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Product of Two Laguerre–Gaussian Beams

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Abstract: We show that a product of two Laguerre–Gaussian (pLG) beams can be expressed as a finite superposition of conventional LG beams with particular coefficients. Based on such an approach, an explicit relationship is derived for the complex amplitude of pLG beams in the Fresnel diffraction zone. Two identical LG beams of the duet produce a particular case of a "squared" Fourier-invariant LG beam, termed as an (LG)^2 beam. For a particular case of pLG beams described by Laguerre polynomials with azimuthal numbers $n - m$ and $n + m$, an explicit expression for the complex amplitude in a Fourier plane is derived. Similar to conventional LG beams, the pLG beams can be utilized for information transmission, as they are characterized by orthogonal azimuthal numbers and carry an orbital angular momentum equal to their topological charge.

Keywords: Laguerre–Gaussian beam; product of complex amplitudes; Fourier-invariant beam; topological charge; orbital angular momentum

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1. Introduction

Among a host of familiar laser beams, the most popular and well understood are Laguerre–Gaussian (LG) beams [1–3]. Initially, these beams were looked upon as intra-cavity modes, but later on, out-of-resonator LG beams were generated from Hermite–Gaussian (HG) beams using an astigmatic converter [4]. Particular interest in the LG beams was provoked by L. Allen's et al. paper [5] in which the LG beams were found to carry the angular orbital momentum (OAM). Extensively studied beams include generalized LG beams in the form of Hermite–Laguerre–Gaussian beams [6,7], elegant [8] and elliptic [9] LG beams. These days have seen no signs of waning interest in studying LG beams thanks to their wide use in telecommunications, micromanipulation, probing atmospheric turbulence, quantum information, and atom cooling. By way of illustration, a comparative analysis of LG beams and Bessel–Gaussian (BG) beams has been conducted [10]. Various approaches to generating LG modes discussed in Refs. [11–13] include the use of a specialized laser utilizing intra-cavity spherical aberration [11], q-plates [12], and a special metasurface [13]. Reciprocal HG-to-LG and LG-to-HG mode conversion was studied in Ref. [14]. Of essential significance is the study of elegant LG beams that have shown outstanding characteristics for many application areas such as optical communications and optical trapping [15]. A method for measuring a topological charge of a partially coherent elegant LG beam has been proposed [16]. The LG beams have formed a basis for developing new types of optical beams with a variety of promising properties. A family of asymmetric LG laser beams has been discussed [17], and a technique for generating high-power asymmetric LG beams has been proposed [18]. Using the LG beams, a vector beam with space dependent transverse polarization has been generated by a method of nonlinear magneto-optic rotation [19]. A new class of composite vortex beams generated by coaxially superimposing LG beams with identical waist location and parameters has been proposed [20]. A new type of a partially coherent beam with a peculiar correlation function, which has been given the name an elliptic correlated Laguerre–Gauss–Shell model, has been theoretically and experimentally studied [21]. Works [22,23] study the asymmetric LG beams in free space, whereas the papers [24–26] investigate these beams as solitons in a nonlinear medium. These beams can be treated as superpositions of two LG beams with different weight coefficients. In the far field, the intensity pattern of such beams is in the form of an ellipse-shaped ring. The LG beams are of great practical value for optical communication [27–30], micromanipulation [31], and photo-induced atom excitation [32].

In this work, we derive exact analytical expressions describing laser beams generated by multiplying two LG beams, also termed as a product of LG (pLG) beams. We show that the complex amplitude of the pLG beams can be decomposed into a finite sum of conventional LG beams, deriving a relationship for a Fresnel transform of such beams. For some particular cases of the pLG beams, an explicit form of a Fourier transform is deduced.

We note that a product of the LG beams was already considered before [33,34]. In [33], an experiment was performed on second harmonic generation of the LG beam, but the analytical expressions for such beams have not been obtained. In [34], the product of two LG beams was investigated both experimentally and theoretically. However, all the obtained equations for the product of two LG beams in [34] are expressed via undefined coefficients. Here, in our work, we obtain for the first time an exact expansion of the product of two Laguerre polynomials into a finite sum of the Laguerre polynomials, and the expansion coefficients are expressed via the Jacobi polynomials of zero argument.

2. Theoretical Background

A conventional LG beam can be described in the source plane (z = 0) by a complex amplitude [1]:

$$\mathrm{LG}_{p,\pm m}(r,\varphi) = \exp\left(-\frac{r^2}{2w^2} \pm im\varphi\right)\left(\frac{r}{w}\right)^m L_p^m\left(\frac{r^2}{w^2}\right), \tag{1}$$

where (r,φ) represents the polar coordinates, w is the waist radius of a Gaussian beam, and $L_p^m(x)$ is an associated Laguerre polynomial, m is the topological charge of the optical vortex, and p is the radial index affecting the number of intensity rings (there are p + 1 rings if $m \neq 0$ and p rings if m = 0). With Equation (1) describing a modal beam, i.e., the one which preserves its intensity pattern upon free-space propagation, the complex amplitude of a LG beam at any plane z is described by a relationship similar to Equation (1):

$$\mathrm{LG}_{p,m}(r,\varphi,z) = \frac{w}{w(z)}\exp\left(\frac{izr^2}{z_0 w^2(z)} + i(2p+m+1)\gamma(z)\right)\mathrm{LG}_{p,m}\left(\frac{wr}{w(z)},\varphi\right), \tag{2}$$

where $w(z) = w\,[1 + (z/z_0)^2]^{1/2}$ is the Gaussian beam radius, $\gamma(z) = \arctan(z/z_0)$, $\gamma(z)$ is the Gough phase, z_0 is the Rayleigh range, and k is the wavenumber of light. The Rayleigh range of the beam from Equations (1) and (2) is given by

$$z_0 = kw^2,$$

and defines a propagation distance to the plane where the area of the beam's cross-section is doubled and where the beam has the largest wavefront curvature (in the initial plane and in the far field, the wavefront is flat). The Rayleigh range is the distance that can be treated as a boundary between the near field ($z << z_0$) and far field ($z >> z_0$).

The product of two LG beams in Equation (1) with identical waist radii, also termed as a pLG beam, is given by

$$\begin{aligned}\mathrm{pLG}_{p,q,m,n}(r,\varphi) &= \mathrm{LG}_{p,m}(r,\varphi)\mathrm{LG}_{q,n}(r,\varphi) = \\ &= \exp\left(-\tfrac{r^2}{w^2} + i(m+n)\varphi\right)\left(\tfrac{r}{w}\right)^{m+n} L_p^m\left(\tfrac{r^2}{w^2}\right) L_q^n\left(\tfrac{r^2}{w^2}\right).\end{aligned} \tag{3}$$

A four-parameter family of beams in Equation (3) presents a generalization of conventional LG beams of Equation (1), as it becomes identical to single-ring LG beams at $p = q = 0$:

$$\mathrm{pLG}_{0,0,m,n}(r,\varphi) = \exp\left(-\frac{r^2}{w^2} + i(m+n)\varphi\right)\left(\frac{r}{w}\right)^{m+n} = 2^{-(n+m)/2}\mathrm{LG}_{0,m+n}\left(\sqrt{2}r,\varphi\right). \quad (4)$$

From the general Equation (3) for the pLG beam in the initial plane, the topological charge (TC) of a pLG beam can be found as a sum of TCs of two constituent LG beams. The power-normalized orbital angular momentum (OAM) of the pLG beam is also found as a sum of two constituent LG beams, i.e., equals $m + n$. Then, we proceed to derive a relationship for the Fresnel transform of function given by Equation (3). The pLG beam can be decomposed into a finite sum of conventional LG beams (Appendix A):

$$L_p^m(x)L_q^n(x) = \sum_{k=0}^{p+q} C_k L_k^{m+n}(2x), \quad (5)$$

where $C_k = 2^{q-k}P_{p+q-k}^{(k+m-q,k+n-p)}(0)P_q^{(k-q,p-k)}(0)$ and $P_\alpha^{\mu,\nu}(0)$ denote Jacobi polynomials at zero [35]. From Equation (5), the amplitude of the pLG beam in Equation (3) is given by a finite sum of conventional LG beams:

$$\begin{aligned}\mathrm{pLG}_{p,q,m,n}(r,\varphi) &= \exp\left(-\tfrac{r^2}{w^2} + i(m+n)\varphi\right)\left(\tfrac{r}{w}\right)^{m+n}\sum_{k=0}^{p+q} C_k L_k^{m+n}\left(\tfrac{2r^2}{w^2}\right) \\ &= 2^{-(m+n)/2}\sum_{k=0}^{p+q} C_k \mathrm{LG}_{k,m+n}(\sqrt{2}r,\varphi).\end{aligned} \quad (6)$$

The relationship in Equation (6) enables the amplitude of field from Equation (3) to be derived at any distance z:

$$\begin{aligned}\mathrm{pLG}_{p,q,m,n}(r,\varphi,z) = \tfrac{w}{2^{(n+m)/2}w(z)}\exp\left(-\tfrac{r^2}{2w^2(z)} + \tfrac{ikr^2}{2R(z)} + i(m+n)\varphi\right)\times \\ \times\left(\tfrac{r}{w(z)}\right)^{n+m}\sum_{s=0}^{p+q} C_s L_s^{n+m}\left(\tfrac{r^2}{w^2(z)}\right)\exp(i(2s+m+n+1)\gamma(z)),\end{aligned} \quad (7)$$

with $R(z) = z\,[1 + (z_0/z)^2]$ being the curvature radius of the wavefront.

With the constituent beams in superposition in Equation (7) having different Gough phases, we can infer that the pLG beam of Equation (3) does not preserve its structure upon free-space propagation. However, with the transverse intensity pattern of beam from Equation (7) given by a set of concentric intensity rings whose maximum number is $p + q + 1$, changes in the intensity pattern upon free-space propagation of beam from Equation (7) are limited to the inter-ring energy flow.

It would be of interest to analyze a particular case of pLG beams whose constituent Laguerre polynomials are identical. In the source plane, the beam, which has been given the name "squared" LG beam, or $(\mathrm{LG})^2$ beam, is given by:

$$\mathrm{pLG}_{p,p,m,m}(r,\varphi) = \left[\mathrm{LG}_{p,m}(r,\varphi)\right]^2 = \exp\left(-\frac{r^2}{w^2} + i2m\varphi\right)\left(\frac{r}{w}\right)^{2m}\left[L_p^m\left(\frac{r^2}{w^2}\right)\right]^2. \quad (8)$$

Similar to the general pLG beam in Equation (3), the beam in Equation (8) is not modal and does not preserve the intensity structure upon free-space propagation. At the same time, unlike the beam from Equation (3), in the far field, the beam from Equation (8) preserves its structure, being Fourier-invariant:

$$\begin{aligned}F_f\left\{\mathrm{pLG}_{p,p,m,m}(r,\varphi)\right\} = \left(\tfrac{iz_0}{f}\right)(-1)^{m+1}\mathrm{pLG}_{p,p,m,m}(\rho,\theta) = \\ = \left(\tfrac{iz_0}{f}\right)(-1)^{m+1}\exp\left(2im\theta - y^2\right)(y)^{2m}\left[L_p^m\left(y^2\right)\right]^2,\end{aligned} \quad (9)$$

where $y = kw\rho/f$, f is the focal length of a spherical lens, (ρ, θ) are the polar coordinates in the Fourier plane, and $F_f\{\ \}$ is the Fourier transform. A comparison of relationships for complex amplitudes in the source plane (Equation (8)) and at the focus of a spherical lens (Equation (9)) shows that they are identical up to a constant. In the Fresnel diffraction zone, the $(\mathrm{LG})^2$ beam is described by a finite superposition of conventional LG beams, which are similar to Equation (7) but have different coefficients:

$$\begin{aligned} \mathrm{pLG}_{p,p,m,m}(r,\varphi,z) = {} & \left(\frac{iz_0}{z}\right)(-1)^{m+1}\frac{(m+p)!}{\pi m!\sigma^{2m+1}}(y)^{2m}\exp\left(\frac{ikr^2}{2z}+2im\varphi\right)\exp\left(-\frac{y^2}{\sigma}\right) \\ & \times\sum_{s=0}^{p}(-1)^s\frac{\Gamma(s+1/2)}{\Gamma(m+s+1)}\left(\frac{\sigma-2}{2}\right)^{2s}L_{2s}^{2m}\left(\frac{y^2}{\sigma(2-\sigma)}\right), \end{aligned} \tag{10}$$

where $y = kw\rho/z$, $\sigma = 1 - iz_0/z$, $\Gamma(x)$ is the gamma function. In the general case, Equation (10) is consistent with Equation (7) because in both cases, the complex amplitude is expressed via a finite sum of conventional LG beams. The difference between Equations (7) and (10) is that the latter only contains LG beams with even radial indices. Equation (10) also suggests that at $p = 0$ (zero-valued radial index), the intensity pattern of the LG beam is a single ring, as $L_0^n(x) = 1$, so in the sum in Equation (10), only the first term is retained, meaning that the beam from Equation (1) with a squared amplitude is preserved upon free-space propagation.

Another particular case of pLG beams in Equation (3) is obtained by considering the product of two LG beams with even TC and tailored values of azimuthal numbers in the Laguerre polynomials:

$$\begin{aligned} \mathrm{pLG}_{p,\,q,\,n-m,\,n+m}(r,\varphi) = {} & \exp\left(-\frac{r^2}{w^2}+i2n\varphi\right)\times \\ & \times\left(\frac{r}{w}\right)^{2n}L_p^{n-m}\left(\frac{r^2}{w^2}\right)L_q^{n+m}\left(\frac{r^2}{w^2}\right), n \geq m \geq 0. \end{aligned} \tag{11}$$

The complex amplitude of the beam from Equation (11) at the focus of a spherical lens (in the Fourier plane) is given by an explicit relationship:

$$\begin{aligned} F_f\left\{\mathrm{pLG}_{p,\,q,\,(n-m),\,(n+m)}(r,\varphi)\right\} = {} & \left(\frac{iz_0}{f}\right)(-1)^{m+1}\exp\left(2in\theta-y^2\right) \\ & \times(y)^{2n}L_p^{n-m+p-q}\left(y^2\right)L_q^{n+m-p+q}\left(y^2\right). \end{aligned} \tag{12}$$

From Equation (12), the product of two beams in Equations (3) and (11) is seen to be Fourier-invariant at $p = q$. From Equations (11) and (12), at $p = q$ and $m = 0$, the conversion of a pLG beam into an $(\mathrm{LG})^2$ beam is seen to occur.

We can derive a relationship for the "energy" of the pLG beam, using it when normalizing such beams:

$$\begin{aligned} W & = \int_0^\infty\int_0^{2\pi}\left|\mathrm{pLG}_{p,\,q,\,m,\,n}(r,\varphi)\right|^2 r\,dr\,d\varphi \\ & = \frac{\pi w^2}{4}\{(m+p)!(n+q)!\}^2\sum_{k=0}^{\min(2m+p,2n+q)}\frac{\left\{P_{m+p}^{m-k,k-m-p}(0)P_{n+q}^{n-k,k-n-q}(0)\right\}^2}{(2m+p-k)!(2n+q-k)!k!^2}. \end{aligned} \tag{13}$$

3. Numerical Simulation

Figure 1 depicts numerically simulated intensity and phase patterns from the standard LG beams (Equation (1)) of two different orders and from the pLG beam (Equation (3)) of the same two orders, shown in the source plane and at the half Rayleigh range. We note that with the waist radius of beam from Equation (1) taken to be $2^{1/2}\,w$, and not w, the Rayleigh range of beam from Equation (1) is twice that of beam from Equation (3). Shown in Figure 1 are intensity and phase patterns at the Rayleigh range for beam from Equation (3), i.e., at $kw^2/2 = z_0/2$. The patterns were numerically simulated using Equations (1) and (3) in the source plane and using a Fresnel transform at the half Rayleigh range.

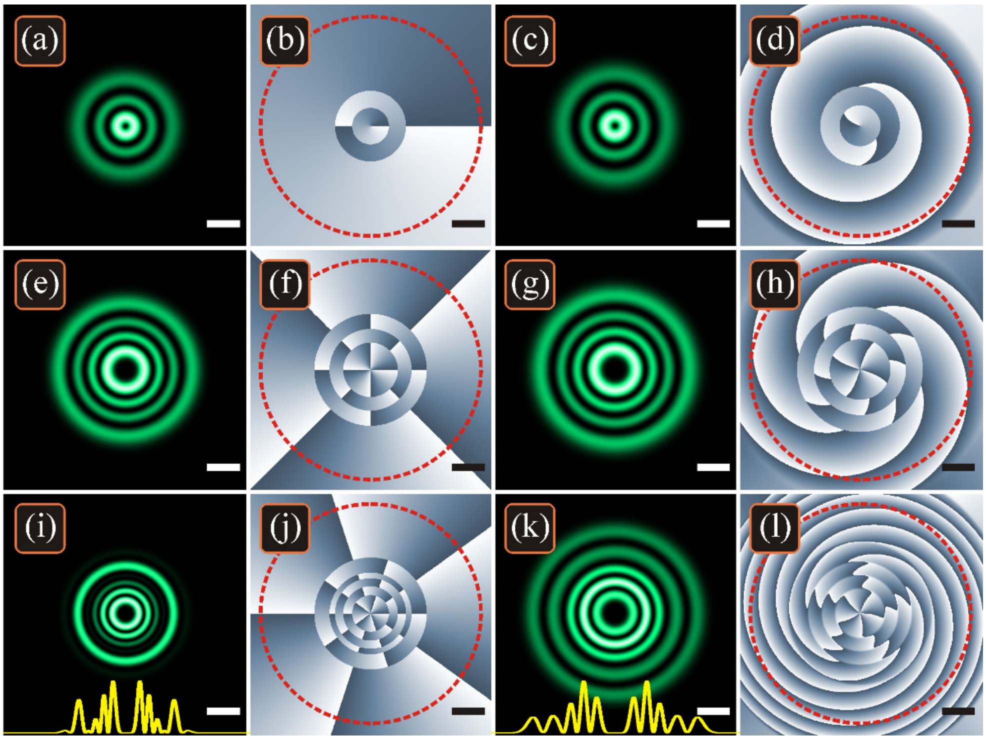

Figure 1. Patterns of intensity (columns 1 (**a**,**e**,**i**) and 3 (**c**,**g**,**k**)) and phase (columns 2 (**b**,**f**,**j**) and 4 (**d**,**h**,**l**), dark—0, white—2π) from conventional LG beams (Equation (1)) of the orders $(p, m) = (2, 1)$ (row 1) and $(q, n) = (3, 4)$ (row 2) and from a pLG beam (Equation (3)) of the order $(p, q, m, n) = (2, 3, 1, 4)$ (row 3) in the source plane $z = 0$ (columns 1 and 2) at the half Rayleigh range $z = z_0/2$ (columns 3 and 4) for the following parameters: wavelength $\lambda = 532$ nm, waist radius $w = 0.5$ mm. The scale bar in all pictures is 1 mm. The topological charge was measured along a dashed circle in the phase patterns. Shown in the bottom row in (**i**,**k**) are intensity profiles.

From Figure 1, the intensity profiles are seen to be preserved upon free-space propagation for both conventional LG beams, while varying for the pLG beam. In particular, in the source plane, the first (from the center) intensity ring is brightest, with the second ring becoming brightest at the Rayleigh range of beam from Equation (3). According to Equation (3), the beam is supposed to have $p + q + 1 = 6$ bright intensity rings. However, just four and five intensity rings can, respectively, be seen in Figures 1i and 1k. Meanwhile, the intensity profile in Figure 1i and phase pattern in Figure 1j confirm that there are six bright rings and five intermitting dark rings (where a phase jump by π occurs). In Figure 1k, the rings are fused, reducing the total number to five.

As predicted by Equation (3), the TC of beam from Equation (3) is found as a sum of TCs of Equation (1): $1 + 4 = 5$. This can be seen from the phase pattern: in Figure 1c,d, TC = 1 (a single phase jump by 2π along the dashed circle); in Figure 1f,h TC = 4 (four phase jumps by 2π along the circle); and in Figure 1j,l TC = 5 (five phase jumps along the circle).

With the Rayleigh range of the Laguerre–Gaussian beams from Equation (1) being twice as large, following the propagation over a distance of $z = kw^2/2$, these diverge not by a factor of $2^{1/2} \approx 1.41$, but by a factor of just $(5/4)^{1/2} \approx 1.12$ (Figure 1a,c,e,g). In the meantime, the beam from Equation (3) expands by a factor of just about 1.5.

Figure 2 depicts the intensity distributions of the pLG beam in different transverse planes (initial plane, near field, Rayleigh range, far field) for other values of the indices p, q, m, and n. In all three cases, the energy is redistributed between the light rings so that in the near field and in the Rayleigh range, the peripheral rings are much brighter than they were in the initial plane. However, the number of bright rings is reduced in the far field. The third row confirms Equations (8) and (9), meaning that the squared LG beam is Fourier-invariant. The intensity in Figure 2l looks the same as in Figure 2i, despite the perturbed patterns in the intermediate planes (Figure 2j,k).

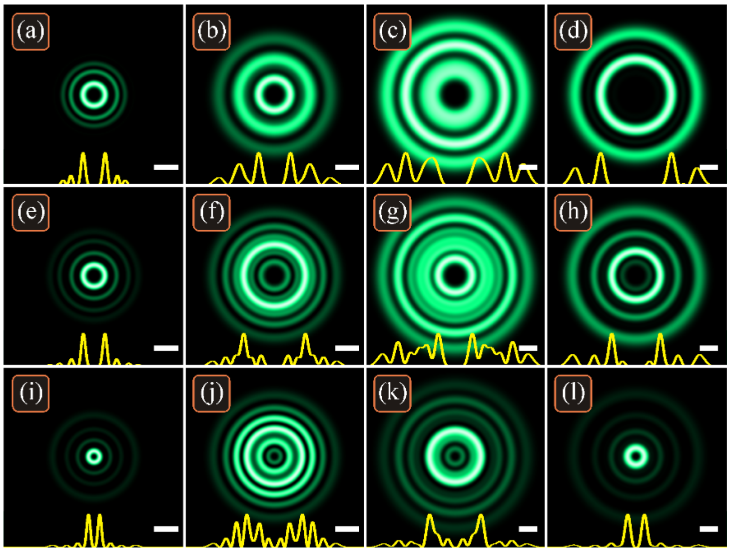

Figure 2. Intensity patterns of a pLG beam (Equation (3)) of the orders $(p, q, m, n) = (1, 4, 1, 4)$ (row 1(**a**–**d**)), $(p, q, m, n) = (2, 3, 2, 3)$ (row 2 (**e**–**h**)), $(p, q, m, n) = (3, 3, 1, 1)$ (row 3 (**i**–**l**)) in four different planes: in the source plane $z = 0$ (column 1), at the half Rayleigh range $z = z_0/2$ (column 2), at the Rayleigh range $z = z_0$ (column 3), and in the far field $z = 10z_0$ (column 4). Other parameters are the following: wavelength $\lambda = 532$ nm, waist radius $w = 0.5$ mm. The scale bar is 1 mm (columns 1–3) and 10 mm (column 4). The yellow curve below each figure shows the intensity profile.

4. Conclusions

We introduced in this paper a four-parameter family of vortex beams overlapping well-known LG beams. The new beams represent a product of two different LG beams with the same radius of waist and can also be termed as a product of LG (pLG) beams. If in a pLG beam, both Laguerre polynomials have the same indices, such a beam may be referred to as a "squared" LG beam, or $(LG)^2$ beam. The proposed pLG beams have been expressed as the superposition of a finite sum of conventional LG beams. For the $(LG)^2$ beams, an explicit Fourier transform has been derived. A particular case of pLG beams whose Laguerre polynomials are described by specially tailored azimuthal indices $n - m$ and $n + m$ has been analyzed, and their Fourier transform has been deduced in an explicit form. The pLG beams are promising for optical communication applications [22,24].

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Appendix A

Here, we briefly describe the derivation of Equation (5):

$$L_p^m(x)L_q^n(x)=\sum_{k=0}^{p+q}C_kL_k^{m+n}(2x),\tag{A1}$$

where $C_k=2^{q-k}P_{p+q-k}^{(k+m-q,k+n-p)}(0)P_q^{(k-q,p-k)}(0)$ and $P_\alpha^{\mu,\nu}(0)$ denote Jacobi polynomials at zero. To achieve this, we use the orthogonality of the Laguerre polynomials:

$$\int\limits_0^\infty e^{-cx}x^\lambda L_p^\lambda(cx)L_q^\lambda(cx)dx=\frac{(q+\lambda)!}{q!c^{\lambda+1}}\delta_{p,q},\tag{A2}$$

with $\delta_{p,q}$ being the Kronecker's delta. Thus, the expansion coefficients in Equation (A1) can be written as

$$C_k=\frac{2^{m+n+1}k!}{(k+m+n)!}\int\limits_0^\infty e^{-2x}x^{m+n}L_k^{m+n}(2x)L_p^m(x)L_q^n(x)dx=\frac{2^{m+n+1}k!}{(k+m+n)!}a_{p,q,k}.\tag{A3}$$

To obtain the coefficients $a_{p,q,k}$, we use the following generating function:

$$\sum_{p=0}^\infty\sum_{q=0}^\infty\sum_{k=0}^\infty a_{p,q,k}s_1^ps_2^qs_3^k=\int\limits_0^\infty e^{-2x}x^{m+n}\frac{\exp\left[-x\left(\frac{s_1}{1-s_1}+\frac{s_2}{1-s_2}+\frac{2s_3}{1-s_3}\right)\right]}{(1-s_1)^{m+1}(1-s_2)^{n+1}(1-s_3)^{m+n+1}}dx.\tag{A4}$$

From Equation (A4), we obtain the following expression for the coefficients $a_{p,q,k}$:

$$a_{p,q,k}=(m+n)!\left\|s_1^ps_2^qs_3^k\right\|\frac{(1-s_1)^m(1-s_2)^n}{(r_1-r_2s_3)^{m+n+1}},\tag{A5}$$

where $r_1=2-s_1-s_2,\ r_2=s_1+s_2-2s_1s_2$, and the following designations are introduced for the derivatives:

$$\left\|s^\lambda\right\|f(s)=\frac{1}{\lambda!}\frac{d^\lambda}{ds^\lambda}f(s)|_{s=0}.$$

After some transformations, the expression (A5) can be rewritten as

$$a_{p,q,k}=\frac{(m+n+k)!}{2^{m+n+p+q+1}k!}\|s^pt^q\|(s+t)^k(1-t+s)^{p+m}(1+t-s)^{q+n}.\tag{A6}$$

We return to the coefficients C_k using Equation (A3). Then, instead of Equation (A6), we obtain:

$$C_k=\frac{1}{2^{p+q}}\|s^pt^q\|(s+t)^k(1-t+s)^{p+m}(1+t-s)^{q+n}.\tag{A7}$$

The derivatives in Equation (A7) can be represented via the Jacobi polynomials of zero argument:

$$\begin{aligned}&\|s^pt^q\|(s+t)^k(1-t+s)^{p+m}(1+t-s)^{q+n}=\\&P_{p+q-k}^{(k+m-q,k+n-p)}(0)2^{p+q-k}\|s^pt^q\|(s+t)^k(s-t)^{p+q-k}=\\&P_{p+q-k}^{(k+m-q,k+n-p)}(0)2^{p+q-k}\|s^p\|(s+1)^k(s-1)^{p+q-k}=\\&\begin{cases}P_{p+q-k}^{(k+n-p,k+m-q)}(0)2^{p+q-k}P_p^{(k-p,q-k)}(0)2^p,\\P_{p+q-k}^{(k+m-q,k+n-p)}(0)2^{p+q-k}P_q^{(k-q,p-k)}(0)2^q.\end{cases}\end{aligned}\tag{A8}$$

From Equation (A8), if $p > q$, we finally obtain the expansion coefficients as

$$C_k = 2^{q-k} P_{p+q-k}^{(k+m-q,k+n-p)}(0) P_q^{(k-q,p-k)}(0). \tag{A9}$$

References

1. Kogelnik, H.; Li, T. Laser beams and resonators. *Appl. Opt.* **1966**, *5*, 1550–1567. [CrossRef] [PubMed]
2. Zauderer, E. Complex argument Hermite-Gaussian and Laguerre-Gaussian beams. *J. Opt. Soc. Am. A* **1986**, *3*, 465–469. [CrossRef]
3. Wünsche, A. Generalized Gaussian beam solutions of paraxial optics and their connection to a hidden symmetry. *J. Opt. Soc. Am. A* **1989**, *6*, 1320–1329. [CrossRef]
4. Abramochkin, E.; Volostnikov, V. Beam transformations and nontransformed beams. *Opt. Commun.* **1991**, *83*, 123–135. [CrossRef]
5. Allen, L.; Beijersbergen, M.W.; Spreeuw, R.J.C.; Woerdman, J.P. Orbital angular momentum of light and the transformation of Laguerre– Gaussian laser modes. *Phys. Rev. A* **1992**, *45*, 8185–8189. [CrossRef] [PubMed]
6. Abramochkin, E.G.; Volostnikov, V.G. Generalized Gaussian beams. *J. Opt. A Pure Appl. Opt.* **2004**, *6*, S157–S161. [CrossRef]
7. Abramochkin, E.; Razueva, E.; Volostnikov, V. General astigmatic transform of Hermite–Laguerre–Gaussian beams. *J. Opt. Soc. Am. A* **2010**, *27*, 2506–2513. [CrossRef]
8. Zhou, G.; Ru, G. Orbital Angular Momentum Density of an Elegant Laguerre-Gaussian Beam. *Prog. Electromagn. Res.* **2013**, *141*, 751–768. [CrossRef]
9. Kotlyar, V.V.; Khonina, S.N.; Almazov, A.A.; Soifer, V.A.; Jefimovs, K.; Turunen, J. Elliptic Laguerre-Gaussian beams. *J. Opt. Soc. Am. A* **2006**, *23*, 43–56. [CrossRef]
10. Mendoza-Hernández, J.; Arroyo-Carrasco, M.L.; Iturbe-Castillo, M.D.; Chávez-Cerda, S. Laguerre–Gauss beams versus Bessel beams showdown: Peer comparison. *Opt. Lett.* **2015**, *40*, 3739–3742. [CrossRef]
11. Wang, M.; Ma, Y.; Sheng, Q.; He, X.; Liu, J.; Shi, W.; Yao, J.; Omatsu, T. Laguerre-Gaussian beam generation via enhanced intracavity spherical aberration. *Opt. Express* **2021**, *29*, 27783–27790. [CrossRef] [PubMed]
12. Rafayelyan, M.; Brasselet, E. Laguerre–Gaussian modal q-plates. *Opt. Lett.* **2017**, *42*, 1966–1969. [CrossRef] [PubMed]
13. Mao, H.; Ren, Y.-H.; Yu, Y.; Yu, Z.; Sun, X.; Zhang, S.; Wong, K.K.Y. Broadband meta-converters for multiple Laguerre-Gaussian modes. *Photon. Res.* **2021**, *9*, 1689–1698. [CrossRef]
14. Liang, G.; Wang, Q. Controllable conversion between Hermite Gaussian and Laguerre Gaussian modes due to cross phase. *Opt. Express* **2019**, *27*, 10684–10691. [CrossRef] [PubMed]
15. Longman, A.; Fedosejevs, R. Optimal Laguerre–Gaussian modes for high-intensity optical vortices. *J. Opt. Soc. Am. A* **2020**, *37*, 841–848. [CrossRef]
16. Dong, M.; Lu, X.-Y.; Zhao, C.; Cai, Y.; Yang, Y. Measuring topological charge of partially coherent elegant Laguerre-Gaussian beam. *Opt. Express* **2018**, *26*, 33035–33043. [CrossRef]
17. Kovalev, A.A.; Kotlyar, V.V.; Porfirev, A.P. Asymmetric Laguerre-Gaussian beams. *Phys. Rev. A* **2016**, *93*, 063858. [CrossRef]
18. Hsieh, Y.H.; Lai, Y.H.; Hsieh, M.X.; Huang, K.F.; Chen, Y.F. Generating high-power asymmetrical Laguerre-Gaussian modes and exploring topological charges distribution. *Opt. Express* **2018**, *26*, 31738–31749. [CrossRef]
19. Ghaderi Goran Abad, M.; Mahmoudi, M. Laguerre-Gaussian modes generated vector beam via nonlinear magneto-optical rotation. *Sci. Rep.* **2021**, *11*, 5972. [CrossRef]
20. Huang, S.; Miao, Z.; He, C.; Pang, F.; Li, Y.; Wang, T. Composite vortex beams by coaxial superposition of Laguerre-Gaussian beams. *Opt. Lasers Eng.* **2016**, *78*, 132–139. [CrossRef]
21. Chen, Y.; Liu, L.; Wang, F.; Zhao, C.; Cai, Y. Elliptical Laguerre-Gaussian correlated Schell-model beam. *Opt. Express* **2014**, *22*, 13975–13987. [CrossRef] [PubMed]
22. Zhong, W.; Yang, Z.; Belic, M.; Zhong, W. Two-dimensional asymmetric Laguerre-Gaussian diffraction-free beams. *Phys. Lett. A* **2022**, *423*, 127818. [CrossRef]
23. Zhong, W.; Zhong, W.; Belic, M.; Zhang, Z. Controllable two-dimensional diffraction-free polygon beams. *Phys. Lett. A* **2022**, *432*, 128009. [CrossRef]
24. Zhong, W.; Yi, L. Two-dimensional Laguerre-Gaussian soliton family in strongly nonlocal nonlinear media. *Phys. Rev. A* **2007**, *75*, 061801. [CrossRef]
25. Zhong, W. Two-dimensional Laguerre-Gaussian asymmetric soliton family in strong nonlocal media. *Chin. Phys. Lett.* **2008**, *25*, 2074–2077.
26. Zhong, W.; Belic, M. Three-dimensional optical vortex and necklace solitons in highly nonlocal nonlinear media. *Phys. Rev. A* **2009**, *79*, 023804. [CrossRef]
27. Pang, K.; Liu, C.; Xie, G.; Ren, Y.; Zhao, Z.; Zhang, R.; Cao, Y.; Zhao, J.; Song, H.; Song, H.; et al. Demonstration of a 10 Mbit/s quantum communication link by encoding data on two Laguerre–Gaussian modes with different radial indices. *Opt. Lett.* **2018**, *43*, 5639–5642. [CrossRef]
28. Doster, T.; Watnik, A.T. Laguerre–Gauss and Bessel–Gauss beams propagation through turbulence: Analysis of channel efficiency. *Appl. Opt.* **2016**, *55*, 10239–10246. [CrossRef]
29. Zeng, J.; Liu, X.; Zhao, C.; Wang, F.; Gbur, G.; Cai, Y. Spiral spectrum of a Laguerre-Gaussian beam propagating in anisotropic non-Kolmogorov turbulent atmosphere along horizontal path. *Opt. Express* **2019**, *27*, 25342–25356. [CrossRef]

30. Cox, M.A.; Maqondo, L.; Kara, R.; Milione, G.; Cheng, L.; Forbes, A. The Resilience of Hermite– and Laguerre–Gaussian Modes in Turbulence. *J. Lightwave Technol.* **2019**, *37*, 3911–3917. [CrossRef]
31. Otsu, T.; Ando, T.; Takiguchi, Y.; Ohtake, Y.; Toyoda, H.; Itoh, H. Direct evidence for three-dimensional off-axis trapping with single Laguerre-Gaussian beam. *Sci. Rep.* **2014**, *4*, 4579. [CrossRef] [PubMed]
32. Peshkov, A.A.; Seipt, D.; Surzhykov, A.; Fritzsche, S. Photoexcitation of atoms by Laguerre-Gaussian beams. *Phys. Rev. A* **2017**, *96*, 023407. [CrossRef]
33. Courtial, J.; Dholakia, K.; Allen, L.; Padgett, M.J. Second-harmonic generation and the conservation of orbital angular momentum with high-order Laguerre-Gaussian modes. *Phys. Rev. A* **1997**, *56*, 4193. [CrossRef]
34. Wu, H.; Mao, L.; Yang, Y.; Rosales-Guzman, C.; Gao, W.; Shi, B.; Zhu, Z. Radial modal transformations of Laguerre-Gauss modes during parametric up-conversion: Towards the full-field selection rule of spatial modes. *Phys. Rev. A* **2020**, *101*, 063805. [CrossRef]
35. Prudnikov, A.P.; Bychkov, Y.; Marychev, O.I. *Integrals and Series: Volume 2: Special Functions*; "Nauka" Publisher: Moscow, Russia, 1983.