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Measuring the *p*th-Order Correlation Function of Light Field via Two-Level Atoms

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Abstract: In this paper, we present a method for measuring arbitrary-order correlation functions of the light field using a two-level atomic system. Theoretically, light field information should be mapped onto the atomic system after the light interacts with the atom. Therefore, we can measure the atomic system and thus obtain information about the light field. We study two typical models, the *p*-photon Jaynes–Cummings model, and the *p*-photon Tavis–Cummings model. In both models, we find that the *p*th-order correlation function of an unknown light field can be obtained by measuring the instantaneous change of energy of the two-level atoms with the aid of a known reference light field. Moreover, we find that the interactions other than the dipole interactions between light and atoms have no effect on the measurement results.

Keywords: correlation functions; indirect measurement; two-level atoms; Jaynes–Cummings model; Tavis–Cummings model

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1. Introduction

In quantum optics, light with quantum correlations is widely used in quantum communication [1–8], quantum computing [9–17], quantum metrology [18–21], quantum imaging [22–26], and quantum sensing [27–30]. In addition, correlation functions of light field proposed by Glauber are also commonly used to distinguish the quantum and classical nature of the light field [31–33]. Unnormalized equal-time *p*th-order correlation functions of light fields proposed by Glauber are defined as $g^{(p)}(0) = \langle \hat{a}^{\dagger p} \hat{a}^{p} \rangle$, where $\hat{a}^{\dagger}$ and $\hat{a}$ are the photon creation and annihilation operators of the light field, respectively. These correlation functions can characterize some properties of the light field. For example, the equal-time first-order correlation function $g^{(1)}(0)$ of the light field characterizes the average photon number of the light field. Additionally, these functions can be used to characterize some quantum phenomena in the quantum light field that are not observed in the classical system. The most well known is the equal-time second-order correlation function $g^{(2)}(0)$, which is commonly used to distinguish between classical and quantum light fields. For example, when $g^{(2)}(0) < [g^{(1)}(0)]^2$, the light field shows a well-known antibunching phenomenon [34–40], which quantifies how the detection of one photon from a source affects the probability of detecting another photon; this is a typical quantum phenomenon. In general, $g^{(2)}(0)/[g^{(1)}(0)]^2 < 0.5$ means that the light field is a good single-photon source [41–46]. Of course, a strictly single-photon source needs to satisfy $g^{(2)}(0)/[g^{(1)}(0)]^2 = 0$. Single-photon sources play a very critical role in quantum networks [47], universal linear quantum computing [48], and boson sampling [49]. Moreover, the resolution of the microscope can be improved by the second-order correlation function of the optical field [50].

The correlation function of the optical field is a significant feature of the optical field, so the measurement of the correlation function of the optical field is a problem that needs to be solved. In 1956, Hanbury Brown and Twiss pioneered a new class of optical interferometry experiments in which the second-order correlation function of the optical field was observed for the first time [51]. From this experiment, theoretical and experimental studies of the nonclassical nature of light became a very hot area of research, which is also known as quantum optics [52]. In addition, methods or devices for measuring the second-order correlation function of the optical field include random phase modulation methods [53], analog detectors [54], linear detectors [55,56], and reversed-wavefront Young interferometers [57].

However, when we study the statistical properties of the optical field in detail, it is not enough to measure the first- and second-order correlation functions of the optical field. For example, when we study two-photon blockades, we need to measure the second-order correlation function and the third-order correlation function. When we study more photon blockades, higher-order correlation function measurements are required. Based on this practical problem, we propose a method to indirectly obtain the pth-order correlation function of the light field by measuring the energy change of atoms. In the following, we investigate this indirect measurement method with an extended p-photon Jaynes–Cummings (JC) model and an extended p-photon Tavis–Cummings (TC) model.

2. Measuring the pth-Order Correlation Function of the Light Field in p-Photon JC Model

We consider an extended p-photon Jaynes–Cummings model (as shown in Figure 1a) with the following Hamiltonian ($\hbar = 1$):

$$\hat{H}_{JC} = \omega_a \hat{a}^\dagger \hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z + g(\hat{a}^{\dagger p}\hat{\sigma}_- + \hat{\sigma}_+\hat{a}^p) + \frac{U}{2}\hat{a}^{\dagger 2}\hat{a}^2 + \gamma\hat{a}^\dagger\hat{a}\hat{\sigma}_z, \tag{1}$$

where p is an integer greater than or equal to 1. ω_a and ω_0 are the eigenfrequency of the optical cavity and the transition frequency of the two-level atom, respectively. g is the dipole interaction strength between the cavity field and the single atom, U is the Kerr-type nonlinear interaction strength of the optical field, and γ is the dispersion interaction between the cavity field and the single atom. $\hat{a}$ and $\hat{a}^\dagger$ are, respectively, the annihilation and creation operators of the single-mode cavity field that satisfy the usual bosonic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. $\hat{\sigma}_{x,y,z}$ are the usual Pauli operators, and $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$. Experimentally, the multiphoton JC model can be implemented in systems such as superconducting circuits and trapped ions [58–62].

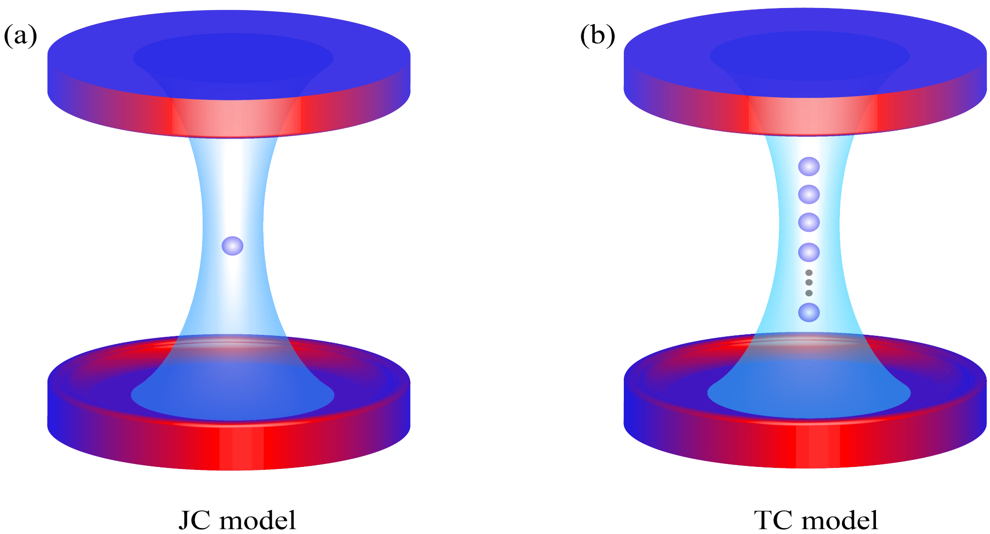

Figure 1. (**a**) Schematic diagram of the JC model with a single two-level atom interacting with a single-mode optical field. (**b**) Schematic diagram of the TC model with multiple two-level atoms interacting with a single-mode optical field.

Here, we assume that the light field is in an unknown state $|\varphi_a(0)\rangle$ and the atom is in the ground state $|g\rangle$, that is, the initial state of the total system is $|\psi(0)\rangle = |\varphi_a(0)\rangle \otimes |g\rangle = \sum_{n=0}^{\infty} c_n|g,n\rangle$, where $c_n = \langle n|\varphi_a(0)\rangle$ and $|n\rangle$ is the number state of the optical field. Since the total excitation number is conserved (the total excitation number operator $\hat{N}_e = \hat{a}^\dagger\hat{a} + p|e\rangle\langle e|$ satisfies the condition $[\hat{N}_e, \hat{H}_{JC}] = 0$), we can assume that the state of the system at time t has the following form:

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n[C^e_{n-p}(t)|e,n-p\rangle + C^g_n(t)|g,n\rangle], \tag{2}$$

where $C^e_{n-p}(0) = 0$ and $C^g_n(0) = 1$. Substituting Equations (1) and (2) into the Schrödinger equation, we can obtain two differential equations:

$$\begin{aligned} i\frac{d}{dt}C^e_{n-p}(t) &= A(n)C^e_{n-p}(t) + B(n)C^g_n(t), &(3)\\ i\frac{d}{dt}C^g_n(t) &= B(n)C^e_{n-p}(t) + D(n)C^g_n(t), &(4)\end{aligned}$$

where

$$\begin{aligned} A(n) &= \frac{\omega_0}{2} + \omega_a(n-p) + \frac{U}{2}(n-p)(n-p-1) + \gamma(n-p), &(5)\\ B(n) &= g\sqrt{n!/(n-p)!}, &(6)\\ D(n) &= -\frac{\omega_0}{2} + \omega_a n - \frac{U}{2}n(n-1) - \gamma n. &(7)\end{aligned}$$

The above two differential equations are easy to solve using the Laplace transform, and we give the specific solution procedure in Appendix A. Then, we can obtain the solutions of these two differential equations as follows:

$$\begin{aligned} C^e_{n-p}(t) &= z_1[\cos(x_2t) - \cos(x_1t)] - i(z_1[\sin(x_2t) - \sin(x_1t)]), &(8)\\ C^g_n(t) &= y_1\cos(x_1t) - y_2\cos(x_2t) - i[y_1\sin(x_1t) - y_2\sin(x_2t)], &(9)\end{aligned}$$

where

$$\begin{aligned} x_1 &= \frac{[A(n)+D(n)] - \sqrt{[A(n)-D(n)]^2 + 4B^2(n)}}{2}, &(10)\\ x_2 &= \frac{[A(n)+D(n)] + \sqrt{[A(n)-D(n)]^2 + 4B^2(n)}}{2}, &(11)\\ y_1 &= \frac{A(n)-x_1}{x_2-x_1}, &(12)\\ y_2 &= \frac{A(n)-x_2}{x_2-x_1}, &(13)\\ z_1 &= \frac{B(n)}{x_2-x_1}. &(14)\end{aligned}$$

When $p = 1$ and $U = \gamma = 0$, Equations (8) and (9) can be reduced to the results of the single-photon JC model [63].

When the total system starts evolving with an initial state $|\psi(0)\rangle = |\varphi_a(0)\rangle \otimes |\phi\rangle$, where $|\varphi_a(0)\rangle$ is an unknown initial state of the light field to be measured, and $|\phi\rangle$ is a known initial state of the prepared probe atom ($|\phi\rangle = |g\rangle$ in here). Then, we can obtain the energy acquired by the atom from the light field at time t as follows:

$$\begin{aligned} \Delta E(|\psi(0)\rangle, t) &= \langle\psi(t)|\frac{\omega_0}{2}\hat{\sigma}_z|\psi(t)\rangle - \langle\psi(0)|\frac{\omega_0}{2}\hat{\sigma}_z|\psi(0)\rangle \\ &= \sum_{n=0}^{\infty}|c_n|^2\left(\left|C^e_{n-p}(t)\right|^2 - \left|C^g_n(t)\right|^2\right) + 1. \end{aligned} \tag{15}$$

Theoretically, since different light fields make different energy changes in the atom, we can distinguish light fields according to the difference in energy changes of the atom. In order to extract information about the light field from the energy change of the atom, we need a scale. Finally, by the ratio of the energy change of the atom to this scale, we can obtain information about the light field. Here, we choose the change of the atomic energy as a scale when the initial state of the total system is a known initial state. Therefore, when the initial state of the total system is $|\psi_r(0)\rangle = |\varphi_r\rangle \otimes |g\rangle$, where $|\varphi_r\rangle$ is a known initial state of the light field. Thus, $|\psi_r(0)\rangle$ is a known initial state, and we call it a known reference state. Then we can similarly obtain the energy obtained by the atom from the light field at time t

$$\begin{aligned} \Delta E(|\psi_r(0)\rangle, t) &= \langle\psi_r(t)|\frac{\omega_0}{2}\hat{\sigma}_z|\psi_r(t)\rangle - \langle\psi_r(0)|\frac{\omega_0}{2}\hat{\sigma}_z|\psi_r(0)\rangle \\ &= \sum_{m=0}^{\infty}|d_m|^2\left(\left|C^e_{m-p}(t)\right|^2 - \left|C^g_m(t)\right|^2\right) + 1, \end{aligned} \tag{16}$$

where $d_m = \langle m|\varphi_r\rangle$ and $|m\rangle$ is the number state of the optical field. Here, we refer to $\Delta E(|\psi_r(0)\rangle, t)$ as the scale of the atomic energy change.

In the following, we start to investigate the ratio of the energy acquired by the atom in the unknown light field and in the known light field, respectively, when $t \to 0$. Thus, we are able to obtain the following expression:

$$\lim_{t\to 0}\frac{\Delta E(|\psi(0)\rangle, t)}{\Delta E(|\psi_r(0)\rangle, t)} = \lim_{t\to 0}\frac{\sum_{n=0}^{\infty}|c_n|^2\left(\left|C^e_{n-p}(t)\right|^2 - \left|C^g_n(t)\right|^2\right) + 1}{\sum_{m=0}^{\infty}|d_m|^2\left(\left|C^e_{m-p}(t)\right|^2 - \left|C^g_m(t)\right|^2\right) + 1}. \tag{17}$$

When $t \to 0$, since $|C^e_{n-p}(t)|^2 = |C^e_{m-p}(t)|^2 = 0$ and $|C^g_n(t)|^2 = |C^g_m(t)|^2 = 1$, according to L'Hôpital's law, we need to consider the first-order derivatives of the numerator and denominator in Equation (20) with respect to time t. These two derivatives are easily derived as follows:

$$\begin{aligned} \frac{d}{dt}\left|C^e_{n-p}(t)\right|^2 &= 2(z_1[\cos(x_2 t) - \cos(x_1 t)])(z_1[-x_2\sin(x_2 t) + x_1\sin(x_1 t)]) \\ &\quad + 2(z_1[\sin(x_2 t) - \sin(x_1 t)])(z_1[x_2\cos(x_2 t) - x_1\cos(x_1 t)]), \end{aligned} \tag{18}$$

$$\begin{aligned} \frac{d}{dt}\left|C^g_n(t)\right|^2 &= 2[y_1\cos(x_1 t) - y_2\cos(x_2 t)][-y_1 x_1\sin(x_1 t) + y_2 x_2\sin(x_2 t)] \\ &\quad + 2[y_1\sin(x_1 t) - y_2\sin(x_2 t)][y_1 x_1\cos(x_1 t) - y_2 x_2\cos(x_2 t)]. \end{aligned} \tag{19}$$

Since $|C^e_{m-p}(t)|^2$ and $|C^g_m(t)|^2$ have the same derivatives with respect to time t as $|C^e_{n-p}(t)|^2$ and $|C^g_n(t)|^2$, we do not have a redundant representation here. Obviously, since

$$\lim_{t\to 0}\frac{d}{dt}|C^e_{n-p}(t)|^2 = 0, \tag{20}$$

$$\lim_{t\to 0}\frac{d}{dt}|C^g_n(t)|^2 = 0, \tag{21}$$

according to L'Hôpital's law, we need to consider the second-order derivatives of the numerator and denominator in Equation (17) with respect to time t, and they are

$$\begin{aligned}\frac{d^2}{dt^2}\left|C^e_{n-p}(t)\right|^2 &= 2\frac{d(z_1[\cos(x_2t)-\cos(x_1t)])}{dt}(z_1[-x_2\sin(x_2t)+x_1\sin(x_1t)]) \\ &\quad +2(z_1[\cos(x_2t)-\cos(x_1t)])\frac{d(z_1[-x_2\sin(x_2t)+x_1\sin(x_1t)])}{dt} \\ &\quad +2(z_1[\sin(x_2t)-\sin(x_1t)])\frac{d(z_1[x_2\cos(x_2t)-x_1\cos(x_1t)])}{dt} \\ &\quad +2(z_1[x_2\cos(x_2t)-x_1\cos(x_1t)])(z_1[x_2\cos(x_2t)-x_1\cos(x_1t)]), \end{aligned} \tag{22}$$

$$\begin{aligned}\frac{d^2}{dt^2}\left|C^g_{n}(t)\right|^2 &= 2[y_1\cos(x_1t)-y_2\cos(x_2t)]\left[-y_1x_1^2\cos(x_1t)+y_2x_2^2\cos(x_2t)\right] \\ &\quad +2[y_1x_1\cos(x_1t)-y_2x_2\cos(x_2t)][y_1x_1\cos(x_1t)-y_2x_2\cos(x_2t)]. \end{aligned} \tag{23}$$

Then, we can obtain

$$\begin{aligned}\lim_{t\to 0}\frac{d^2}{dt^2}\left|C^e_{n-p}(t)\right|^2 &= 2[z_1(x_2-x_1)]^2 \\ &= 2B(n)^2, \end{aligned} \tag{24}$$

$$\begin{aligned}\lim_{t\to 0}\frac{d^2}{dt^2}\left|C^g_{n}(t)\right|^2 &= 2(y_1-y_2)\left(-y_1x_1^2+y_2x_2^2\right)+2(y_1x_1-y_2x_2)^2 \\ &= -2B(n)^2. \end{aligned} \tag{25}$$

By L'Hôpital's law, substituting Equations (24) and (25) into Equation (17), we obtain

$$\lim_{t\to 0}\frac{\Delta E(|\psi(0)\rangle,t)}{\Delta E(|\psi_r(0)\rangle,t)} = \frac{\sum_{n=0}^{\infty}|c_n|^2B^2(n)}{\sum_{m=0}^{\infty}|d_m|^2B^2(m)} = \frac{\sum_{n=0}^{\infty}|c_n|^2\sqrt{n!/(n-p)!}}{\sum_{m=0}^{\infty}|d_m|^2\sqrt{m!/(m-p)!}}. \tag{26}$$

And since $\langle\psi(0)|\hat{a}^{\dagger p}\hat{a}^p|\psi(0)\rangle = \sum_{n=0}^{\infty}|c_n|^2\sqrt{n!/(n-p)!}$, the above equation actually represents the following equation

$$\lim_{t\to 0}\frac{\Delta E(|\psi(0)\rangle,t)}{\Delta E(|\psi_r(0)\rangle,t)} = \frac{\langle\psi(0)|\hat{a}^{\dagger p}\hat{a}^p|\psi(0)\rangle}{\langle\psi_r(0)|\hat{a}^{\dagger p}\hat{a}^p|\psi_r(0)\rangle}. \tag{27}$$

We rewrite the above equation as

$$\langle\psi(0)|\hat{a}^{\dagger p}\hat{a}^p|\psi(0)\rangle = \langle\psi_r(0)|\hat{a}^{\dagger p}\hat{a}^p|\psi_r(0)\rangle \lim_{t\to 0}\frac{\Delta E(|\psi(0)\rangle,t)}{\Delta E(|\psi_r(0)\rangle,t)}. \tag{28}$$

Since $|\psi_r(0)\rangle$ is a light field initial state known to us, in effect, both $\langle\psi_r(0)|\hat{a}^{\dagger p}\hat{a}^p|\psi_r(0)\rangle$ and $\Delta E(|\psi_r(0)\rangle,t)$ are known. Therefore, we only need to measure the change in energy of the atom over a short period of time to obtain the pth-order correlation function of the unknown light field $|\psi(0)\rangle$.

In order to verify the above conclusions from the actual dynamical evolution of the atom, we plot the variation of the energy obtained by the atom from the light field with time for the single-photon JC model and the two-photon JC model in Figure 2a,b, respectively, when the system takes different parameter values and different initial states. It is worth noting, in particular, that the red dashed line indicates the variation of the energy acquired by the atom from the light field with time for a known reference state. To extract information about the unknown light field from the variation of the atomic energy, we compare the energy acquired by the atom from the unknown light field with the energy acquired by the atom from the known light field to obtain information about the unknown light field. For example, we plot in Figure 2c,d the relative ratios of the energy obtained by the atom from the unknown light field in the single-photon JC model and the two-photon JC model with the aid of the reference state as a function of time, respectively. Obviously, when the interaction time between the atom and the light field is short enough, we can obtain the first-order correlation function and the second-order correlation function of the unknown light field by measuring the energy change of the atom with the assistance of a known reference state. Of course, we can also obtain higher-order correlation functions for the unknown light field, which we will not redundantly discuss here.

Furthermore, we can see from Figure 2c,d that when $t \to 0$, different values of γ and U do not affect the value of this correlation function of the light field obtained by measuring the energy change of the atoms. That is, the elastic collision interactions between the atom and the light field and between photon and photon do not affect the results obtained by this measurement method.

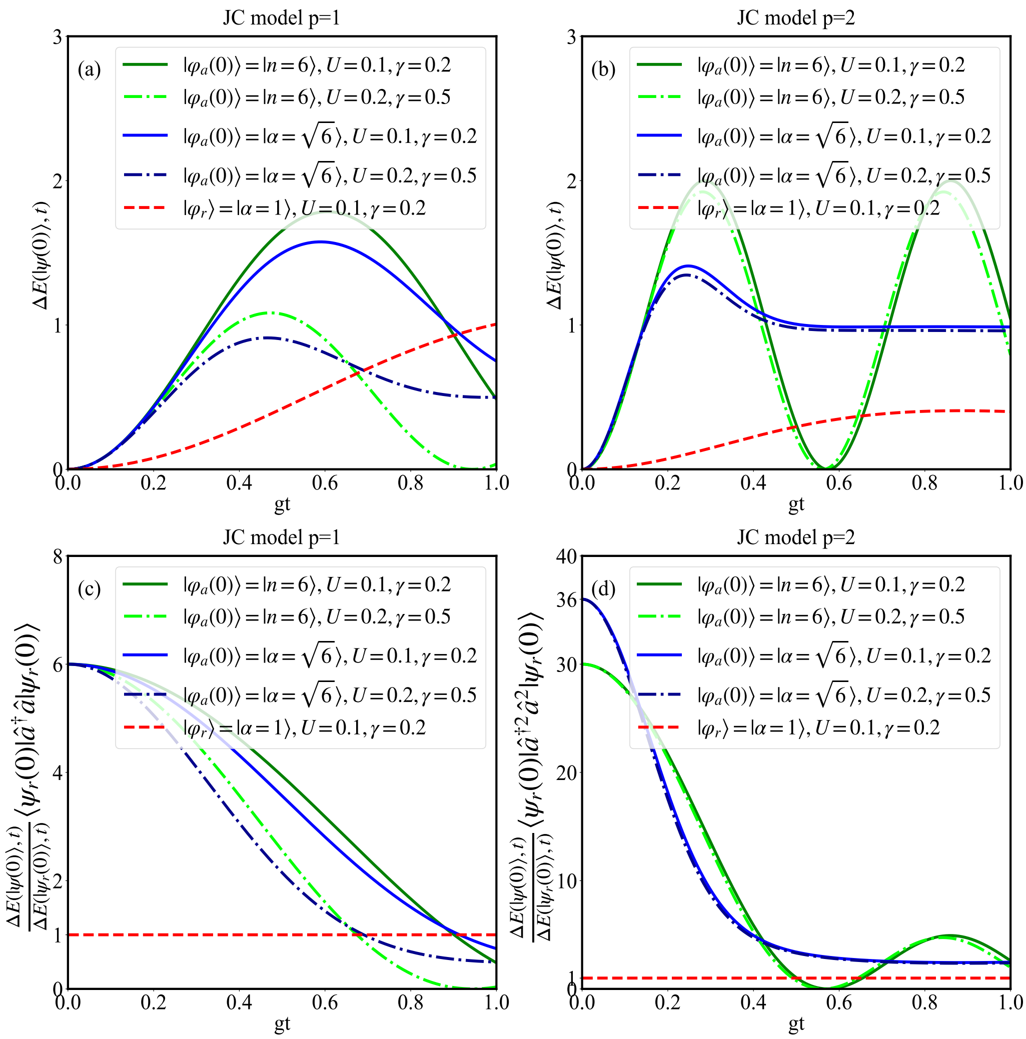

Figure 2. (**a**) Variation with time of the energy acquired by atoms from the light field in the single-photon JC model for different parameters and different initial states. (**b**) Variation with time of the energy acquired by atoms from the light field in the two-photon JC model for different parameters and different initial states., respectively. (**c**) The ratio of the energy acquired by an atom from an unknown light field and a known reference light field in the single-photon JC model as a function of time. (**d**) The ratio of the energy acquired by an atom from an unknown light field and a known reference light field in the two-photon JC model as a function of time. The initial state of the total system is $|\psi(0)\rangle = |\varphi_a(0)\rangle \otimes |g\rangle$ and the known reference state is $|\psi_r(0)\rangle = |\varphi_r\rangle \otimes |g\rangle$. The values of other parameters are $\omega_a = 1$, $\omega_0 = \omega_a$, and the values of U and γ in the figure are all ratios to ω_a.

3. Measuring the pth-Order Correlation Function of the Optical Field in p-Photon TC Model

Above, we have investigated how to obtain the correlation function of the unknown light field in the JC model using the measurement of the energy change of a single atom. Here, we study how to obtain the correlation function of the unknown light field in the TC model using the energy change of multiple atoms. The Hamiltonian of an extended p-photon TC model (as shown in Figure 1b) is as follows:

$$\hat{H}_{TC} = \omega_a \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + g(\hat{a}^{\dagger p} \hat{J}_- + \hat{J}_+ \hat{a}^p) + \frac{U}{2} \hat{a}^{\dagger 2} \hat{a}^2 + \gamma \hat{a}^\dagger \hat{a} \hat{J}_z + \chi \hat{J}_z^2, \tag{29}$$

where $\hat{J}_\alpha$ ($\alpha = x, y, z$) is the collective angular momentum operator for the spin ensemble consisting of N identical two-level atoms; these operators $\hat{J}_x, \hat{J}_y, \hat{J}_z$ satisfy the commutation relation of $SU(2)$ algebra and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. χ is the strength of the interaction between the atoms, and the other parameters are the same as in the JC model.

When we were studying single-photon TC quantum batteries before [64], we found an interesting phenomenon, which is expressed by the following equation:

$$\lim_{t \to 0} \frac{\Delta E(|m\rangle, t)}{\Delta E(|M\rangle, t)} = \frac{m}{M}, \tag{30}$$

where $|m\rangle$ and $|M\rangle$ denote the two number states, respectively, and $\Delta E(|m\rangle, t)$ (or $\Delta E(|M\rangle, t)$) denotes the variation of the energy obtained by all the atoms from the number state light field $|m\rangle$ (or $|M\rangle$) with time t. Furthermore, when the initial state of the total system is $|\psi(0)\rangle = |\varphi_a(0)\rangle \otimes |J, -J\rangle$ ($J = N/2$), where $|\varphi_a(0)\rangle$ denotes that the light field is in an arbitrary state as well as $|J, -J\rangle$ denoting that the atoms are in the ground state, the energy that the atoms acquire from the light field at time t is [64]

$$\Delta E(|\varphi_a(0)\rangle, t) = \sum_{m=0}^{\infty} |c(|m\rangle, |\varphi_a(0)\rangle)|^2 \Delta E(|m\rangle, t), \tag{31}$$

where $|c(|m\rangle, |\varphi_a(0)\rangle)|^2$ is the probability distribution of the initial state $|\varphi_a(0)\rangle$ in the number states.

Similar to the idea of the study in the JC model, we choose a number state $|M\rangle$ as a known reference light field, then we can obtain the following expression:

$$\lim_{t \to 0} \frac{\Delta E(|\varphi_a(0)\rangle, t)}{\Delta E(|M\rangle, t)} \langle M|\hat{a}^\dagger \hat{a}|M\rangle. \tag{32}$$

Substituting Equation (31) into Equation (32), we can obtain

$$\begin{aligned}
& \lim_{t \to 0} \frac{\Delta E(|\varphi_a(0)\rangle, t)}{\Delta E(|M\rangle, t)} \langle M|\hat{a}^\dagger \hat{a}|M\rangle \\
= & \lim_{t \to 0} \frac{\sum_{m=0}^{\infty} |c(|m\rangle, |\varphi_a(0)\rangle)|^2 \Delta E(|m\rangle, t)}{\Delta E(|M\rangle, t)} \langle M|\hat{a}^\dagger \hat{a}|M\rangle \\
= & \frac{\sum_{m=0}^{\infty} |c(|m\rangle, |\varphi_a(0)\rangle)|^2 m}{M} M \\
= & \langle \varphi_a(0)|\hat{a}^\dagger \hat{a}|\varphi_a(0)\rangle.
\end{aligned} \tag{33}$$

For the above equation, it is also possible to obtain information on the average photon number of the unknown light field by measuring the change in the energy acquired by the atoms from the unknown light field with the assistance of a known reference state.

When the light field takes different initial states and the Hamiltonian of the total system takes different parameter values, in Figure 3a,b, we plot the variation of the energy acquired by the atoms from the light field with time in the single-photon TC model and the two-photon TC model, respectively. Furthermore, in Figure 3c,d, we plot the variation of the relative ratio of the energy acquired by the atoms in the single-photon TC model and the two-photon TC model with respect to time, respectively. We find that when the time is sufficiently short, i.e., $t \to 0$, with the assistance of a known reference state, we can completely obtain the pth-order correlation function of the optical field by measuring the change of atomic energy in the p-photon TC model.

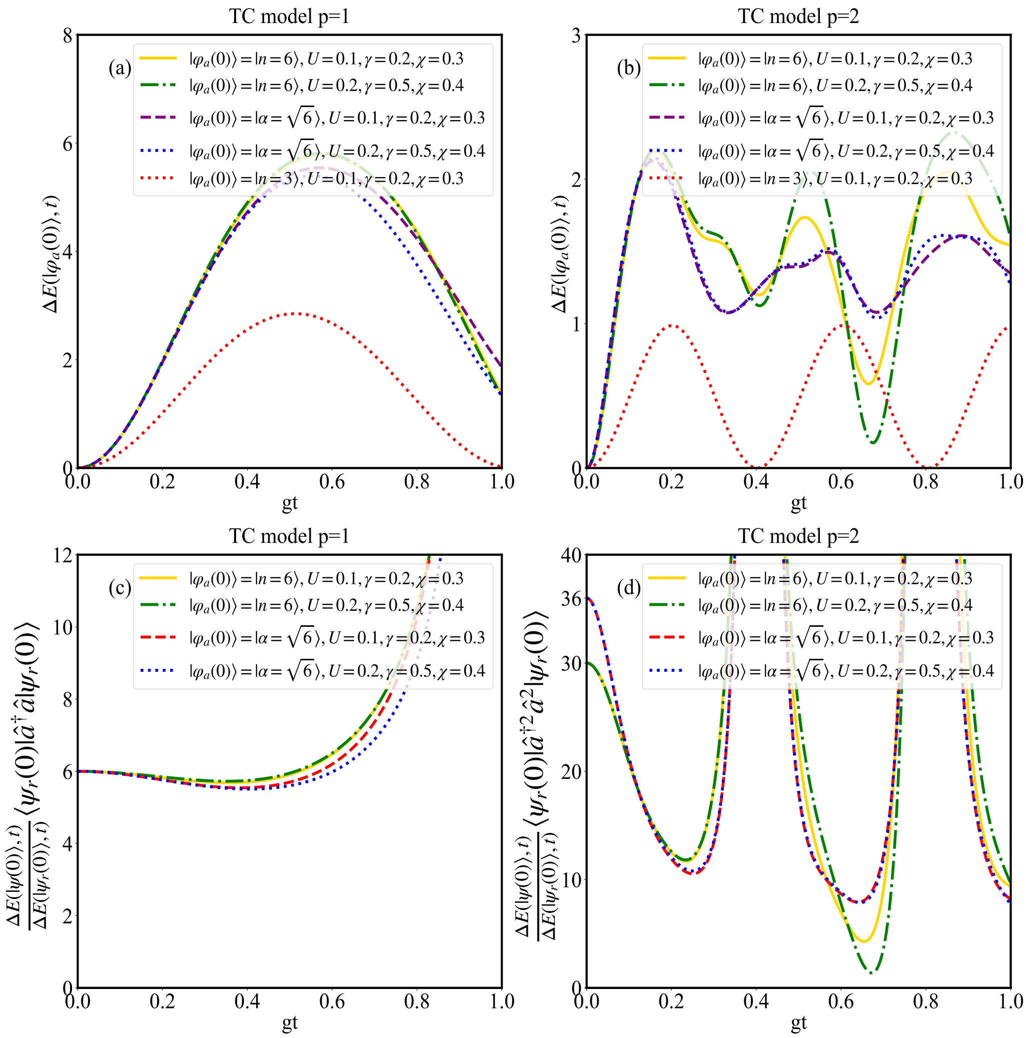

Figure 3. (**a**) Variation with time of the energy acquired by the all atoms from the light field in the single-photon TC model for different parameters and different initial states. (**b**) Variation with time of the energy acquired by the all atoms from the light field in the two-photon TC model for different parameters and different initial states, respectively. (**c**) The ratio of the energy acquired by the all atoms from an unknown light field and a known reference light field in the single-photon TC model as a function of time. (**d**) The ratio of the energy acquired by the all atoms from an unknown light field and a known reference light field in the two-photon TC model as a function of time. The initial state of the total system is $|\psi(0)\rangle = |\varphi_a(0)\rangle \otimes |J, -J\rangle$ and the known reference state is $|\psi_r(0)\rangle = |M = 3\rangle \otimes |J, -J\rangle$. The values of other parameters are $\omega_a = 1$, $\omega_0 = \omega_a$, and the values of U , γ and χ in the figure are all ratios to ω_a.

Here, we can compare the p-photon TC model with the p-photon JC model. When $t \to 0$, we find that using the change of atomic energy in the p-photon TC model to obtain the pth-order correlation function of the light field is the same as that using the p-photon JC model. That is, this indirect measurement is independent of the specific atomic number and is only related to the form of the dipole interaction between the atom and the light field.

All the data obtained from the numerical calculations in Figure 3 were performed by QuTIP [65], a quantum toolbox.

4. Conclusions

In conclusion, we propose a method to indirectly measure the pth-order correlation function of the optical field using a two-level atomic system. We find that with the aid of a known reference state, the information of the pth-order correlation function of the optical field can be extracted by measuring the energy change of the atoms. We investigate the advantage of this measurement method using the p-photon JC model and the p-photon TC model, respectively, that is, the ability to measure arbitrary order correlation functions compared to other methods. We find that any interaction in the system other than the dipole interaction between light and atoms does not affect the measurement results, i.e., this measurement method is robust to other possible interactions. Finally, we compare this indirect measurement method in both models, and we find that this measurement method depends only on the dipole interaction between light and atoms, independent of the specific number of atoms involved in the measurement.

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Appendix A. The Specific Solution Procedure for the System of Differential Equations (3) and (4)

Here, we give the specific solution procedure for Equations (3) and (4). Substituting Equations (1) and (2) into the Schrödinger equation, we can obtain two differential equations:

$$i\frac{d}{dt}C^{e}_{n-p}(t) = A(n)C^{e}_{n-p}(t) + B(n)C^{g}_{n}(t), \tag{A1}$$

$$i\frac{d}{dt}C^{g}_{n}(t) = B(n)C^{e}_{n-p}(t) + D(n)C^{g}_{n}(t), \tag{A2}$$

where

$$A(n) = \frac{\omega_0}{2} + \omega_a(n-p) + \frac{U}{2}(n-p)(n-p-1) + \gamma(n-p), \tag{A3}$$

$$B(n) = g\sqrt{n!/(n-p)!}, \tag{A4}$$

$$D(n) = -\frac{\omega_0}{2} + \omega_a n - \frac{U}{2}n(n-1) - \gamma n. \tag{A5}$$

We make a Laplace transform of $C^e_{n-p}(t)$ and $C^g_n(t)$, i.e., $X(s) = L\left[C^e_{n-p}(t)\right], Y(s) = L\left[C^g_n(t)\right]$. Then, Equations (A1) and (A2) become the following expressions:

$$
\begin{aligned}
sX(s) - C^e_{n-p}(0) &= -iA(n)X(s) - iB(n)Y(s) && \text{(A6)}\\
sY(s) - C^g_n(0) &= -iB(n)X(s) - iD(n)Y(s) && \text{(A7)}
\end{aligned}
$$

From the initial condition, we have $C^e_{n-p}(0) = 0, C^g_n(0) = 1$, then we can obtain

$$
\begin{aligned}
X(s) &= \frac{-iB(n)}{s + iA(n)}Y(s) && \text{(A8)}\\
Y(s) &= \frac{s + iA(n)}{(s + iD(n))(s + iA(n)) + B(n)^2} && \text{(A9)}
\end{aligned}
$$

Since $(s + iC(n))(s + iA(n)) + B^2(n) = (s + ix_1)(s + ix_2)$, where

$$
\begin{aligned}
x_1 &= \frac{(A(n) + D(n)) - \sqrt{(A(n) - D(n))^2 + 4B^2(n)}}{2}, && \text{(A10)}\\
x_2 &= \frac{(A(n) + D(n)) + \sqrt{(A(n) - D(n))^2 + 4B^2(n)}}{2}, && \text{(A11)}
\end{aligned}
$$

Then

$$
\begin{aligned}
Y(s) &= \frac{s + iA(n)}{(s + ix_1)(s + ix_2)}\\
&= \frac{y_1}{s + ix_1} - \frac{y_2}{s + ix_2}\\
&= y_1\left[\frac{s}{s^2 + x_1^2} - i\frac{x_1}{s^2 + x_1^2}\right] - y_2\left[\frac{s}{s^2 + x_2^2} - i\frac{x_2}{s^2 + x_2^2}\right] && \text{(A12)}
\end{aligned}
$$

where $y_1 = \frac{A(n)-x_1}{x_2-x_1}, y_2 = \frac{A(n)-x_2}{x_2-x_1}$. Finally, we perform an inverse Laplace transform on $Y(s)$ to obtain

$$
C^g_n(t) = L^{-1}[Y(s)] = y_1\cos(x_1t) - y_2\cos(x_2t) - i[y_1\sin(x_1t) - y_2\sin(x_2t)] \qquad \text{(A13)}
$$

Similarly, we can obtain

$$
\begin{aligned}
X(s) &= \frac{-iB(n)}{s + iA(n)}Y(s)\\
&= \frac{-iB(n)y_1}{(s + iA(n))(s + ix_1)} + \frac{iB(n)y_2}{(s + iA(n))(s + ix_2)}\\
&= z_1\left(\frac{s}{s^2 + A^2(n)} - i\frac{A(n)}{s^2 + A^2(n)}\right) - z_1\left(\frac{s}{s^2 + x_1^2} - i\frac{x_1}{s^2 + x_1^2}\right)\\
&\quad + z_2\left(\frac{s}{s^2 + A^2(n)} - i\frac{A(n)}{s^2 + A^2(n)}\right) - z_2\left(\frac{s}{s^2 + x_2^2} - i\frac{x_2}{s^2 + x_2^2}\right) && \text{(A14)}
\end{aligned}
$$

where $z_1 = \frac{-B(n)y_1}{x_1-A(n)} = \frac{-B(n)}{x_1-A(n)}\frac{A(n)-x_1}{x_2-x_1} = \frac{B(n)}{x_2-x_1}, z_2 = \frac{B(n)}{x_2-A(n)}\frac{A(n)-x_2}{x_2-x_1} = \frac{-B(n)}{x_2-x_1} = -z_1$. Similarly, we perform an inverse Laplace transform on $X(s)$ to obtain

$$
\begin{aligned}
C^e_{n-p}(t) &= L^{-1}[X(s)] = z_1[\cos(A(n)t) - \cos(x_1t)] + z_2[\cos(A(n)t) - \cos(x_2t)]\\
&\quad -i(z_1[\sin(A(n)t) - \sin(x_1t)] + z_2[\sin(A(n)t) - \sin(x_2t)])\\
&= z_1[\cos(x_2t) - \cos(x_1t)] - i(z_1[\sin(x_2t) - \sin(x_1t)]) && \text{(A15)}
\end{aligned}
$$

Substituting Equations (A13) and (A15) into Equation (2), we obtain the quantum state of the total system at time t.

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