereas the named set* **CP** *= (Pn, c, B), which represents the positions of all pawns in* **CH***, is a named subset of* **CH**.

Remark 5. *In general, a named set can have different levels in another named set as the following example demonstrates.*

Example 7. *Let us consider a list* **L** *that represents people and their characteristics. For instance, each element of* **L** *consists of the name of a person, her/his height, weight, age, and the number of this element in the list. In essence,* **L** *is a named set with characteristics forming the support and names of people constituting the set of names. Moreover, it is a nested named set because each characteristic is also a named set (fundamental triad), for example, height equal to 6 ft. These characteristics have level 1 in* **L**. *The considered characteristics are named numbers, for example, 6 ft or 150 lb. Consequently, these characteristics are named sets of level 2 in* **L**. *The (abstract) numbers, which form the support of the later named sets, have level 3 in* **L**. *At the same time, the same number, e.g., 6, can be also the number of the top named set, i.e., the list* **L**, *and thus have the level 1 in* **L**.

Proposition 1. *If a named set* **Y** *has level* n *in a nested named set* **X** *and a named set* **Z** *has level* m *in a nested named set* **Y**, *then the named set* **Z** *has level* $n + m$ *in the nested named set* **X**.

Proof is performed by induction on the levels of named sets.

Proposition 2. *If a named set* **Y** *is a named subset of a nested named set* **X** *and a named set* **Z** *has level* m *in a nested named set* **Y**, *then the named set* **Z** *has level* m *in the nested named set* **X**.

Proof is performed by induction on the levels of named sets.

4. Operations with Nested Named Sets

Here, we study operations with set-theoretical completely amply nested named sets. In this case, reflections in named sets are binary relations, and all elements from the support and reflectors of these named sets are also set-theoretical named sets [1].

The intricate structure of nested named sets implies that there are three types of operations with nested named sets:

Outer operations are operations with named sets without taking into account their nested structure, i.e., without changing named sets that are nested.

Inner operations are operations with named sets that are applied to the named sets nested within them, i.e., operations that cause changes (transformations) of named sets are induced by changes (transformations) of named sets nested within them.

Combined operations are operations with named sets based on their nested structure, i.e., operations that cause direct changes (transformations) of named sets and changes (transformations)induced by changes (transformations) of named sets nested within them.

Example 8. *Let us consider such an operation as the unification* υ *of a set-theoretical named set. In it, all elements from the set of names are changed for one element, i.e., the set of names becomes a one-element set. The result of unification is a singlenamed set, which is, in essence, an ordinary set* [1]. *By its definition, unification* υ *is an outer operation with named sets.*

Example 9. *As an example of an inner operation, it is possible to take the inner unification* ν *of a named set. In it, if* **X** *is a nested named set, then the first-level named sets nested in* **X** *are unified.*

Example 10. *Let us consider the following operation. Taking a nested named set* **X**, *we delete all elements from the support of* **X** *that are not singlenamed sets and connections of these elements. This is a combined operation with named sets.*

There are two classes of outer operations with nested named sets.

Free outer operations with nested named sets do not depend on the structure of nesting.

Conditional outer operations with nested named sets are dependent on the structure of nesting.

First-level conditional outer operations with nested named sets are dependent on the first-level nested named sets of the operated named sets.

n-level conditional outer operations with nested named sets are dependent on the nested named sets of level n of the operated named sets.

Flat first-level conditional outer operations with nested named sets are dependent only on the first-level nested named sets of the operated named sets.

Flat n-level conditional outer operations with nested named sets are dependent only on the nested named sets of level *n* of the operated named sets.

Free outer operations with nested named sets are operations with plain named sets, which are studied in [1].

Here, we consider some inner operations with amply nested set-theoretical named sets, i.e., we assume that all elements of the supports and sets of names in given named sets are also set-theoretical named sets. For simplicity here, we define and study only first-level inner and combined operations, i.e., operations that involve only the top named sets and named sets that are elements of the support and the set of names of the top named set, i.e., first-level named sets. First, we introduce and study set-theoretical operations. The basic operations with sets are union and intersection. Analyzing the counterparts of these operations for named sets, we find that there are six such operations [1]. For nested named sets, there are even more unions and intersections, which are introduced and studied below.

Definition 8. *The first-level disjunctive union* $\mathbf{X} \cup_{\mathrm{d}} \mathbf{Y}$ *of two nested named sets* $\mathbf{X} = (X, f, N)$ *and* $\mathbf{Y} = (Y, g, M)$ *is defined as the named sets* $\mathbf{Z} = (Z, h, Q)$*, in which:*

$$Z = \{Z_{ij} = X_i \cup Y_j;\ X_i \in X\ \&\ Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cup M_r;\ N_k \in N\ \&\ M_r \in M\}$$

The constructed relation *h* has the following property.

Condition D. *The named set* $X_i \cup Y_j$ *is connected to the named set* $N_k \cup M_r$ *by h if and only if* X_i *is connected to* N_k *by f or* Y_j *is connected to* M_r *by g.*

Note that all elements, X_i from X; Y_j from Y; N_k from N; and M_r from M, are named sets. Here and in what follows, ∪ is the union of (flat) named sets studied in [1].

Let us study properties of the first-level disjunctive union of nested named sets. It has many properties similar to but not identical to those of union of sets.

By definition, the empty named set $\mathbf{\Lambda} = (\varnothing, \varnothing, \varnothing)$ is a nested named set, and we have the following result.

Theorem 1 (Identity Law)**.** $\mathbf{X} \cup_d \mathbf{\Lambda} = \mathbf{\Lambda} \cup_d \mathbf{X} = \mathbf{X}$ *for any nested named set* $\mathbf{X} = (X, f, N)$.

Indeed, $\mathbf{\Lambda} = (\varnothing, \varnothing, \varnothing)$, whereas $X_i \cup \varnothing = X_i$, $r \cup \varnothing = r$, and $N_k \cup \varnothing = N_k$.

This result shows that the empty named set $\mathbf{\Lambda}$ is the identity element with respect to first-level disjunctive union.

Theorem 2 (Commutative Law). *The first-level disjunctive union of named sets is commutative, i.e.,* $\mathbf{X} \cup_d \mathbf{Y} = \mathbf{Y} \cup_d \mathbf{X}$ *for any nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$.

Proof. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, building their first-level disjunctive unions $\mathbf{X} \cup_{\mathrm{d}} \mathbf{Y} = (Z, h, Q)$ and $\mathbf{Y} \cup_{\mathrm{d}} \mathbf{X} = (V, k, R)$. Then, according to Definition 8, we have:

$$Z = \{Z_{ij} = X_i \cup Y_j\,;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cup M_r;\ N_k \in N \text{ and } M_r \in M\}$$

$$\mathrm{V} = \{\mathrm{V_{ij}} = \mathrm{Y_i} \cup \mathrm{X_j};\ \mathrm{X_i} \in \mathrm{X} \text{ and } \mathrm{Y_j} \in \mathrm{Y}\}$$

$$R = \{R_{kr} = M_k \cup N_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h satisfies Condition D.

The named set $X_i \cup Y_j$ is connected to the named set $N_k \cup M_r$ by h if and only if X_i is connected to N_k by f or Y_j is connected to M_r by g.

The relation k also satisfies Condition D.

The named set $X_i \cup Y_j$ is connected to the named set $N_k \cup M_r$ by h if and only if Y_j is connected to M_r by g or X_i is connected to N_k by f.

The union of (plain) named sets is a commutative operation [1]. This means that for all named sets $X_i \in X$ and $Y_j \in Y$, we have $Z_{ij} = V_{ij}$, and for all named sets $M_i \in M$ and $N_j \in N$, we have $Q_{ij} = R_{ij}$. Consequently, we have $Z = V$ and $Q = R$

In addition, relations h and k are constructed in the same way and therefore coincide. Then, by definition, we have:

$$\mathbf{X} \cup_{\mathrm{d}} \mathbf{Y} = \mathbf{Y} \cup_{\mathrm{d}} \mathbf{X}$$

Theorem is proved. □

Theorem 3 (Associative Law)**.** *The first-level disjunctive union is associative, i.e.,* $\mathbf{X} \cup_d (\mathbf{Y} \cup_d \mathbf{Z}) = (\mathbf{X} \cup_d \mathbf{Y}) \cup_d \mathbf{Z}$ *for any nested named sets* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$.

Proof is similar to the proof of Theorem 3. However, instead of the commutativity of the union of (plain) named sets, we use the associativity of the union of (plain) named sets [1].

Theorem 4 (Normalization Law). *For any non-empty nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$, *their first-level disjunctive union* $\mathbf{Y} \cup_d \mathbf{X}$ *is a normalized named set if and only if at least one of the named sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *is a normalized named set*.

Proof. *Sufficiency*. Let us take two non-empty named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that the named set $\mathbf{X}$ is normalized. Building their first-level disjunctive union $\mathbf{X} \cup_{\mathrm{d}} \mathbf{Y} = (Z, h, Q)$, we have:

$$Z = \{Z_{ij} = X_i \cup Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cup M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h satisfies Condition D.

The named set $X_i \cup Y_j$ is connected to the named set $N_k \cup M_r$ by h if and only if X_i is connected to N_k by r or Y_j is connected to M_r by q.

As the named set $\mathbf{X}$ is normalized, for any element $b = N_k$ from N, there is an element $a = X_i$ from X connected to b by r. Consequently, for any element $N_k \cup M_r$ from Q, the element $X_i \cup Y_j$ from Z is connected to $N_k \cup M_r$ by h. As b is an arbitrary element from N, the named set $N_k \cup M_r$ is an arbitrary element from Q, and it has an element from Z connected to it by h. This means that $\mathbf{Y} \cup_{\mathrm{d}} \mathbf{X}$ is a normalized named set.

When $\mathbf{Y}$ is a normalized named set, it is treated in the same way.

Sufficiency is proved.

Necessity is proved by contradiction. To do this, we take two non-empty named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that neither $\mathbf{X}$ nor $\mathbf{Y}$ is normalized. By definition, this means that there is an element $b = N_k$ from N for which there is no element $a = X_i$ from X connected to b. Besides, there is an element $d = M_r$ from N for which there is no element $c = Y_j$ from Y connected to d. Consequently, there is no element $X_i \cup Y_j$ from Z connected to $N_k \cup M_r$. This means that $\mathbf{Y} \cup_{\mathrm{d}} \mathbf{X}$ is not a normalized named set.

Thus, if the first-level disjunctive union $\mathbf{Y} \cup_{\mathrm{d}} \mathbf{X}$ is a normalized named set, then at least one of the named sets $\mathbf{X}$ and $\mathbf{Y}$ must be normalized.

Theorem is proved. □

This theorem is complemented by the following result, which is implied by Theorem 1.

Proposition 3. *The named set* $\mathbf{X} \cup_d \mathbf{\Lambda} = \mathbf{\Lambda} \cup_d \mathbf{X}$ *is normalized if and only if* $\mathbf{X}$ *is a normalized named set.*

It is possible to characterize conormalization in a similar way to normalization.

Theorem 5 (Conormalization Law)**.** *For any non-empty nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$*, their first-level disjunctive union* $\mathbf{X} \cup_d \mathbf{Y}$ *is a conormalized named set if and only if at least one of the named sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *is a conormalized named set.*

Proof is similar to the proof of Theorem 4.

Remark 6. *It is also possible to prove Theorem 5 using the duality relation between a named set and its inverse* [1].

Theorem 5 is complemented by the following result, which is directly implied by Theorem 1.

Proposition 4. *The named set* $\mathbf{X} \cup_d \mathbf{\Lambda} = \mathbf{\Lambda} \cup_d \mathbf{X}$ *is conormalized if and only if* $\mathbf{X}$ *is a conormalized named set.*

Theorems 4 and 5 imply the following result.

Corollary 1 (Binormalization Law)**.** *For any non-empty nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$*, their first-level disjunctive union* $\mathbf{Y} \cup_d \mathbf{X}$ *is a binormalized named set if and only if at least one of the named sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *is a binormalized named set.*

Propositions 3 and 4 imply the following result.

Corollary 2. *The named set* $\mathbf{X} \cup_d \mathbf{\Lambda} = \mathbf{\Lambda} \cup_d \mathbf{X}$ *is binormalized if and only if* $\mathbf{X}$ *is a binormalized named set.*

Remark 7. *The analogues of Theorems 4 and 5 are not true for functional named sets because in a general case, the first-level disjunctive union of two nested named sets can be not functional (cofunctional) even when both named sets are functional (cofunctional), as the following example demonstrates.*

Example 11. *Let us consider nested named sets* $\mathbf{X} = (X, r, N)$ *and* $\mathbf{Y} = (Y, q, M)$*, where* $X = \{X_0\}$*,* $N = \{N_0\}$*,* $Y = \{Y_0\}$*,* $M = \{M_0\}$*, r connects* X_0 *with* N_0*, q connects* Y_0 *with* M_0*,* $X_0 = (T, p, P)$*,* $Y_0 = (V, t, U)$*,* $T = \{a\}$*,* $V = \{b\}$*, and* $N_0 = M_0 = (\{1\}, e, \{1\})$*, where e is the identity mapping.*

Both named sets $\mathbf{X}$ and $\mathbf{Y}$ are functional. However, taking the first-level disjunctive union $\mathbf{Y} \cup_\mathrm{d} \mathbf{X} = (Z, h, Q)$, we see that it is not a functional named set because by construction, the element $X_0 \cup Y_0$ from its support Z is connected by the relation h to three elements: $N_0 \cup N_0 = N_0$, $M_0 \cup M_0 = M_0$, and $N_0 \cup M_0$ from Q.

Theorem 6 (Idempotent Law)**.** $\mathbf{X} \cup_d \mathbf{X} = \mathbf{X}$ *for any nested named set* $\mathbf{X}$ *if and only if the support* $S(\mathbf{X})$ *and the set of names* $N(\mathbf{X})$ *are closed with respect to the unions of their elements while the naming relation* $r(\mathbf{X})$ *satisfies Condition D.*

Proof. *Sufficiency*. Let us take a nested named set $\mathbf{X} = (X, r, N)$ and suppose that it satisfies the conditions of the theorem. Building the first-level disjunctive union $\mathbf{X} \cup_\mathrm{d} \mathbf{X} = (Z, h, Q)$, we have $Z = \{X_0 \cup X_0 = X_0\}$ and $Q = \{N_0 \cup N_0 = N_0\}$. Then the support $\mathrm{S}(\mathbf{X})$ is closed with respect to the unions of its elements. Consequently, by Definition 8, we have:

$$\mathrm{S}(\mathbf{X} \cup_\mathrm{d} \mathbf{X}) = \mathrm{S}(\mathbf{X})$$

Besides, the set of names N(**X**) is also closed with respect to the unions of its elements. Consequently, by Definition 8, we have:

$$\mathrm{N}(\mathbf{X} \cup_{\mathrm{d}} \mathbf{X}) = \mathrm{N}(\mathbf{X})$$

As the relation h satisfies Condition D, by construction, the relation r satisfies this condition and thus, the relation h coincides with the relation r. As a result, we obtain the necessary equality $\mathbf{X} \cup_{\mathrm{d}} \mathbf{X} = \mathbf{X}$.

Sufficiency is proved.

Necessity. If $\mathbf{X} = (X, r, N)$, $\mathbf{X} \cup_{\mathrm{d}} \mathbf{X} = (Z, h, Q)$, and $\mathbf{X} \cup_{\mathrm{d}} \mathbf{X} = \mathbf{X}$, then $X = Z$, i.e., the support, S(**X**) = X is closed with respect to the unions of its elements, $N = Q$, i.e., the set of names N(**X**) = N is closed with respect to the unions of its elements, and relations h and r coincide, and thus, r satisfies Condition C

Theorem is proved. □

Corollary 3. $\mathbf{X} \cup_{\mathrm{d}} \mathbf{X} = \mathbf{X}$ *if the support* S(**X**) *consists of one non-empty element and the set of names* N(**X**) *also consists of only one non-empty element*.

Remark 8. *In a general case, the first-level disjunctive union of two nested named sets is not an idempotent operation, i.e., Theorem 6 is not true for all nested named sets, as the following example demonstrates.*

Example 12. *Let us take a nested named set* $\mathbf{X} = (X, r, N)$*, in which the set X contains exactly two non-empty named sets,* X_1 *and* X_2*, and the set of names N contains only one named set. Building the first-level disjunctive union* $\mathbf{X} \cup_{\mathrm{d}} \mathbf{X} = (Z, h, Q)$*, we see that the support Z contains three named sets,* X_1*,* X_2 *and* $X_1 \cup X_2$*, which are equal neither to* X_1 *nor to* X_2*. Thus, Z is not equal to X, and* $\mathbf{X} \cup_{\mathrm{d}} \mathbf{X}$ *is not equal to* **X**.

One more binary operation with nested named sets is first-level strict disjunctive union.

Definition 9. *The first-level strict disjunctive union of two nested named sets* $\mathbf{X} = (X, f, N)$ *and* $\mathbf{Y} = (Y, g, M)$, is defined as the named sets $\mathbf{Z} = (Z, h, Q) = \mathbf{X} \cup_{\mathrm{sd}} \mathbf{Y}$*, in which*:

$$Z = \{Z_{ij}=X_i \cup Y_j;\ X_i \in X,\ Y_j \in \&X_i \cap Y_j \neq \varnothing\} \cup \{X_i \in X;\ \forall Y_j \in Y (X_i \cap Y_j = \varnothing)\} \cup \{Y_i \in Y;\ \forall X_j \in X (Y_i \cap X_j = \varnothing)\}$$

$$Q = \{Q_{kr} = N_k \cup M_r;\ N_k \in N,\ M_r \in M \ \&\ N_k \cap M_r \neq \varnothing\} \cup \{N_i \in N;\ \forall M_j \in M\ (N_i \cap M_j = \varnothing)\} \cup \{M_i \in M;\ \forall N_j \in N\ (M_i \cap N_j = \varnothing)\}$$

The relation h is constructed in the following way.

If $X_i \cap Y_j \neq \varnothing$, X_i is connected to N_k or Y_j is connected to M_r and $N_k \cap M_r \neq \varnothing$, then $X_i \cup Y_j$ is connected to $N_k \cup M_r$ by h.

If $X_i \cap Y_j \neq \varnothing$, X_i is connected to N_k, Y_j is connected to M_r, and $N_k \cap M_r = \varnothing$, then $X_i \cup Y_j$ is connected to N_k and to M_r by h.

If for any $Y_j \in Y$ $(X_i \cap Y_j = \varnothing)$, X_i is connected to N_k by f and $\forall M_j \in M$ $(N_i \cap M_j = \varnothing)$, then X_i is connected to N_k by h.

If for any $X_j \in X$ $(Y_i \cap X_j = \varnothing)$,$Y_i$ is connected to M_k by f and$\forall N_j \in N$ $(M_i \cap N_j = \varnothing)$, then Y_i is connected to M_k by h.

If for any $Y_j \in Y$ $(X_i \cap Y_j = \varnothing)$, X_i is connected to N_k by f and $N_k \cap M_r \neq \varnothing$, then X_i is connected to $N_k \cup M_r$ by h.

If for any $X_j \in X$ $(Y_i \cap X_j = \varnothing)$, Y_i is connected to M_k by f and $N_k \cap M_r \neq \varnothing$, then Y_i is connected to $N_k \cup M_r$ by h.

Some properties of first-level strict disjunctive union are similar to properties of first-level disjunctive union, whereas others are dissimilar. For instance, Definition 9 implies the following results.

Theorem 7 (Identity Law). $\mathbf{X} \cup_{\mathrm{sd}} \mathbf{\Lambda} = \mathbf{\Lambda} \cup_{\mathrm{sd}} \mathbf{X} = \mathbf{X}$ *for any nested named set* $\mathbf{X} = (X, f, N)$.

This means that the empty named set $\mathbf{\Lambda}$ is the identity element with respect to strict first-level disjunctive union.

Theorem 8 (Commutative Law)**.** $\mathbf{X} \cup_{sd} \mathbf{Y} = \mathbf{Y} \cup_{sd} \mathbf{X}$ *for any nested named sets* **X** *and* **Y**.

First-level strict disjunctive union is intrinsically connected to the union of plain named sets. To show this, we use the following concept.

Definition 10. *Two named sets* **X** *and* **Y** *are disjunctive on the first level if any two named sets* X_i *from* **X** *and* Y_j *from* **Y** *do not intersect.*

Let us consider two nested named sets $\mathbf{X} = (X, f, N)$ and $\mathbf{Y} = (Y, g, M)$.

Theorem 9. *The first-level strict disjunctive union* $\mathbf{X} \cup_{sd} \mathbf{Y} = \mathbf{Y} \cup_{sd} \mathbf{X}$ *of two nested named sets* **X** *and* **Y** *coincides with their union as plain named sets if X and Y, as well as N and M, are disjunctive on the first level.*

One more binary operation with nested named sets is first-level conjunctive union.

Definition 11. *The first-level conjunctive union of two nested named sets* $\mathbf{X} = (X, f, N)$ *and* $\mathbf{Y} = (Y, g, M)$, is defined as the named sets $\mathbf{Z} = (Z, h, Q) = \mathbf{X} \cup_c \mathbf{Y}$, *in which:*

$$Z = \{X_i \cup Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{N_k \cup M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h is constructed in such a way that it satisfies the following condition.

Condition C. *The named set* $X_i \cup Y_j$ *is connected to the named set* $N_k \cup M_r$ *by h if and only if* X_i *is connected to* N_k *by f and* Y_j *is connected to* M_r *by g*.

Let us study the properties of the first-level conjunctive union of nested named sets.

Definition 12. *A disconnected named set* $\mathbf{\Lambda}_{X,N}$ *has the following form:*

$$\mathbf{\Lambda}_{X,N} = (X, \varnothing, N)$$

Theorem 10. $\mathbf{X} \cup_c \mathbf{\Lambda} = \mathbf{\Lambda} \cup_c \mathbf{X} = \mathbf{\Lambda}_{X,N}$ *for any nested named set* $\mathbf{X} = (X, f, N)$.

Proof is similar to the proof of Theorem 1.

Theorem 11 (Commutative Law). $\mathbf{X} \cup_c \mathbf{Y} = \mathbf{Y} \cup_c \mathbf{X}$ *for any nested named sets* **X** *and* **Y**.

Proof is similar to the proof of Theorem 2.

Theorem 12 (Associative Law). *The first-level conjunctive union is associative, i.e.,* $\mathbf{X} \cup_c (\mathbf{Y} \cup_c \mathbf{Z}) = (\mathbf{X} \cup_c \mathbf{Y}) \cup_c \mathbf{Z}$ *for any nested named sets* **X**, **Y**, *and* **Z**.

Proof is similar to the proof of Theorem 3.

Theorem 13 (Normalization Law)**.** *For any non-empty nested named sets* **X** *and* **Y**, *their first-level conjunctive union* $\mathbf{Y} \cup_c \mathbf{X}$ *is a normalized named set if and only if both named sets* **X** *and* **Y** *are normalized named sets*.

Proof. *Sufficiency*. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that both of them are normalized. Building their first-level conjunctive union $\mathbf{X} \cup_c \mathbf{Y} = (Z, h, Q)$, we have:

$$Z = \{Z_{ij} = X_i \cup Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cup M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h satisfies Condition D.

The named set $X_i \cup Y_j$ is connected to the named set $N_k \cup M_r$ by h if and only if X_i is connected to N_k by r or Y_j is connected to M_r by q.

As the named set **X** is normalized, for any element $b = N_k$ from N, there is an element $a = X_i$ from X connected to b by r. As the named set **Y** is normalized, for any element $d = M_r$ from M, there is an element $c = Y_j$ from Y connected to d by q. Consequently, for any element $N_k \cup M_r$ from Q, the element $X_i \cup Y_j$ from Z is connected to $N_k \cup M_r$. As b is an arbitrary element from N, the named set $N_k \cup M_r$ is an arbitrary element from Q, and it has an element from Z connected to it by h. This means that $\mathbf{Y} \cup_c \mathbf{X}$ is a normalized named set.

Sufficiency is proved.

We prove *necessity* by contradiction. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that either **X** or **Y** is not normalized. For convenience, we assume that **X** is not normalized. By definition, this means that there is an element $b = N_k$ from N, for which there is no element $a = X_i$ from X connected to b. Then, by the construction of the relation h for any element $d = M_r$ from M, there is no element $X_i \cup Y_j$ from Z connected to $N_k \cup M_r$. This means that $\mathbf{Y} \cup_c \mathbf{X}$ is not a normalized named set.

Thus, if the first-level conjunctive union $\mathbf{Y} \cup_c \mathbf{X}$ is a normalized named set, then both named sets **X** and **Y** must be normalized.

Theorem is proved. □

Theorem 14 (Conormalization Law)**.** *For any nested named sets* **X** *and* **Y***, their first-level conjunctive union* $\mathbf{Y} \cup_d \mathbf{X}$ *is a conormalized named set if and only if both named sets* **X** *and* **Y** *are conormalized*.

Proof is similar to the proof of Theorem 14.

Theorem 14 and 15 imply the following result.

Corollary 4 (Binormalization Law)**.** *For any nested named sets* **X** *and* **Y***, their first-level conjunctive union* $\mathbf{Y} \cup_d \mathbf{X}$ *is a binormalized named set if and only if both named sets* **X** *and* **Y***, are binormalized.*

To continue with properties of first-level conjunctive union, we remind that if $\mathbf{X} = (X, f, N)$ is a named set, $a \in X$, and f connects a with b, then b is called a name of a [1].

Theorem 15 (Functionality Law)**.** *For any non-empty nested named sets* **X** *and* **Y***, their first-level conjunctive union* $\mathbf{Y} \cup_c \mathbf{X}$ *is a functional named set if and only if both* **X** *and* **Y** *are functional named sets*.

Proof. *Necessity*. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that **X** is not functional and both named sets are not empty. By definition, this means that there is an element $a = X_i$ from X that has two names b and c, i.e., a is connected to elements b and c by the relation f. Then, by the construction of the conjunctive union $\mathbf{Y} \cup_c \mathbf{X}$, any element $X_i \cup Y_j$ from Z is connected to two elements from the set of names, $N(\mathbf{Y} \cup_d \mathbf{X})$. This means that $\mathbf{Y} \cup_d \mathbf{X}$ is not a functional named set.

The case when **Y** is not a functional named set is treated in the same way.

Necessity is proved.

Sufficiency. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that both of them are functional. First, we also suppose that these named sets are not empty. Building their first-level conjunctive union $\mathbf{X} \cup_d \mathbf{Y} = (Z, h, Q)$, we have:

$Z = \{Z_{ij} = X_i \cup Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$

$Q = \{Q_{kr} = N_k \cup M_r;\ N_k \in N \text{ and } M_r \in M\}$

The relation h satisfies Condition D.

The named set $X_i \cup Y_j$ is connected to the named set $N_k \cup M_r$ by h if and only if X_i is connected to N_k by f or Y_j is connected to M_r by g.

As the named set **X** is functional, then any element $a = X_i$ from X is connected to not more than one element $b = N_k$ from N by the relation r. As the named set **Y** is functional, then any element $d = Y_j$ from Y is connected to not more than one element $c = M_r$ from M by the relation q.

Consequently, the element $X_i \cup Y_j$ is connected to not more than one element, $N_k \cup M_r$ from Q by the relation h. This means that the first-level conjunctive union $\mathbf{X} \cup_c \mathbf{Y}$ is a functional named set.

Theorem is proved. □

Note that the empty named set $\mathbf{\Lambda}$ is functional. Therefore, if we have the first-level conjunctive union $\mathbf{X} \cup_c \mathbf{\Lambda}$, and **X** is functional, then $\mathbf{X} \cup_c \mathbf{\Lambda}$ is functional because according to Theorem 1, it is equal to $\mathbf{\Lambda}$.

Theorem 16 (Cofunctionality Law)**.** *For any nested named sets* **X** *and* **Y**, *their first-level conjunctive union* $\mathbf{Y} \cup_c \mathbf{X}$ *is a cofunctional named set if and only if both* **X** *and* **Y** *are cofunctional named sets*.

Proof is similar to the proof of Theorem 15.

Remark 9. *It is also possible to prove Theorem 16 using the duality relation between a named set and its inverse* [1].

Theorems 15 and 16 imply the following result.

Corollary 5 (Individualization Law)**.** *For any nested named sets* **X** *and* **Y**, *their first-level conjunctive union* $\mathbf{Y} \cup_c \mathbf{X}$ *is an individually named set if and only if both* **X** *and* **Y** *are individually named sets*.

We see that some properties of first-level conjunctive union are similar to properties of first-level disjunctive union, whereas others are dissimilar.

One more binary operation with nested named sets is strict first-level conjunctive union.

Definition 13. *The first-level strict conjunctive union of two nested named sets* $\mathbf{X} = (X, f, N)$ *and* $\mathbf{Y} = (Y, g, M)$ *is defined as the named sets* $\mathbf{Z} = (Z, h, Q) = \mathbf{X} \cup_{sc} \mathbf{Y}$, *in which*:

$$Z = \{Z_{ij} = X_i \cup Y_j;\ X_i \in X,\ Y_j \in Y \text{ and } X_i \cap Y_j \neq \varnothing\}$$

$$\mathrm{Q} = \{\mathrm{Q_{kr}} = \mathrm{N_k} \cup \mathrm{M_r};\ \mathrm{N_k} \in \mathrm{N},\ \mathrm{M_r} \in \mathrm{M} \text{ and } \mathrm{N_k} \cap \mathrm{M_r} \neq \varnothing\}$$

The relation,h, is constructed in the following way.

If $X_i \cap Y_j \neq \varnothing$, X_i is connected to N_k, Y_j is connected to M_r, and $N_k \cap M_r \neq \varnothing$, then $X_i \cup Y_j$ is connected to $N_k \cup M_r$ by h.

If $X_i \cap Y_j \neq \varnothing$, X_i is connected to N_k,Y_j is connected to M_r, and $N_k \cap M_r = \varnothing$, then $X_i \cup Y_j$ is connected to N_k and to M_r by h.

If for any $Y_j \in Y$ $(X_i \cap Y_j = \varnothing)$, X_i is connected to N_k by f and $\forall M_j \in M$ $(N_i \cap M_j = \varnothing)$, then X_i is connected to N_k by h.

If for any $X_j \in X$ $(Y_i \cap X_j = \varnothing)$, Y_i is connected to M_k by f and $\forall N_j \in N$ $(M_i \cap N_j = \varnothing)$, then Y_i is connected to M_k by h.

If for any $Y_j \in Y$ $(X_i \cap Y_j = \varnothing)$, X_i is connected to N_k by f and $N_k \cap M_r \neq \varnothing$, then X_i is connected to $N_k \cup M_r$ by h.

If for any $X_j \in X$ $(Y_i \cap X_j = \varnothing)$, Y_i is connected to M_k by f and $N_k \cap M_r \neq \varnothing$, then Y_i is connected to $N_k \cup M_r$ by h.

Some properties of first-level strict conjunctive union are similar to properties of first-level conjunctive union, whereas others are dissimilar. In particular, Definition 13 directly implies the following result.

Theorem 17 (Identity Law)**.** $\mathbf{X} \cup_{\mathrm{sc}} \mathbf{\Lambda} = \mathbf{\Lambda} \cup_{\mathrm{sc}} \mathbf{X} = \mathbf{X}$ *for any nested named set* $\mathbf{X} = (X, f, N)$.

This means that the empty named set $\mathbf{\Lambda}$ is the identity element with respect to strict first-level conjunctive union.

Theorem 18 (Commutative Law)**.** $\mathbf{X} \cup_{sc} \mathbf{Y} = \mathbf{Y} \cup_{sc} \mathbf{X}$ *for any nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$.

Proof is similar to the proof of Theorem 2.

One more binary operation with nested named sets is first-level disjunctive intersection.

Definition 14. *The first-level disjunctive intersection of two nested named sets* $\mathbf{X} = (X, f, N)$ *and* $\mathbf{Y} = (Y, g, M)$ is defined as the named sets $\mathbf{Z} = (Z, h, Q)$, *in which:*

$$Z = \{Z_{ij} = X_i \cap Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cap M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h is constructed in the following way.

If X_i is connected to N_k by f or Y_j is connected to M_r by g, then when $X_i \cap Y_j$ is not empty, it is connected to $N_k \cap M_r$ by h.

Here and in what follows, $\cap$ is the intersection of (flat) named sets studied in [1].

Theorem 19 (Identity Law)**.** $\mathbf{X} \cap_d \mathbf{\Lambda} = \mathbf{\Lambda} \cap_d \mathbf{X} = \mathbf{\Lambda}$ *for any nested named set* $\mathbf{X} = (X, f, N)$.

Indeed, $\mathbf{\Lambda} = (\varnothing, \varnothing, \varnothing)$, while as $X_i \cap \varnothing = \varnothing$, $r \cap \varnothing = \varnothing$, and $N_k \cap \varnothing = \varnothing$.

This result shows that the empty named set $\mathbf{\Lambda}$ is the null element with respect to first-level disjunctive intersection.

Theorem 20 (Commutative Law)**.** $\mathbf{X} \cap_d \mathbf{Y} = \mathbf{Y} \cap_d \mathbf{X}$ *for any nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$.

Proof. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, building their first-level disjunctive intersections $\mathbf{X} \cap_{\mathrm{d}} \mathbf{Y} = (Z, h, Q)$ and $\mathbf{Y} \cap_{\mathrm{d}} \mathbf{X} = (V, k, R)$. Then, according to Definition 8, we have:

$$Z = \{Z_{ij} = X_i \cap Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cap M_r;\ N_k \in N \text{ and } M_r \in M\}$$

$$V = \{V_{ij} = Y_i \cap X_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$R = \{R_{kr} = M_k \cap N_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h is constructed in the following way.

If X_i is connected to N_k by r or Y_j is connected to M_r by q, then when $X_i \cap Y_j$ is not empty, it is connected to $N_k \cap M_r$ by h.

The relation k is constructed in the following way.

If X_i is connected to N_k by r or Y_j is connected to M_r by q, then $X_i \cap Y_j$ is connected to $N_k \cap M_r$ by k.

The intersection of (plain) named sets is a commutative operation [1]. This means that for all named sets $X_i \in X$ and $Y_j \in Y$, we have $Z_{ij} = V_{ij}$, and for all named sets $M_i \in M$ and $N_j \in N$, we have $Q_{ij} = R_{ij}$. Consequently, we obtain the equalities $Z = V$ and $Q = R$

In addition, relations h and k are constructed in the same way and therefore coincide. Then by definition, we have:

$$\mathbf{X} \cap_{\mathrm{d}} \mathbf{Y} = \mathbf{Y} \cap_{\mathrm{d}} \mathbf{X}$$

Theorem is proved. □

Theorem 21 (Associative Law)**.** *The first-level disjunctive intersection is associative, i.e.,* $\mathbf{X} \cap_d (\mathbf{Y} \cap_d \mathbf{Z}) = (\mathbf{X} \cap_d \mathbf{Y}) \cap_d \mathbf{Z}$ *for any nested named sets* **X**, **Y**, *and* **Z**.

Proof is similar to the proof of Theorem 3. However, instead of the associativity of the union of (plain) named sets, we use the associativity of the intersection of (plain) named sets [1].

Theorem 22 (Normalization Law)**.** *For any non-empty nested named sets* **X** *and* **Y**, *their first-level disjunctive intersection* $\mathbf{Y} \cap_d \mathbf{X}$ *is a normalized named set if at least one of the named sets* **X** *and* **Y** *is a normalized named set and each of the elements from its support has a non-empty intersection with some elements from the support of the other named set.*

Proof. Let us take two non-empty named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, supposing that the named set **X** is normalized. Building their first-level disjunctive intersection $\mathbf{X} \cap_{\mathrm{d}} \mathbf{Y} = (Z, h, Q)$, we have:

$$Z = \{Z_{ij} = X_i \cap Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cap M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h is constructed in the following way.

If X_i is connected to N_k by r or Y_j is connected to M_r by q, then when $X_i \cap Y_j$ is not empty, it is connected to $N_k \cap M_r$ by h.

As the named set **X** is normalized, for any element $b = N_k$ from N, there is an element $a = X_i$, from X connected to b. By the initial conditions, there is an element Y_j from Y such that the element $X_i \cap Y_j$ from Z is not empty. Consequently, for any element $N_k \cap M_r$ from Q, the element $X_i \cap Y_j$ from Z is connected to $N_k \cap M_r$. As b is an arbitrary element from N, $N_k \cap M_r$ is an arbitrary element from Q. This means that $\mathbf{Y} \cap_{\mathrm{d}} \mathbf{X}$ is a normalized named set.

The case when **Y** is a normalized named set is treated in the same way.

Theorem is proved. □

Remark 10. *In a general case, the first-level disjunctive intersection* $\mathbf{Y} \cap_d \mathbf{X}$ *is not always a normalized named set, even if both named sets* **X** *and* **Y** *are normalized, as the following example demonstrates.*

Example 13. *Let us consider nested named sets* $\mathbf{X} = (X, r, N)$ *and* $\mathbf{Y} = (Y, q, M)$, *where* $X = \{X_0\}$; $N = \{N_0\}$; $Y = \{Y_0\}$; $M = \{M_0\}$; *r connects* X_0 *with* N_0; *q connects* Y_0 *with* M_0; $X_0 = (T, p, P)$; $Y_0 = (V, t, U)$; $T = \{a\}$, $V = \{b\}$; *and* $N_0 = M_0 = (\{1\}, e, \{1\})$, *where e is the identity mapping. Bothnamed sets* **X** *and* **Y** *are normalized. However, taking the first-level disjunctive intersection* $\mathbf{Y} \cap_{\mathrm{d}} \mathbf{X} = (\mathrm{Z, h, Q})$, *we see that it is not a normalized named set because its support is empty, as* $X_0 \cap Y_0 = \mathbf{\Lambda}$, *whereas its set of names, Q, contains the named set* N_0 *and is therefore not empty.*

It is possible to characterize conormalization in a similar way to normalization.

Theorem 23 (Conormalization Law)**.** *For any non-empty nested named sets* **X** *and* **Y**, *their first-level disjunctive intersection* $\mathbf{Y} \cap_d \mathbf{X}$ *is a conormalized named set if at least one of the named sets* **X** *and* **Y** *is a conormalized named set and each of the elements from its set of names has a non-empty intersection with some elements from the set of names of the other named set.*

Proof is similar to the proof of Theorem 22.

Remark 11. *It is also possible to prove Theorem 23 using the duality relation between a named set and its inverse* [1].

Remark 12. *In a general case, the first-level disjunctive intersection* $\mathbf{Y} \cap_d \mathbf{X}$ *is not always a conormalized named set, even if both named sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *are conormalized.*

Theorems 22 and 23 imply the following result.

Corollary 6 (Binormalization Law)**.** *For any non-empty nested named sets* $\mathbf{X}$ *and* $\mathbf{Y}$*, their first-level disjunctive intersection* $\mathbf{Y} \cap_d \mathbf{X}$ *is a binormalized named set if and only if at least one of the named sets* $\mathbf{X}$ *and* $\mathbf{Y}$ *is a binormalized named set, each of the elements from its support has a non-empty intersection with some elements from the support of the other named set, and each of the elements from its set of names has a non-empty intersection with some elements from the set of names of the other named set.*

Remark 13. *The first-level disjunctive intersection of two nested named sets can be non-cofunctional, even if both nested named sets are cofunctional.*

Remark 14. *In a general case, the first-level disjunctive intersection of two nested named sets is not an idempotent operation, i.e., an analogue of Theorem 6 is not true for first-level disjunctive intersection and all nested named sets, as the following example demonstrates.*

Example 14. *Let us take a nested named set* $\mathbf{X} = (X, r, N)$*, in which the set* X *contains exactly two different non-empty named sets,* X_1 *and* X_2 *with anon-empty intersection, whereas the set of names* N *contains only one named set. Building the first-level disjunctive intersection* $\mathbf{X} \cap_d \mathbf{X} = (Z, h, Q)$*, we see that the support,* Z*, contains three named sets,* X_1*,* X_2*, and* $X_1 \cap X_2$*, which is equal to neither to* X_1 *nor* X_2*. Therefore,* Z *is not equal to* X*, and* $\mathbf{X} \cap_d \mathbf{X}$ *is not equal to* $\mathbf{X}$.

One more binary operation with nested named sets is first-level conjunctive union.

Definition 15. *The first-level conjunctive intersection of two nested named sets* $\mathbf{X} = (X, f, N)$ *and* $\mathbf{Y} = (Y, g, M)$ *is defined as the named sets* $\mathbf{Z} = (Z, h, Q)$*, in which:*

$$Z = \{X_i \cap Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{N_k \cap M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h, is constructed in the following way.

If $X \cap Y_j$ if it is not empty, X_i is connected to N_k by f and Y_j is connected to M_r by g, then $X_i \cap Y_j$ is connected to $N_k \cap M_r$ by h.

Let us suppose that in non-empty named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, each of the elements from X has a non-empty intersection with some elements from Y, and each of the elements from Y has a non-empty intersection with some elements from X.

Theorem 24 (Functionality Law)**.** *The first-level conjunctive intersection* $\mathbf{Y} \cap_c \mathbf{X}$ *is a functional named set if and only if both* $\mathbf{X}$ *and* $\mathbf{Y}$ *are functional named sets.*

Proof. *Sufficiency*. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, and suppose that both of them are functional. First, we also suppose that these named sets are not empty. Building their first-level conjunctive intersection $\mathbf{X} \cap_c \mathbf{Y} = (Z, h, Q)$, we have:

$$Z = \{Z_{ij} = X_i \cap Y_j;\ X_i \in X \text{ and } Y_j \in Y\}$$

$$Q = \{Q_{kr} = N_k \cap M_r;\ N_k \in N \text{ and } M_r \in M\}$$

The relation h is constructed in the following way.

If X_i is connected to N_k by r and Y_j is connected to M_r by q, then when $X_i \cap Y_j$ is not empty, it is connected to $N_k \cap M_r$ by h.

As the named set **X** is functional, any element $a = X_i$, from X is connected to not more than one element $b = N_k$ from N by the relation r. As the named set **Y** is functional, any element $d = Y_j$ from Y is connected to not more than one element $c = M_r$ from M by the relation q.

Consequently, the element $X_i \cap Y_j$ is connected to not more than one element $N_k \cap M_r$, from Q by the relation h. This means that the first-level conjunctive intersection $\mathbf{X} \cap_c \mathbf{Y}$ is a functional named set.

Sufficiency is proved.

Necessity. Let us take two named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$ and suppose that **X** is not functional, whereas both named sets are not empty. By definition, this means that there is an element $a = X_i$ from X that has two names b and c, i.e., a is connected to b and c by the relation r. By the initial assumption, there is an element $d = Y_j$ from Y such that $a \cap d \neq \varnothing$. Then, by the construction of their first-level conjunctive intersection $\mathbf{X} \cap_c \mathbf{Y} = (Z, h, Q)$, the element $a \cap d = X_i \cap Y_j$ from Z is connected to $b \cap c$ from Q. As a is an arbitrary element from X, this means that $\mathbf{Y} \cap_c \mathbf{X}$ is not a functional named set.

The case when **Y** is not a functional named set is treated in the same way.

Thus, if the first-level conjunctive intersection $\mathbf{Y} \cap_c \mathbf{X}$ is a functional named set, then both of the named sets **X** and **Y** must be functional.

Theorem is proved. □

Let us suppose that in non-empty named sets $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, each of the elements from N has a non-empty intersection with some elements from M, and each of the elements from M has a non-empty intersection with some elements from N.

Theorem 25 (Cofunctionality Law)**.** *For any nested named sets* **X** *and* **Y**, *their first-level conjunctive intersection* $\mathbf{Y} \cap_c \mathbf{X}$ *is a cofunctional named set if and only if both* **X** *and* **Y** *are cofunctional named sets.*

Proof is similar to the proof of Theorem 6.

Remark 15. *It is also possible to prove Theorem 25 using the duality relation between a named set and its inverse* [1].

Let us assume that in non-empty named sets, $\mathbf{X} = (X, r, N)$ and $\mathbf{Y} = (Y, q, M)$, each of the elements from X has a non-empty intersection with some elements from Y, each of the elements from Y has a non-empty intersection with some elements from X, each of the elements from N has a non-empty intersection with some elements from M, and each of the elements from M has a non-empty intersection with some elements from N. Then, Theorems 24 and 25 imply the following result.

Corollary 7 (Individualization Law)**.** *For any nested named sets* **X** *and* **Y**, *their first-level conjunctive intersection* $\mathbf{Y} \cup_c \mathbf{X}$ *is an individually named set if and only if both* **X** *and* **Y** *are individually named sets.*

5. Conclusions

Information processing, as the base for natural and artificial intelligence, consists of transformations of data and knowledge. As data and knowledge are represented by named sets, these transformations are represented by operations transforming named sets. Here, various properties of binary operations on systems of nested named sets are obtained because nesting structures are important for data and knowledge representations, as well as for programming languages and algorithms. The goal is to form a base of operations for algorithmic systems of artificial intelligence.

When we built here binary operations with nested named sets, unions, and intersections, we used only one operation of the union of (plain) named sets and only one operation of the intersection of (plain) named sets introduced and studied in [1]. These operations are used when people combine data and knowledge or find common information from different sources. Naturally, AI systems must also perform such operations, and knowing their properties allows us to build more efficient software and hardware for AI systems.

At the same time, it is necessary to understand that there are other operations with (plain) named sets that are performed by people and could be useful for AI systems. Thus, an interesting and practical problem for future research is to use these operations to build operations for nested named sets and study their properties.

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