Synchronization of mutually delay-coupled nonidentical quantum cascade lasers: supplementary informations

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1 Derivation of the coupled Adler delay differential equations

We assume that both lasers are operating above threshold. The laser equations consist of six equations for the amplitudes and phases of the laser electrical fields, $E_j = R_j \exp(i\Phi_j)$ $(j = 1, 2)$, and the carrier densities N_1 and N_2 [1]. They are given by

$$
\begin{aligned}
R_1' &= N_1 R_1 + \varepsilon R_2(t-\tau)\cos(\theta + \Phi_2(t-\tau) - \Phi_1 - C), && (1)\\
\Phi_1' &= -\frac{\Delta}{2} + \alpha N_1 + \varepsilon \frac{R_2(t-\tau)}{R_1}\sin(\theta + \Phi_2(t-\tau) - \Phi_1 - C), && (2)\\
TN_1' &= P_1 - N_1 - (1+2N_1)R_1^2, && (3)\\
R_2' &= N_2 R_2 + \varepsilon R_1(t-\tau)\cos(\theta + \Phi_1(t-\tau) - \Phi_2 - C), && (4)\\
\Phi_2' &= \frac{\Delta}{2} + \alpha N_2 + \varepsilon \frac{R_1(t-\tau)}{R_2}\sin(\theta + \Phi_1(t-\tau) - \Phi_2 - C), && (5)\\
TN_2' &= P_2 - N_2 - (1+2N_2)R_2^2. && (6)
\end{aligned}
$$

We assume weak detunings and large delays and scale Δ and τ as

$$\Delta = \varepsilon \Delta_1 \text{ and } \tau = \varepsilon^{-1}\tau_1 \tag{7}$$

where Δ_1 and τ_1 are $O(1)$ quantities. If $\varepsilon = 0$, we recover the rate equations for two solitary lasers. If $P_j > 0$, they approach their steady state values $(R_j, N_j) = (\sqrt{P_j}, 0)$ on the t time scale. If $0 < \varepsilon << 1$, we note from Eq. (2) and (5) with $N_j = O(\varepsilon)$ that $\Phi_j' = O(\varepsilon)$. This means that the evolution of the phases is slow compared to R_j and N_j. It motivates to look for an asymptotic solution that depends on two distinct time scales, namely t and $s \equiv \varepsilon t$. Specifically, we seek a solution of the form

$$\begin{aligned} R_j(t,s,\varepsilon) &= \sqrt{P_j} + \varepsilon R_{j1}(t,s) + ..., & (8)\\ N_j(t,s,\varepsilon) &= \varepsilon N_{j1}(t,s) + ... & (9)\\ \Phi_j(t,s,\varepsilon) &= \Phi_{j0}(t,s) + \varepsilon \Phi_{j1}(t,s) + ... & (10)\end{aligned}$$

The assumption of two independent times requires the chain rule

$$\frac{dY}{dt} = Y_t + \varepsilon Y_s \tag{11}$$

where the subscripts t and s mean partial derivatives with respect to t and s. The leading order problem is $O(1)$ and provides an equation for Φ_{j0} given by

$$\Phi_{j0t} = 0 \ (j = 1, 2) \tag{12}$$

which implies that $\Phi_{j0} = \Phi_{j0}(s)$ is an unknown function of s. The next problem is $O(\varepsilon)$ and is $(j = 1, 2)$

$$R_{j1t} - N_{j1}\sqrt{P_j} = \sqrt{P_{3-j}} \cos(\theta + \Phi_{3-j0}(s - \tau_1) - \Phi_{j0}(s) - C) \tag{13}$$

$$TN_{j1t} + 2\sqrt{P_j} R_{j1} + (1 + 2P_j)N_{j1} = 0. \tag{14}$$

Because the right hand side of Eq. (13) is a slowly varying function of s, the solution of Eqs. (13) and (14) quickly approach their quasi steady state values on the t time scale:

$$\begin{aligned} N_{j1} &= -\sqrt{\frac{P_{3-j}}{P_j}} \cos(\theta + \Phi_{3-j0}(s - \tau_1) - \Phi_{j0}(s) - C), & (15)\\ R_{j1} &= -\frac{1}{2\sqrt{P_j}}(1 + 2P_j)N_{j1}. & (16)\end{aligned}$$

We next consider the equation for Φ_{j1} $(j = 1, 2)$

$$\begin{aligned} \Phi_{j1t} &= F_j \equiv \mp\frac{\Delta_1}{2} + \alpha N_{j1} \\ &\quad + \sqrt{\frac{P_{3-j}}{P_j}} \sin(\theta + \Phi_{3-j}(s - \tau_1) - \Phi_j(s) - C) - \Phi_{j0s}. & (17)\end{aligned}$$

The left hand side admits a constant as a nontrivial solution. In order to obtain a bounded solution for Φ_{j1}, the right hand side needs to satisfy the solvability condition ($j = 1, 2$)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t F_j(t,s)ds = 0. \tag{18}$$

This condition leads to two delay-coupled Adler equations given by

$$\begin{aligned} \frac{d\Phi_{10}}{ds} &= -\frac{\Delta_1}{2} - \alpha\sqrt{\frac{P_2}{P_1}}\cos(\theta + \Phi_{20}(s-\tau_1) - \Phi_{10}(s) - C) \\ &\quad + \sqrt{\frac{P_2}{P_1}}\sin(\theta + \Phi_{20}(s-\tau_1) - \Phi_{10}(s) - C), \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{d\Phi_{20}}{ds} &= \frac{\Delta_1}{2} - \alpha\sqrt{\frac{P_1}{P_2}}\cos(\theta + \Phi_{10}(s-\tau_1) - \Phi_{20}(s) - C) \\ &\quad + \sqrt{\frac{P_1}{P_2}}\sin(\theta + \Phi_{10}(s-\tau_1) - \Phi_{20}(s) - C). \end{aligned} \tag{20}$$

Combining the trigonometric functions and using the original variables and parameters, Eqs. (19)-(20) become

$$\frac{d\Phi_1}{dt} = -\frac{\Delta}{2} + \varepsilon\sqrt{\frac{P_2}{P_1}(1+\alpha^2)}\sin(\theta_0 + \Phi_2(t-\tau) - \Phi_1), \tag{21}$$

$$\frac{d\Phi_2}{dt} = \frac{\Delta}{2} + \varepsilon\sqrt{\frac{P_1}{P_2}(1+\alpha^2)}\sin(\theta_0 + \Phi_1(t-\tau) - \Phi_2). \tag{22}$$

where

$$\theta_0 \equiv \theta - C - \arctan(\alpha). \tag{23}$$

2 Compound Laser Modes if $P_2/P_1 << 1$

The compound laser modes (CLMs) are the solutions of Eqs. (21) and (22) of the form

$$\Phi_1 = \omega t, \ \ \Phi_2 = \omega t + \sigma. \tag{24}$$

Inserting (24) into Eqs. (21) and (22) leads to two equations for ω and σ given by

$$\omega = -\frac{\Delta}{2} + \varepsilon\sqrt{\frac{P_2}{P_1}(1+\alpha^2)}\sin(\theta_0 - \omega\tau + \sigma), \tag{25}$$

$$\omega = \frac{\Delta}{2} + \varepsilon\sqrt{\frac{P_1}{P_2}(1+\alpha^2)}\sin(\theta_0 - \omega\tau - \sigma). \tag{26}$$

Eliminating σ, the solution for $\omega = \omega(\Delta)$ is obtained by solving the following quadratic equation for $\Delta = \Delta(\omega)$

$$\frac{\Delta^2}{4}\left(F_-^2 C_1 + F_+^2 C_2\right) + \Delta\omega F_+ F_-(C_1 + C_2) + \omega^2(F_+^2 C_1 + F_-^2 C_2) - 4\varepsilon^2(1+\alpha^2) = 0 \tag{27}$$

where

$$F_{\pm} \equiv \sqrt{\frac{P_1}{P_2}} \pm \sqrt{\frac{P_2}{P_1}},\ C_1 \equiv \frac{1}{\sin^2(\theta_0 - \omega\tau)},\ \text{and } C_2 \equiv \frac{1}{\cos^2(\theta_0 - \omega\tau)}. \tag{28}$$

The condition for having real roots is provided by the discriminant

$$\omega^2 F_+^2 F_-^2 (C_1 + C_2)^2 - (F_-^2 C_1 + F_+^2 C_2) \begin{bmatrix} \omega^2 (F_+^2 C_1 + F_-^2 C_2) \\ -4\varepsilon^2 (1+\alpha^2) \end{bmatrix} \geq 0 \tag{29}$$

Eq. (29) sequentially simplifies as

$$\begin{aligned}
\left\{ \begin{array}{c} \omega^2 \left[\begin{array}{c} F_+^2 F_-^2 (C_1^2 + C_2^2 + 2C_1C_2) - F_-^2 F_+^2 C_1^2 \\ -F_-^4 C_1 C_2 - F_+^4 C_2 C_1 - F_-^2 F_+^2 C_2^2 \end{array} \right] \\ +4\varepsilon^2(1+\alpha^2)\left(F_-^2 C_1 + F_+^2 C_2\right) \end{array} \right\} &\geq 0, \\
\left\{ \begin{array}{c} \omega^2 \left[F_+^2 F_-^2 (2C_1C_2) - F_-^4 C_1 C_2 - F_+^4 C_2 C_1 \right] \\ +4\varepsilon^2(1+\alpha^2)\left(F_-^2 C_1 + F_+^2 C_2\right) \end{array} \right\} &\geq 0, \\
-C_1 C_2 \omega^2 \left[F_+^2 - F_-^2 \right]^2 + 4\varepsilon^2 (1+\alpha^2) \left(F_-^2 C_1 + F_+^2 C_2\right) &\geq 0, \\
-16 C_1 C_2 \omega^2 + 4\varepsilon^2 (1+\alpha^2) \left(F_-^2 C_1 + F_+^2 C_2\right) &\geq 0.
\end{aligned} \tag{30}$$

Introducing

$$x \equiv P_2/P_1 \tag{31}$$

into Eq. (30), we find

$$-16 C_1 C_2 \omega^2 + 4\varepsilon^2(1+\alpha^2)\left((\frac{1}{x} - 2 + x) C_1 + (\frac{1}{x} + 2 + x) C_2 \right) \geq 0, \tag{32}$$

or equivalently, the following quadratic equation for x

$$\begin{aligned}
x^2 + x \left[\frac{4\varepsilon^2(1+\alpha^2) 2 (C_2 - C_1) - 16 C_1 C_2 \omega^2}{4\varepsilon^2 (1+\alpha^2)(C_1 + C_2)} \right] + 1 &\geq 0, \\
x^2 + x \left[\frac{2(C_2 - C_1)}{C_1 + C_2} - \frac{16 C_1 C_2 \omega^2}{4\varepsilon^2(1+\alpha^2)(C_1+C_2)} \right] + 1 &\geq 0, \\
x^2 + x \left[2(\sin^2(\theta_0 - \omega\tau)) - \cos^2(\theta_0 - \omega\tau)) - \frac{4\omega^2}{\varepsilon^2(1+\alpha^2)} \right] + 1 &\geq 0, \\
x^2 + x \left[-2\cos(2(\theta_0 - \omega\tau)) - \frac{4\omega^2}{\varepsilon^2(1+\alpha^2)} \right] + 1 &\geq 0.
\end{aligned} \tag{33}$$

By gradually changing $\omega\tau$ from $\omega\tau = -1.5$ to $\omega\tau = 1.5$, we determine the real roots of the quadratic equation

$$x^2 + x \left[-2\cos(2(\theta_0 - \omega\tau)) - \frac{4\omega^2}{\varepsilon^2(1+\alpha^2)} \right] + 1 = 0. \tag{34}$$

They delimit the domain of real solutions for $\Delta = \Delta(\omega)$. From Eq. (34) and assuming $\omega^2 = O(x^{-1})$, we find the limit

$$\omega_{\pm}\tau \rightarrow \pm \frac{\varepsilon\tau}{2} \sqrt{(1+\alpha^2)\frac{P_1}{P_2}} \text{ as } x \rightarrow 0. \tag{35}$$

The corresponding values of Δ are given by

$$\Delta_{\pm} = -2\omega_{\pm}. \tag{36}$$

Solving the quadratic equation (27) and then taking the limit $P_2/P_1 \to 0$ leads to

$$\Delta \to -2\omega \pm \frac{4P_2}{P_1}\sqrt{\frac{4}{C_1 C_2}\left(\omega_+ - \omega\right)\left(\omega - \omega_-\right)} \tag{37}$$

which implies that the CLM frequency is

$$\omega = -\frac{\Delta}{2} \quad \left(2\omega_- \leq \Delta \leq 2\omega_+\right) \tag{38}$$

in first approximation.

3 Stability of the CLMs if $P_2/P_1 << 1$

From Eqs. (21) and (22), we determine the linearized equations for the CLMs (24). By considering small perturbations proportional to $\exp(\lambda t)$, we find that the growth rate λ satisfies the characteristic equation

$$\left\{ \begin{array}{c} \lambda^2 + \lambda\varepsilon\sqrt{(1+\alpha^2)} \left[\begin{array}{c} \sqrt{\frac{P_2}{P_1}}\cos(\theta_0 - \omega\tau + \sigma) \\ +\sqrt{\frac{P_1}{P_2}}\cos(\theta_0 - \omega\tau - \sigma) \end{array} \right] \\ + \left[\begin{array}{c} \varepsilon^2(1+\alpha^2)\cos(\theta_0 - \omega\tau + \sigma) \\ \times\cos(\theta_0 - \omega\tau - \sigma)\left(1 - \exp(-2\lambda\tau)\right) \end{array} \right] \end{array} \right\} = 0 \tag{39}$$

We determine the solution of Eq. (39) in the limit $P_2/P_1 \to 0$. We know from Eq. (35) that the frequency ω scales like $\varepsilon\sqrt{P_1/P_2}$. It motivates to scale λ as

$$\lambda = \varepsilon\sqrt{\frac{P_1}{P_2}}\Lambda. \tag{40}$$

In terms of Λ, Eq. (39) can be rewritten as

$$\left\{ \begin{array}{c} \varepsilon^2\frac{P_1}{P_2}\Lambda^2 + \varepsilon^2\frac{P_1}{P_2}\Lambda\sqrt{(1+\alpha^2)} \left[\begin{array}{c} \frac{P_2}{P_1}\cos(\theta_0 - \omega\tau + \sigma) \\ +\cos(\theta_0 - \omega\tau - \sigma) \end{array} \right] \\ + \left[\begin{array}{c} \varepsilon^2(1+\alpha^2)\cos(\theta_0 - \omega\tau + \sigma) \\ \times\cos(\theta_0 - \omega\tau - \sigma)\left(1 - \exp(-2\varepsilon\sqrt{\frac{P_1}{P_2}}\Lambda\tau)\right) \end{array} \right] \end{array} \right\} = 0 \tag{41}$$

In the limit $P_2/P_1 \to 0$, the two first terms of Eq. (41) dominate and we obtain

$$\Lambda = -\sqrt{(1+\alpha^2)}\left[\cos(\theta_0 - \omega\tau - \sigma)\right], \tag{42}$$

in first approximation. Stability (instability) now means the condition

$$\cos(\theta_0 - \omega\tau - \sigma) > 0 \ (\cos(\theta_0 - \omega\tau - \sigma) < 0). \tag{43}$$

The condition

$$\cos(\theta_0 - \omega\tau - \sigma) = 0 \tag{44}$$

corresponds to $\Lambda = 0$ and characterizes the limit points of the CLM orbits. We next wish to relate this condition to the bifurcation diagram of the CLMs. Taking the derivative of Eqs. (25) and (26) with respect to Δ, we find

$$\begin{aligned}
\frac{d\omega}{d\Delta} &= -\frac{1}{2} + \sqrt{\frac{P_2}{P_1}} H_1(-\tau\frac{d\omega}{d\Delta} + \frac{d\sigma}{d\Delta}), && (45)\\
\frac{d\omega}{d\Delta} &= \frac{1}{2} + \sqrt{\frac{P_1}{P_2}} H_2(-\tau\frac{d\omega}{d\Delta} - \frac{d\sigma}{d\Delta}). && (46)
\end{aligned}$$

where

$$\begin{aligned}
H_1 &\equiv \varepsilon\sqrt{\frac{P_2}{P_1}(1+\alpha^2)}\cos(\theta_0 - \omega\tau + \sigma,) && (47)\\
H_2 &\equiv \varepsilon\sqrt{\frac{P_1}{P_2}(1+\alpha^2)}\cos(\theta_0 - \omega\tau - \sigma). && (48)
\end{aligned}$$

Multiplying Eqs. (45) and (46) by $d\Delta/d\omega$ and reorganizing lead to two linear equations for $d\Delta/d\omega$ and $d\sigma/d\omega$

$$\begin{aligned}
\frac{1}{2}\frac{d\Delta}{d\omega} &= -1 + H_1(-\tau + \frac{d\sigma}{d\omega}), && (49)\\
\frac{1}{2}\frac{d\Delta}{d\omega} &= 1 - H_2(-\tau - \frac{d\sigma}{d\omega}). && (50)
\end{aligned}$$

Solving those equations, we obtain for $d\Delta/d\omega$

$$\frac{d\Delta}{d\omega} = \frac{2}{H_1 - H_2}\left[H_1 + H_2 + 2\varepsilon\tau H_1 H_2\right] \tag{51}$$

Eq. (51) is valid for arbitrary values of the pump parameters. It provide the slope of the CLM curve $\omega = \omega(\Delta)$. A limit point (or saddle-node bifurcation point) of the CLM orbit satisfies the condition $d\Delta/d\omega = 0$. In order to relate this condition to Eq. (44), we analyze Eq. (51) in the limit $P_2/P_1 \to 0$. In this limit, $H_2 \sim \sqrt{P1/P2} \to \infty$ and (51) simplifies as

$$\frac{d\Delta}{d\omega} = -2. \tag{52}$$

But a different limit is however possible if

$$\cos(\theta_0 - \omega\tau - \sigma) = O(\frac{P_2}{P_1}) \to 0 \text{ as } P_2/P_1 \to 0. \tag{53}$$

In this limit, both H_1 and H_2 are $O(\sqrt{P_2/P_1})$ small quantities and (51) simplifies as

$$\frac{d\Delta}{d\omega} = \frac{2}{H_1 - H_2}(H_1 + H_2) \tag{54}$$

The condition $d\Delta/d\omega = 0$ now implies $H_1 + H_2 = 0$. The necessary condition is however (53) which clearly matches (44) in first approximation.

4 The critical case if $P_2/P_1 \to 0$

We now consider the case $P_1 = O(1)$ and $|P_2|$ small. Physically, we expect that Laser 1 acts as a Master laser injecting its signal into Laser 2 being the Slave laser. In this section, we first show that there is a critical scaling between P_2 and ε for which our previous asymptotic solution assuming $P_j = O(1)$ ($j = 1, 2$) becomes invalid. After evaluating the critical scalings of all variables and parameters with respect to ε, we plan to develop a new asymptotic theory valid in the limit $\varepsilon \to 0$.

Inserting (16) into (8) for $j = 2$, we find that the long time solution for R_2 is given by

$$R_2 = P_2^{1/2} + \varepsilon \frac{(1+2P_2)P_1^{1/2}}{2P_2} \cos(\theta + \Phi_{10}(s-\tau_1) - \Phi_{20}(s) - C). \tag{55}$$

If $|P_2| \to 0$, the two terms perturbation expansion becomes non uniform as soon as

$$P_2^{3/2} \sim \varepsilon, \tag{56}$$

or equivalently, if

$$P_2 \sim \varepsilon^{2/3}. \tag{57}$$

With (57), Eq. (55) then indicates that

$$R_2 \sim P_2^{1/2} \sim \varepsilon^{1/3}. \tag{58}$$

Using (15) and (9), we determine the scalings for the N_j as

$$N_1 \sim \varepsilon\sqrt{P_2} \sim \varepsilon^{4/3}, \;\; N_2 \sim \frac{\varepsilon}{\sqrt{P_2}} \sim \varepsilon^{2/3}. \tag{59}$$

Last, we learn from (35) and (36) that ω and Δ follow the scalings

$$\omega \sim \varepsilon^{2/3} \text{ and } \Delta \sim \varepsilon^{2/3}. \tag{60}$$

We are now ready for a new asymptotic analysis. We first define the new parameters p_2, and Δ_2 as

$$P_2 = \varepsilon^{2/3} p_2, \;\; \Delta = \varepsilon^{2/3}\Delta_2, \tag{61}$$

and introduce the new variables r_1, r_2, n_1, and n_2 defined by

$$R_1 = \sqrt{P_1} + \varepsilon^{4/3} r_1, \;\; R_2 = \varepsilon^{1/3} r_2, \;\; N_1 = \varepsilon^{4/3} n_1, \text{ and } N_2 = \varepsilon^{2/3} n_2 \tag{62}$$

Inserting (61) and (62) into Eqs. (1)-(6), we obtain

$$r_1' = \left[n_1\sqrt{P_1} + r_2(t-\tau)\cos(\theta + \Phi_2(t-\tau) - \Phi_1 - C)\right] + ... \tag{63}$$

$$\Phi_1' = \varepsilon^{2/3}(-\frac{\Delta_2}{2}) + \varepsilon^{4/3}\left[\begin{array}{c} \alpha n_1 \\ +\frac{r_2(t-\tau)}{\sqrt{P_1}}\sin(\theta + \Phi_2(t-\tau) - \Phi_1 - C) \end{array}\right] + ..., \tag{64}$$

$$Tn_1' = -2\sqrt{P_1}r_1 - n_1(1+2P_1) + ..., \tag{65}$$

$$r_2' = \varepsilon^{2/3}\left[n_2 r_2 + \sqrt{P_1}\cos(\theta + \Phi_1(t-\tau) - \Phi_2 - C),\right] + ..., \tag{66}$$

$$\Phi_2' = \varepsilon^{2/3}\left[\frac{\Delta_2}{2} + \alpha n_2 + \frac{\sqrt{P_1}}{r_2}\sin(\theta + \Phi_1(t-\tau) - \Phi_2 - C)\right] + ..., \tag{67}$$

$$Tn_2' = p_2 - n_2 - r_2^2 + ... \tag{68}$$

We now seek a two time solution of the form

$$r_1 = r_{10}(t,s) + \varepsilon^{2/3} r_{11}(t,s) + ... \tag{69}$$

$$r_2 = r_{20}(t,s) + \varepsilon^{2/3} r_{21}(t,s) + ... \tag{70}$$

$$n_1 = n_{10}(t,s) + \varepsilon^{2/3} n_{11}(t,s) + ... \tag{71}$$

$$n_2 = n_{20}(t,s) + \varepsilon^{2/3} n_{21}(t,s) + ... \tag{72}$$

$$\Phi_j = \Phi_{j0}(t,s) + \varepsilon^{2/3}\Phi_{j1}(t,s) + ... \tag{73}$$

where $s \equiv \varepsilon^{2/3}t$ is defined as a new slow time variable. This variable is motived by the right hand sides of Eqs. (64), (66) and (67) indicating that r_2 and the phases Φ_j are, in first approximation, functions of s.

We now proceed as in Section 1. The leading order problem is $O(1)$ and is given by

$$r_{10t} = n_{10}\sqrt{P_1} + r_{20}(t-\tau)\cos(\theta + \Phi_{20}(t-\tau) - \Phi_{10} - C), \tag{74}$$

$$\Phi_{10t} = r_{20t} = \Phi_{20t} = 0 \tag{75}$$

$$Tn_{10t} = -2\sqrt{P_1}r_{10} - n_{10}(1+2P_1), \tag{76}$$

$$Tn_{20t} = p_2 - n_{20} - r_{20}^2. \tag{77}$$

Eq. (75) tells us that Φ_{10}, r_{20}, and Φ_{20} are all three functions of the slow time s. Recall that the delay τ was scaled like $\tau = \varepsilon^{-1}\tau_1$ in Section 1. Eq. (74) then becomes

$$r_{10t} = n_{10}\sqrt{P_1} + r_{20}(s - \varepsilon^{-1/3}\tau_1)\cos(\theta + \Phi_{20}(s - \varepsilon^{-1/3}\tau_1) - \Phi_{10}(s) - C). \tag{78}$$

Eqs. (78) and (76) are two coupled equations for r_1 and n_1. Because the second term in the right hand side of Eq. (78) is slowly varying, r_1 and n_1 quickly approach their quasi steady state values on the t time scale. They are

$$n_{10} = -\frac{r_{20}(s - \varepsilon^{-1/3}\tau_1)}{\sqrt{P_1}}\cos(\theta + \Phi_{20}(s - \varepsilon^{-1/3}\tau_1) - \Phi_{10}(s) - C), \tag{79}$$

$$r_{10} = -\frac{n_{10}(1+2P_1)}{2\sqrt{P_1}}. \tag{80}$$

Similarly, r_{20}^2 in the right hand side of Eq. (77) is a slowly varying function of s and n_{20} is quickly converging to its quasi steady state on the t time scale. The latter is given by

$$n_{20} = p_2 - r_{20}^2. \tag{81}$$

The next problem for Φ_{11}, r_{21}, and Φ_{21} is $O(\varepsilon^{2/3})$ and is given by

$$\begin{aligned}
\Phi_{11t} &= -\Phi_{10s} - \frac{\Delta_2}{2}, && (82)\\
r_{21t} &= -r_{20s} + n_2 r_2 + \sqrt{P_1}\cos(\theta + \Phi_1(s - \varepsilon^{-1/3}\tau_1) - \Phi_2(s) - C), && (83)\\
\Phi_{21t} &= -\Phi_{20s} + \frac{\Delta_2}{2} + \alpha n_2 + \frac{\sqrt{P_1}}{r_2}\sin\left(\begin{array}{c}(\theta + \Phi_1(s - \varepsilon^{-1/3}\tau_1)\\ -\Phi_2(s) - C\end{array}\right). && (84)
\end{aligned}$$

Solvability of these equations requires the three conditions

$$\begin{aligned}
\frac{d\Phi_{10}}{ds} &= -\frac{\Delta_2}{2}, && (85)\\
\frac{dr_{20}}{ds} &= n_2 r_2 + \sqrt{P_1}\cos(\theta + \Phi_1(s - \varepsilon^{-1/3}\tau_1) - \Phi_2(s) - C), && (86)\\
\frac{d\Phi_{20}}{ds} &= \frac{\Delta_2}{2} + \alpha n_2 + \frac{\sqrt{P_1}}{r_2}\sin(\theta + \Phi_1(s - \varepsilon^{-1/3}\tau_1) - \Phi_2(s) - C). && (87)
\end{aligned}$$

The solution of Eq. (85) with $\Phi_{10}(0) = 0$ is

$$\Phi_{10} = -\frac{\Delta_2}{2}s. \tag{88}$$

With (88), Eqs. (86) and (87) become

$$\begin{aligned}
\frac{dr_{20}}{ds} &= n_2 r_2 + \sqrt{P_1}\cos(\theta_1 - \overline{\Phi}_{20}) && (89)\\
\frac{d\overline{\Phi}_{20}}{ds} &= \Delta_2 + \alpha n_2 + \frac{\sqrt{P_1}}{r_2}\sin(\theta_1 - \overline{\Phi}_{20}) && (90)
\end{aligned}$$

where

$$\begin{aligned}
\theta_1 &\equiv \theta + \frac{\Delta_2}{2}\varepsilon^{-1/3}\tau_1 - C && (91)\\
\overline{\Phi}_{20} &\equiv \Phi_{20} + \frac{\Delta_2}{2}s && (92)
\end{aligned}$$

Together with (81), Eqs. (89) and (90) are the equations of a laser subject to an injected signal.

We now verify that the steady state solutions of Eqs. (89) and (90) corresponds to the CLMs, previously defined by (24). The steady state solutions of Eqs. (89) and (90) corresponds to r_2 constant and $\overline{\Phi}_{20} = \sigma$ constant. Using (92), we determine Φ_{20} as

$$\Phi_{20} = -\frac{\Delta_2}{2}s + \sigma = -\frac{\Delta t}{2} + \sigma. \tag{93}$$

The original phase Φ_{20} exhibits the frequency

$$\omega = -\frac{\Delta}{2} \tag{94}$$

which is the CLM frequency as $P_2/P_1 \to 0$ given by (38).

References

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