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A New Approach to Off-Line Robust Model Predictive Control for Polytopic Uncertain Models

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Abstract: Concerning the robust model predictive control (MPC) for constrained systems with polytopic model characterization, some approaches have already been given in the literature. One famous approach is an off-line MPC, which off-line finds a state-feedback law sequence with corresponding ellipsoidal domains of attraction. Originally, each law in the sequence was calculated by fixing the infinite horizon control moves as a single state feedback law. This paper optimizes the feedback law in the larger ellipsoid, foreseeing that, if it is applied at the current instant, then better feedback laws in the smaller ellipsoids will be applied at the following time. In this way, the new approach achieves a larger domain of attraction and better control performance. A simulation example shows the effectiveness of the new technique.

Keywords: polytopic model; model predictive control; off-line approach; domain of attraction

1. Introduction

Model predictive control (MPC) has attracted considerable attention, since it is an effective control algorithm to deal with multivariable constrained control problems. The nominal MPC for constrained linear systems has been solved in systematic ways around 2000 [1,2]. Recently, some approaches have been extended to distributed implementations [3,4]. Synthesizing robust MPC for constrained uncertain systems has attracted great attention after the nominal MPC. This has become a significant branch of MPC. The lack of robustness in MPC, based on a nominal model [5], calls for a robust MPC technique based on uncertainty models. Up to now, robust MPC has been solved in several ways [6–8]. A good technique for robust MPC, however, requires not only guaranteed stability, but also low computational burden, big (at least desired) domain of attraction, and low performance cost value [9].

The authors in [6] firstly solved a min-max optimization problem in an infinite horizon for systems with polytopic description, by fixing the control moves as a state feedback law that was on-line calculated. The authors in [9] off-line calculated a feedback law sequence with corresponding ellipsoidal domains of attraction, and on-line interpolated the control law at each sampling time applying this sequence. The computational burden is largely reduced. In addition, nominal performance cost is used in [9] to the take place of the "worst-case" one so that feasibility can be improved. A heuristic varying-horizon formulation is used and the feedback gains can be obtained in a backward manner.

In this paper, the off-line technique in [9] is further studied. Originally, each off-line feedback law was calculated where the infinite horizon control move was treated as a single state feedback law. This paper, instead, optimizes the feedback law in the larger ellipsoid by considering that the feedback laws in the smaller ellipsoids will be applied at the next time if it is applied at the current time. In a sense, the new technique in this paper is equivalent to a varying-horizon MPC, i.e., the control horizon (say M) gradually changes from $M > 1$ to $M = 1$, while the technique in [9] can be taken as a fixed horizon MPC with $M = 1$. Hence, the new technique achieves better control performance and can control a wider class of systems, i.e., improve control performance and feasibility.

So far, the state feedback approach is popular in most of the robust MPC problem, and the full state is assumed to be exactly measured to act as the initial condition for future predictions [10–14]. However, in many control problems, not all states can be measured exactly, and only the output information is available for feedback. In this case, an output feedback RMPC design is necessary (e.g., [3,15]). The output feedback MPC approach, parallel to that in [6], has been proposed in [16], and the off-line robust MPC has been studied in [17]. The new approach in this paper may be applied to improve the procedure in [17], which will be studied in the near future. **Notation**: The notations are fairly standard. $\Re^n$ is the n-dimensional space of real valued vectors. $W > 0$ ($W \geq 0$) means that W is symmetric positive-definite (symmetric non-negative-definite). For a vector x and positive-definite matrix $\mathcal{W}$, $\|x\|^2_{\mathcal{W}} = x^T \mathcal{W} x$. $x(k+i|k)$ is the value of vector x at a future time $k + i$ predicted at time k. The symbol * induces a symmetric structure, e.g., when H and R are symmetric matrices, then $\begin{bmatrix} H+S+* & * \\ T & R \end{bmatrix} := \begin{bmatrix} H+S+S^T & T^T \\ T & R \end{bmatrix}$.

2. Problem Statement

Consider the following time varying model:

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{1}$$

with input and state constraints, i.e.,

$$-\overline{u} \leq u(k+i) \leq \overline{u}, \forall i \geq 0 \tag{2a}$$

$$-\overline{\psi} \leq \Psi x(k+i+1) \leq \overline{\psi}, \forall i \geq 0 \tag{2b}$$

where $u \in \Re^m$ and $x \in \Re^n$ are input and measurable state, respectively; $\overline{u} := [\overline{u}_1, \overline{u}_2, \cdots, \overline{u}_m]^T$, and $\overline{\psi} := \left[\overline{\psi}_1, \overline{\psi}_2, \cdots, \overline{\psi}_q\right]^T$ with $\overline{u}_i > 0$, $i = 1 \cdots m$ and $\overline{\psi}_j > 0$, $j = 1 \cdots q$; $\ominus \in \Re^{q \times n}$. Input constraint is common in practice, which arises from physical and technological limitations. It is well known that the negligence of control input constraints usually leads to limit cycles, parasitic equilibrium points, or even causes instability. Moreover, we assume that $[A(k)|B(k)] \in \Omega, \forall k \geq 0$, where $\Omega = Co\{A_1|B_1, A_2|B_2, \cdots, A_L|B_L\}$, i.e., there exist L nonnegative coefficients $\omega_l(k)$, $l = 1 \cdots L$ such that

$$\sum_{l=1}^{L} \omega_l(k) = 1, [A(k)|B(k)] = \sum_{l=1}^{L} \omega_l(k)[A_l|B_l] \tag{3}$$

A predictive controller is proposed to drive systems (1) and (2) to the origin $(x, u) = (0, 0)$, and at each time k, to solve the following optimization problem:

$$\min_{\vec{u}(k)} \max_{[A(k+i)|B(k+i)] \in \Omega, i \geq 0} J_\infty(k) = \sum_{i=0}^{\infty} \left[\| x(k+i|k) \|^2_{\mathcal{Q}} + \| u(k+i|k) \|^2_{\mathcal{R}} \right] \tag{4a}$$

The following constraints are imposed on Equation (4a):

$$x(k+i+1|k) = A(k+i)x(k+i|k) + B(k+i)u(k+i|k), x(k|k) = x(k) \tag{4b}$$

$$-\overline{u} \leq u(k+i|k) \leq \overline{u}, -\overline{\psi} \leq \Psi x(k+i+1|k) \leq \overline{\psi} \tag{4c}$$

for all $i \geq 0$. In (4), $\mathcal{Q} > 0$ and $\mathcal{R} > 0$ are weighting matrices and $\vec{u}(k) = \left[u(k|k)^T, u(k+1|k)^T, u(k+2|k)^T, \cdots\right]^T$ are the decision variables. At time k, $u(k) = u(k|k)$ is implemented and the optimization (4) is repeated at time $k + 1$.

The authors in [9] simplified problem (4) by fixing $\vec{u}(k)$ as a state feedback law, i.e., $u(k+i|k) = F(k)x(k+i|k), \forall i \geq 0$. Define a quadratic function

$$V(i,k) = x\,(k+i|k)^T P(k)x(k+i\big|k), P(k) > 0, \forall k \geq 0 \tag{5}$$

with the robust stability constraint as follows:

$$V(i+1,k) - V(i,k) \leq -\|\, x(k+i|k)\,\|_{\mathcal{Q}}^2 - \|\, u(k+i|k)\,\|_{\mathcal{R}}^2, \forall [A(k+i)\big|B(k+i)] \in \Omega,\, i \geq 0 \tag{6}$$

For a stable closed-loop system, $x(\infty|k) = 0$ and $V(\infty,k) = 0$. Summing (6) from 0 to ∞ obtains

$$\max_{[A(k+i)|B(k+i)]\in\Omega, i\geq 0} J_\infty(k) \leq V(0,k) \leq \gamma \tag{7}$$

where $\gamma > 0$. Define $Q = \gamma P(k)^{-1}$ and $F(k) = YQ^{-1}$, then Equations (4c), (6) and (7) are satisfied if

$$\begin{bmatrix} 1 & * \\ x(k) & Q \end{bmatrix} \geq 0, Q > 0 \tag{8}$$

$$\begin{bmatrix} Q & * & * & * \\ A_l Q + B_l Y & Q & * & * \\ \mathcal{Q}^{1/2} Q & 0 & \gamma I & * \\ \mathcal{R}^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0,\ l = 1 \cdots L \tag{9}$$

$$\begin{bmatrix} Z & Y \\ Y^T & Q \end{bmatrix} \geq 0,\ Z_{jj} \leq \bar{u}_j^2,\ j = 1 \cdots m \tag{10}$$

$$\begin{bmatrix} Q & * \\ \Psi(A_l Q + B_l Y) & \Gamma \end{bmatrix} \geq 0, \Gamma_{ss} \leq \bar{\Psi}_s^2,\ l = 1 \cdots L;\ s = 1 \cdots q \tag{11}$$

where Z_{jj} (Γ_{ss}) is the jth (sth) diagonal element of Z (Γ) [18]. In this way, problem (4) is approximated by

$$\min_{\gamma, Q, Y, Z, \Gamma} \gamma \tag{12}$$

s.t. Equations (8)–(11).

The closed-loop system is exponentially stable if (12) is feasible at the initial time $k = 0$.

Based on [6], the authors in [9] off-line determined a look-up table of feedback laws with corresponding ellipsoidal domains of attraction. The control law was determined on-line from the look-up table. A linear interpolation of the two corresponding off-line feedback laws was chosen when the state stayed between two ellipsoids and an additional condition was satisfied.

Algorithm 1 The Basic Off-Line MPC [9]

1: Off-line, generate state points $x_1, x_2, \cdots, x_N$ where x_{h-1},
2: $h = N \cdots 2$ is nearer to the origin than x_h.
3: Substitute $x(k)$ in (8) by x_h,
4: $h = N \cdots 1$
5: Solve (12) to obtain Q_h, Y_h, γ_h,
6: The ellipsoids $\varepsilon_h = \left\{ x \in \Re^n \middle| x^T Q_h^{-1} x \le 1 \right\}$
7: The feedback laws $F_h = Y_h Q_h^{-1}$.
8: Take appropriate choices to ensure $\varepsilon_h \supset \varepsilon_{h-1}$, $\forall h = N \cdots 2$.
9: On-line, **if for each** x_h,
10: The following condition is satisfied:
11:

$$Q_h^{-1} - (A_l + B_l F_{h-1})^T Q_h^{-1} (A_l + B_l F_{h-1}) > 0,\ l = 1, 2, \cdots, L \tag{13}$$

12: Then each time k adopt the following control law:
13:

$$F(k) = \begin{cases} F(\alpha_h(k)), & x(k) \in \varepsilon_h,\ x(k) \notin \varepsilon_{h-1}, \\ F_1 & x(k) \in \varepsilon_1, \end{cases} \tag{14}$$

14: where $F(\alpha_h(k)) = \alpha_h(k) F_h + (1 - \alpha_h(k)) F_{h-1}$, $x(k)^T \left[\alpha_h(k) Q_h^{-1} + (1 - \alpha_h(k)) Q_{h-1}^{-1} \right] x(k) = 1$ and $0 \le \alpha_h(k) \le 1$.

Compared with [6], the on-line computational burden is reduced, but the optimization problem gives worse control performance. In this paper, we propose a new algorithm with better control performance and larger domains of attraction.

3. The Improved Off-Line Technique

In calculating F_h, Algorithm 1 does not consider F_i, $\forall i < h$. However, for smaller ellipsoids F_i, $\forall i < h$ are better feedback laws than F_h. In the following, the selection of Q_1, F_1, γ_1 is the same with Algorithm 1, but a different technique is adopted in this paper to calculate Q_h, F_h, γ_h, $\forall h \ge 2$. For x_h, $\forall h \ge 2$, we choose Q_h, F_h such that, for all $x(k) \in \varepsilon_h$, at the following time $\{F_{h-1}, \cdots, F_2, F_1\}$ is applied and $x(k + h - 1|k) \in \varepsilon_1$.

3.1. Calculating Q_2, F_2

Define an optimization problem

$$\min_{u(k+i|k), i \ge 1} \max_{[A(k+i)|B(k+i)] \in \Omega, i \ge 1} J_{2,\text{tail}}(k) = \sum_{i=1}^{\infty} \left[\| x(k+i|k) \|_{\mathcal{Q}}^2 + \| u(k+i|k) \|_{\mathcal{R}}^2 \right],\ \text{s.t. (4b), (4c) for all } i \ge 1 \tag{15}$$

and solve this problem by

$$u(k+i|k) = F_1 x(k+i|k), \forall i \ge 1 \tag{16}$$

By analogy to Equation (7),

$$\max_{[A(k+i)|B(k+i)] \in \Omega, i \ge 1} J_{2,\text{tail}}(k) \le x(k+1|k)^T P_1 x(k+1 \Big| k) \le \gamma_1 \tag{17}$$

where $P_1 = \gamma_1 Q_1^{-1}$. In this way, problem (4a) is turned into min-max optimization of (also refer to [18])

$$\bar{J}_2(k) := \bar{J}(k) = \|x(k)\|_{\mathcal{Q}}^2 + \|u(k)\|_{\mathcal{R}}^2 + \| x(k+1|k) \|_{P_1}^2 \tag{18}$$

Solve $u(k)$ by $u(k) = F_2 x(k)$ and define

$$\overline{J}_2(k) = x(k)^T \Big\{ \mathcal{Q} + F_2^T \mathcal{R} F_2 + [A(k) + B(k)F_2]^T P_1 [A(k) + B(k)F_2] \Big\} x(k) \leq \gamma_2 \tag{19}$$

Introduce a slack variable P_2 such that

$$\gamma_2 - x_2^T P_2 x_2 \geq 0 \tag{20}$$

and

$$\mathcal{Q} + F_2^T \mathcal{R} F_2 + [A(k) + B(k)F_2]^T P_1 [A(k) + B(k)F_2] \leq P_2 \tag{21}$$

Moreover, $u(k) = F_2 x(k)$ should satisfy hard constraints

$$-\underline{u} \leq F_2 x(k) \leq \overline{u}, \ -\underline{\psi} \leq \Psi[A(k) + B(k)F_2]x(k) \leq \overline{\psi}, \forall x(k) \in \varepsilon_2 \tag{22}$$

and terminal constraint

$$x(k+1|k) \in \varepsilon_1, \forall x(k) \in \varepsilon_2 \tag{23}$$

Equation (23) is equivalent to $[A(k) + B(k)F_2]^T Q_1^{-1} [A(k) + B(k)F_2] \leq Q_2^{-1}$. Define $Q_2 = \gamma_2 P_2^{-1}$ and $F_2 = Y_2 Q_2^{-1}$, then the following LMIs can be obtained:

$$\begin{bmatrix} 1 & * \\ x_2 & Q_2 \end{bmatrix} \geq 0, \ Q_2 > 0 \tag{24}$$

$$\begin{bmatrix} Q_2 & * & * & * \\ A_l Q_2 + B_l Y_2 & \gamma_2 P_1^{-1} & * & * \\ \mathcal{Q}^{1/2} Q_2 & 0 & \gamma_2 I & * \\ \mathcal{R}^{1/2} Y_2 & 0 & 0 & \gamma_2 I \end{bmatrix} \geq 0, \ l = 1 \cdots L \tag{25}$$

$$\begin{bmatrix} Q_2 & * \\ A_l Q_2 + B_l Y_2 & Q_1 \end{bmatrix} \geq 0, \ \ l = 1 \cdots L \tag{26}$$

Constraint (22) is satisfied if [6]

$$\begin{bmatrix} Z_2 & Y_2 \\ Y_2^T & Q_2 \end{bmatrix} \geq 0, \ Z_{2,jj} \leq z_{j,\inf}^2, \ j = 1 \cdots m \tag{27}$$

$$\begin{bmatrix} Q_2 & * \\ \Psi(A_l Q_2 + B_l Y_2) & \Gamma_2 \end{bmatrix} \geq 0, \Gamma_{2,ss} \leq \psi_{s,\inf}^2, \ l = 1 \cdots L; \ s = 1 \cdots q \tag{28}$$

Thus, Y_2, Q_2 and γ_2 can be obtained by solving

$$\min_{\gamma_2, Y_2, Q_2, Z_2, \Gamma_2} \gamma_2, \ \text{s.t. Equations (24)–(28)} \tag{29}$$

3.2. *Calculating Q_h, F_h, $\forall h \geq 3$*

The rationale in Section 3.1 is applied, with a little change. Define an optimization problem

$$\min_{u(k+i|k), i \geq h-1} \ \max_{[A(k+i)|B(k+i)] \in \Omega, i \geq h-1} J_{h,\text{tail}}(k) = \sum_{i=h-1}^{\infty} \Big[\| x(k+i|k) \|_{\mathcal{Q}}^2 + \| u(k+i|k) \|_{\mathcal{R}}^2 \Big] \tag{30}$$

s.t. Equations (4b) and (4c) for all $i \geq h-1$

By analogy to Equation (18), problem (4a) is turned into min-max optimization of

$$\bar{J}_h(k) := \bar{J}(k) = \sum_{i=0}^{h-2}\left[\| x(k+i|k) \|_{\mathcal{Q}}^2 + \| u(k+i|k) \|_{\mathcal{R}}^2\right] + \| x(k+h-1|k) \|_{P_1}^2 \tag{31}$$

which is solved by

$$u(k+i|k) = F_{h-i}x(k+i|k), i = 0 \cdots h-2; u(k+i|k) = F_1 x(k+i|k), \forall i \geq h-1 \tag{32}$$

By analogy to Equation (19), define

$$\bar{J}_h(k) = x(k)^T\left\{\mathcal{Q} + F_h^T \mathcal{R} F_h + [A(k)+B(k)F_h]^T P_{1,l_2\cdots l_{h-1}}[A(k)+B(k)F_h]\right\}x(k) \leq \gamma_h \tag{33}$$

where, by induction, for $\forall h \geq 3$,

$$\begin{gathered} P_{1,l_2\cdots l_{h-1}} = \left[\prod_{i=2}^{h-1}(A_{l_i}+B_{l_i}F_i)\right]^T P_1\left[\prod_{i=2}^{h-1}(A_{l_i}+B_{l_i}F_i)\right] + \\ \sum_{i=3}^{h-1}\left\{\left[\prod_{j=i}^{h-1}(A_{l_j}+B_{l_j}F_j)\right]^T(\mathcal{Q}+F_{i-1}^T\mathcal{R}F_{i-1})\left[\prod_{j=i}^{h-1}(A_{l_j}+B_{l_j}F_j)\right]\right\} + \mathcal{Q} + F_{h-1}^T\mathcal{R}F_{h-1} \end{gathered} \tag{34}$$

and

$$P_{1,l_2\cdots l_h} = (A_{l_h}+B_{l_h}F_h)^T P_{1,l_2\cdots l_{h-1}}(A_{l_h}+B_{l_h}F_h) + \mathcal{Q} + F_h^T\mathcal{R}F_h \tag{35}$$

By Equation (33), introduce a slack variable $P_h = \gamma_h Q_h^{-1}$ and define $F_h = Y_h Q_h^{-1}$ such that

$$\begin{bmatrix} 1 & * \\ x_h & Q_h \end{bmatrix} \geq 0 \tag{36}$$

and

$$\begin{bmatrix} Q_h & * & * & * \\ A_{l_h}Q_h + B_{l_h}Y_h & \gamma_h P_{1,l_2\cdots l_{h-1}}^{-1} & * & * \\ \mathcal{Q}^{1/2}Q_h & 0 & \gamma_h I & * \\ \mathcal{R}^{1/2}Y_h & 0 & 0 & \gamma_h I \end{bmatrix} \geq 0,\ l_i = 1\cdots L;\ i = 2\cdots h \tag{37}$$

Moreover, the terminal constraint should be equivalent to

$$[A(k)+B(k)F_2]^T S_{1,l_2\cdots l_{h-1}}[A(k)+B(k)F_2] \leq Q_h^{-1} \tag{38}$$

where, by induction,

$$S_{1,l_2\cdots l_{h-1}} = \left[\prod_{i=2}^{h-1}(A_{l_i}+B_{l_i}F_i)\right]^T Q_1^{-1}\left[\prod_{i=2}^{h-1}(A_{l_i}+B_{l_i}F_i)\right] \tag{39}$$

Equation (38) means that, for $\forall x(k) \in \varepsilon_h$, if at the following time $\{F_{h-1}, \cdots, F_2, F_1\}$ is applied, then $x(k+h-1|k) \in \varepsilon_1$. Equation (38) can be transformed into

$$\begin{bmatrix} Q_h & * \\ \prod_{i=2}^{h-1}(A_{l_i}+B_{l_i}F_i)(A_{l_h}Q_h+B_{l_h}Y_h) & Q_1 \end{bmatrix} \geq 0,\ l_i = 1\cdots L; i = 2\cdots h \tag{40}$$

Again, $u(k|k) = F_h x(k|k)$ should satisfy

$$\begin{bmatrix} Z_h & Y_h \\ Y_h^T & Q_h \end{bmatrix} \geq 0,\ Z_{h,jj} \leq z_{j,\inf}^2,\ j = 1 \cdots m \tag{41}$$

and

$$\begin{bmatrix} Q_h & * \\ \Psi\left(A_{l_h} Q_h + B_{l_h} Y_h\right) & \Gamma_h \end{bmatrix} \geq 0, \Gamma_{h,ss} \leq \psi_{s,\inf}^2,\ l_h = 1 \cdots L;\ s = 1 \cdots q \tag{42}$$

Thus, Y_h, Q_h and γ_h can be obtained by solving

$$\min_{\gamma_h, Y_h, Q_h, Z_h, \Gamma_h} \gamma_h \tag{43}$$

s.t. Equations (36), (37) and (40)–(42).

Algorithm 2 The Improved Off-Line MPC

1: Off-line, generate state points $x_1, x_2, \cdots, x_N$ where x_{h-1},
2: $h = N \cdots 2$ is nearer to the origin than x_h.
3: Substitute $x(k)$ in (8) by x_1
4: Solve (12) to obtain Y_1,
5: $Q_1, \gamma_1, F_1 = Y_1 Q_1^{-1}$
6: $P_1 = \gamma_1 Q_1^{-1}$.
7: For x_2, solve (29)
8: Obtain Q_2, Y_2 and $F_2 = Y_2 Q_2^{-1}$.
9: For x_h, $h = N \cdots 3$, solve (43)
10: Obtain Q_h, Y_h and $F_h = Y_h Q_h^{-1}$.
11: $\varepsilon_h \supset \varepsilon_{h-1}, \forall h = N \cdots 2$.
12: On-line, at each time k adopt the control law (14).

Theorem 1. *For systems* (1) *and* (2)*, and an initial state $x(0) \in \varepsilon_N$, the off-line constrained robust MPC in Algorithm 2 robustly asymptotically stabilizes the closed-loop system.*

Proof. Similar to [9], when $x(k)$ satisfies $\|x(k)\|_{Q_h^{-1}}^2 \leq 1$ and $\|x(k)\|_{Q_{h-1}^{-1}}^2 \geq 1$, $h \neq 1$, let $Q(\alpha_h(k))^{-1} = \alpha_h(k) Q_h^{-1} + (1 - \alpha_h(k)) Q_{h-1}^{-1}$, $Z(\alpha_h(k)) = \alpha_h(k) Z_h + (1 - \alpha_h(k)) Z_{h-1}$ and $\Gamma(\alpha_h(k)) = \alpha_h(k)\Gamma_h + (1 - \alpha_h(k))\Gamma_{h-1}$. By linear interpolation, $\begin{bmatrix} Z(\alpha_h(k)) & * \\ F(\alpha_h(k))^T & Q(\alpha_h(k))^{-1} \end{bmatrix} \geq 0$ and $\begin{bmatrix} Q(\alpha_h(k))^{-1} & * \\ \Psi(A_l + B_l F(\alpha_h(k))) & \Gamma(\alpha_h(k)) \end{bmatrix} \geq 0$, which means that $F(\alpha_h(k))$ satisfies the input and state constraints. Since $\{F_{h-1}, F_{h-1}, F_{h-2}, \cdots, F_1\}$ is a stable feedback law sequence for all initial state inside of ε_{h-1}, it is shown that

$$\begin{bmatrix} Q_{h-1}^{-1} & * \\ \prod_{i=2}^{h-1} (A_{l_i} + B_{l_i} F_i)(A_{l_h} + B_{l_h} F_{h-1}) & Q_1 \end{bmatrix} \geq 0 \tag{44}$$

Moreover, Equation (40) is equivalent to $\begin{bmatrix} Q_h^{-1} & * \\ \prod_{i=2}^{h-1} (A_{l_i} + B_{l_i} F_i)(A_{l_h} + B_{l_h} F_h) & Q_1 \end{bmatrix} \geq 0$. Hence, by linear interpolation,

$$\begin{bmatrix} Q(\alpha_h(k))^{-1} & * \\ \prod_{i=2}^{h-1}(A_{l_i}+B_{l_i}F_i)(A_{l_h}+B_{l_h}F(\alpha_h(k))) & Q_1 \end{bmatrix} \geq 0 \tag{45}$$

which means that $\{F(\alpha_h(k)), F_{h-1}, \cdots, F_1\}$ is a stable control law sequence for all $x(k) \in \varepsilon_{h,\alpha_h(k)} = \left\{x \in \Re^n \middle| x^T Q(\alpha_h(k))^{-1} x \leq 1\right\}$ and is guaranteed to drive $x(k+h-1|k)$ into ε_1, with the constraints satisfied. Inside of ε_1, F_1 is applied, which is stable and drives the state to the origin. □

If Equation (38) is made to be automatically satisfied, more off-line feedback laws may be needed in order for ε_N to cover a desired space region. However, with this automatic satisfaction, better control performance can be obtained. Hence, we give the following alternative algorithm.

Algorithm 3 The Improved Off-Line MPC with an Automatic Condition

1: Off-line, as in Algorithm 2,
2: Obtain Q_h, Y_h, γ_h and F_h, $h = N \cdots 1$.
3: $\varepsilon_h \supset \varepsilon_{h-1}$, $h = N \cdots 2$.
4: On-line, **if for each** x_h, $h = N \cdots 3$,
5: The following condition is satisfied:
6:

$$(A_{l_h}+B_{l_h}F_h)^T S_{1,l_2\cdots l_{h-1}}(A_{l_h}+B_{l_h}F_h) \leq Q_h^{-1},\ l_i = 1\cdots L;\ i = 2\cdots h \tag{46}$$

7: **for** x_2
8: The following condition is satisfied:
9:

$$(A_l+B_lF_2)^T Q_1^{-1}(A_l+B_lF_2) \leq Q_2^{-1},\ l = 1\cdots L \tag{47}$$

10: Then at each time k adopt the control law (14).

4. Numerical Example

4.1. Example 1

Consider the system $\begin{bmatrix} x^{(1)}(k+1) \\ x^{(2)}(k+1) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ \beta(k) & 0.8 \end{bmatrix}\begin{bmatrix} x^{(1)}(k) \\ x^{(2)}(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$, where $\beta(k)$ is an uncertain parameter. Use $0.5 \leq \beta(k) \leq 2.5$ to form polytopic description and $\beta(k) = 1.5$ to calculate the state evolution. The constraints are $|u(k+i|k)| \leq 2$, $\forall i \geq 0$. Choose the weighting matrices as $\mathcal{Q} = I$ and $\mathcal{R} = 1$. Consider the following two cases.

Case 1. Choose $x_h = [\xi_h,\ 0]^T$, $h = 1\cdots 4$. Choose $\xi_1 = 1$ and $\xi_h = \xi_{h-1} + \Delta\xi_h$, $h = 2\cdots 4$. Choose large $\Delta\xi_h$ such that: (i) condition (13) is satisfied, (ii) optimization problem (12) is feasible for x_h, and (iii) optimization problem (29) or (43) is feasible for x_h. Thus, we obtain $\xi_2 = 1.1$, $\xi_3 = 1.5$ and $\xi_4 = 1.9$. The initial state lies at $x(0) = [1.9,\ 0]^T$.

Apply Algorithms 1 and 2. The state and input responses for these two algorithms are shown in Figures 1 and 2, respectively. The upper bounds of the cost value γ_h for these two algorithms are [15.4987, 18.7479, 35.5955, 66.6209] and [15.4987, 18.7425, 34.9721, 58.1560], respectively. Moreover, denote

$$\hat{J} = \sum_{i=0}^{\infty}\left[\|x(i)\|_{\mathcal{Q}}^2 + \|u(i)\|_{\mathcal{R}}^2\right] \tag{48}$$

then $\hat{J}^* = 16.3221$ for Algorithm 1 and $\hat{J}^* = 15.1033$ for Algorithm 2. The simulation results show that Algorithm 2 achieves better control performance.

Case 2. Increase N. Algorithm 1 becomes infeasible at $x_h = 3.4658$. However, Algorithm 2 is still feasible by choosing $x_1 = 1$, $x_2 = 1.1$, $x_3 = 1.5$, $x_4 = 1.9$, $x_5 = 3.8$, $x_6 = 9.0$, $x_7 = 12.8$, etc.

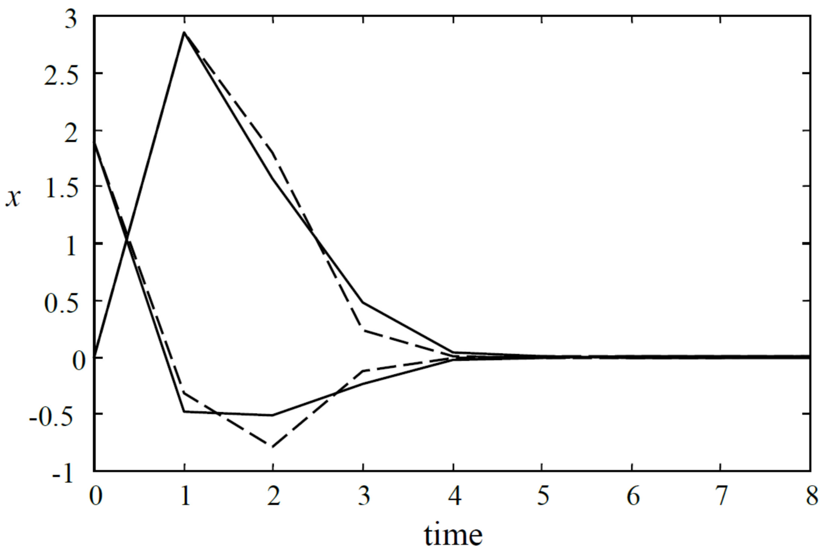

Figure 1. The state responses of the closed-loop systems (dashed line for Algorithm 1, solid Algorithm 2).

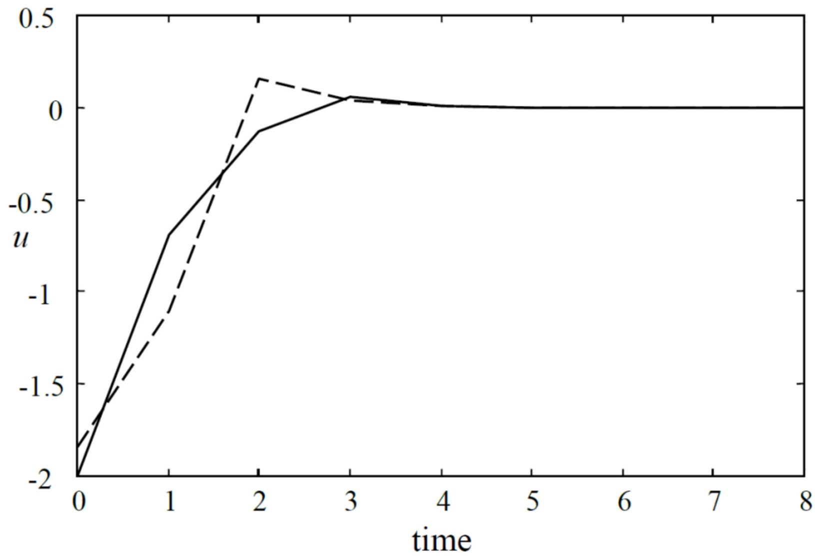

Figure 2. The input responses of the closed-loop system (dashed line for Algorithm 1, solid Algorithm 2).

4.2. Example 2

Directly consider the system in [9]: $\begin{bmatrix} x^{(1)}(k+1) \\ x^{(2)}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1-\beta(k) \end{bmatrix} \begin{bmatrix} x^{(1)}(k) \\ x^{(2)}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k)$, where $\beta(k)$ is an uncertain parameter. Use $0.1 \leq \beta(k) \leq 10$ to form a polytopic description and use $\beta(k) = 9$ to calculate the state evolution. The constraints are $|u(k+i|k)| \leq 0.2$, $\left|x^{(2)}(k+i+1\middle|k)\right| \leq 0.03$, $\forall i \geq 0$. Choose the weighting matrices as $\mathcal{Q} = I$ and $\mathcal{R} = 0.00002$.

Case 1. Choose $x_h = \begin{bmatrix} 0.01 + 0.00025(h-1) & 0 \end{bmatrix}^T$ and $x_N = 0.05$. The initial state lies at $x_0 = [0.05, 0]^T$. Apply Algorithms 1 and 3. The state trajectories, the state responses and the input responses for these two algorithms are shown in Figures 3–5, respectively. Moreover, denote $\hat{J} = \sum_{i=0}^{\infty}\left[\|x(k+i)\|_{\mathcal{Q}}^2 + \|u(k+i)\|_{\mathcal{R}}^2\right]$, then $\hat{J}^* = 0.0492$ for Algorithm 1 and $\hat{J}^* = 0.0975$ for Algorithm 3.

Case 2. Choose $x_h = \begin{bmatrix} 0.01 + 0.00025(h-1) & 0 \end{bmatrix}^T$, $h = 1 \cdots NN$ such that the optimization problem is infeasible for x_{NN}. Then, for Algorithm 1, x_{NN} is 19.6006, and for Algorithm 3, 0.0505.

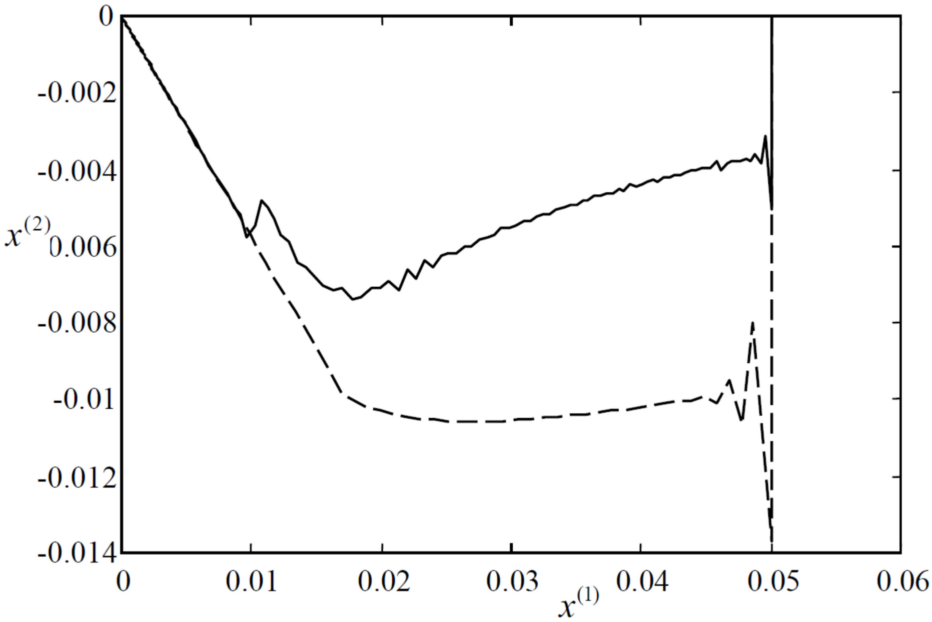

Figure 3. The state trajectories of the closed-loop systems (dashed line for Algorithm 1, solid Algorithm 3).

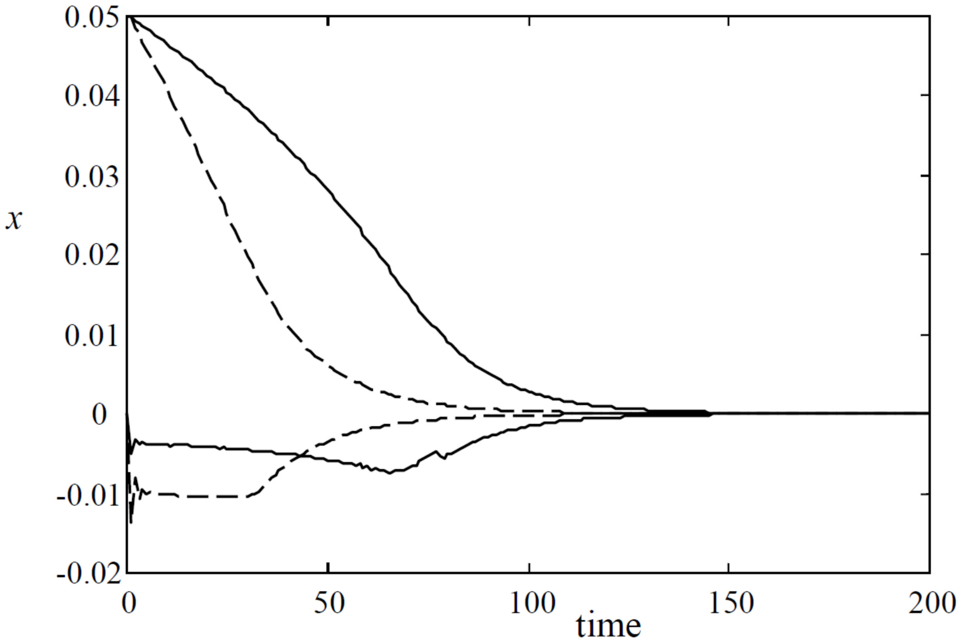

Figure 4. The state responses of the closed-loop systems (dashed line for Algorithm 1, solid Algorithm 3).

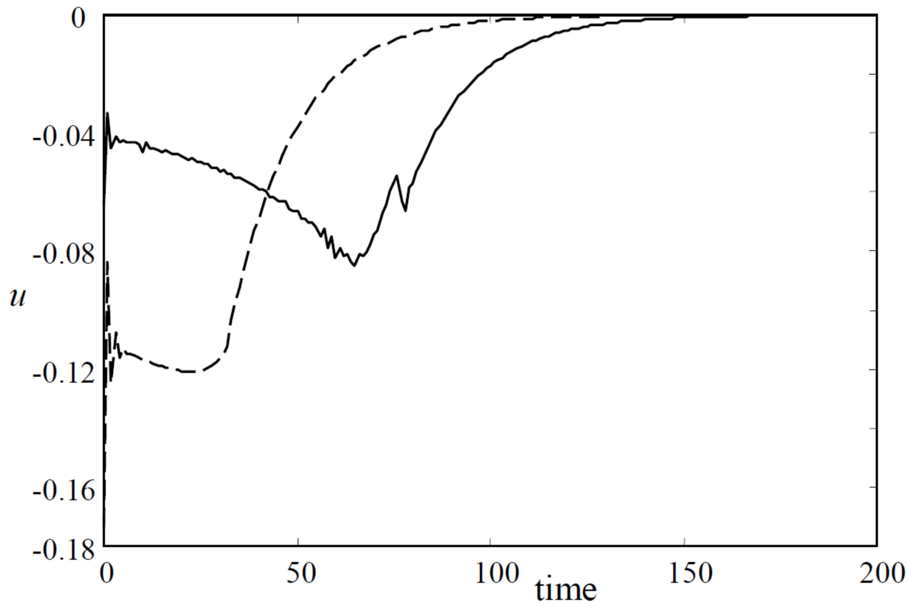

Figure 5. The input responses of the closed-loop systems (dashed line for Algorithm 1, solid Algorithm 3).

5. Conclusions

In this paper, we have given a new algorithm for off-line robust MPC. The off-line state feedback law is optimized instead, such that each single state feedback law is fixed by the infinite-horizon control moves. This new algorithm consists of MPC with a varying horizon, i.e., the control horizon (say M) varies from $M > 1$ to $M = 1$, while the original Algorithm 1 can be taken as an approach with $M = 1$. Simulation results show that the new algorithm achieves better control performance. Our future research on this topic will be extending it to the output feedback MPC approaches.

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