Supplementary Material
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Title: **Higher-Order Interactions in Quantum Optomechanics: Analytical Solution of Nonlinearity**
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In this Supplementary Material, we show by numerical solution of a stochastic nonlinear operator differential equation, and also taking its expectation values within the Mean Field Approximation, that the proposed analytical scheme in the above referenced manuscript works very well, is convergent, and uniformly converges to the accurate solution.

First, consider the infinitely-ordered nonlinear operator equation

$$\tau\frac{d}{dt}\hat{u}(t)=-\mu\hat{u}(t)-\kappa\left[e^{\hat{u}(t)}-1\right]+v(t)-\hat{n}(t),\tag{S1}$$

which models the voltage operator of an RC circuit shunted by a nonlinear ideal diode, driven by a sinusoidal voltage source $v(t)=V_0e^{-\alpha t}\sin(\omega t)$, and stochastic noise $\hat{n}(t)$. We suppose that the noise $\hat{n}(t)$ is governed by a Weiner process. Here, and without loss of generality, both κ and μ are taken to be positive real parameters. Hence, this model does not include an oscillating part due to an imaginary μ, which could have been otherwise absorbed into $\hat{u}(t)$ by a rotating frame transformation. This particular choice also eliminates the imaginary part of $\hat{u}(t)$. We also assume here, for the illustrative purpose of this example, that $\mu=V_0=\tau=1$, $\omega=2\pi\alpha$, and $\omega=2\pi\times1\mathrm{kHz}$.

Using the proposed method in the paper under consideration, this above operator equation can be first put into the infinitely-ordered linear system of ordinary differential equations as

$$\tau\frac{d}{dt}\begin{Bmatrix}\hat{u}(t)\\\hat{u}^2(t)\\\hat{u}^3(t)\\\hat{u}^4(t)\\\hat{u}^5(t)\\\vdots\end{Bmatrix}=-\begin{bmatrix}\kappa+1 & \frac{\kappa}{2!} & \frac{\kappa}{3!} & \frac{\kappa}{4!} & \frac{\kappa}{5!} & \cdots\\ 0 & 2(\kappa+1) & \frac{2\kappa}{2!} & \frac{2\kappa}{3!} & \frac{2\kappa}{4!} & \cdots\\ 0 & 0 & 3(\kappa+1) & \frac{3\kappa}{2!} & \frac{3\kappa}{3!} & \cdots\\ 0 & 0 & 0 & 4(\kappa+1) & \frac{4\kappa}{2!} & \cdots\\ 0 & 0 & 0 & 0 & 5(\kappa+1) & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}\begin{Bmatrix}\hat{u}(t)\\\hat{u}^2(t)\\\hat{u}^3(t)\\\hat{u}^4(t)\\\hat{u}^5(t)\\\vdots\end{Bmatrix}+\begin{Bmatrix}v(t)\\2\hat{u}(t)v(t)\\3\hat{u}^2(t)v(t)\\4\hat{u}^3(t)v(t)\\5\hat{u}^4(t)v(t)\\\vdots\end{Bmatrix}-\begin{Bmatrix}\hat{n}(t)\\2\hat{u}(t)\hat{n}(t)\\3\hat{u}^2(t)\hat{n}(t)\\4\hat{u}^3(t)\hat{n}(t)\\5\hat{u}^4(t)\hat{n}(t)\\\vdots\end{Bmatrix}.\tag{S2}$$

Subsequently, the input terms can be linearized using the proposed method in the paper under consideration. Doing this results in

$$\tau \frac{d}{dt}\begin{Bmatrix} \hat{u}(t) \\ \hat{u}^2(t) \\ \hat{u}^3(t) \\ \hat{u}^4(t) \\ \hat{u}^5(t) \\ \vdots \end{Bmatrix} = -\begin{bmatrix} \kappa+1 & \frac{\kappa}{2!} & \frac{\kappa}{3!} & \frac{\kappa}{4!} & \frac{\kappa}{5!} & \cdots \\ 0 & 2(\kappa+1) & \frac{2\kappa}{2!} & \frac{2\kappa}{3!} & \frac{2\kappa}{4!} & \cdots \\ 0 & 0 & 3(\kappa+1) & \frac{3\kappa}{2!} & \frac{3\kappa}{3!} & \cdots \\ 0 & 0 & 0 & 4(\kappa+1) & \frac{4\kappa}{2!} & \cdots \\ 0 & 0 & 0 & 0 & 5(\kappa+1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{Bmatrix} \hat{u}(t) \\ \hat{u}^2(t) \\ \hat{u}^3(t) \\ \hat{u}^4(t) \\ \hat{u}^5(t) \\ \vdots \end{Bmatrix} + \begin{Bmatrix} v(t) \\ 2\bar{u}v(t) \\ 3\bar{u}^2 v(t) \\ 4\bar{u}^3 v(t) \\ 5\bar{u}^4 v(t) \\ \vdots \end{Bmatrix} - \begin{Bmatrix} \hat{n}(t) \\ 2\bar{u}\hat{n}(t) \\ 3\bar{u}^2 \hat{n}(t) \\ 4\bar{u}^3 \hat{n}(t) \\ 5\bar{u}^4 \hat{n}(t) \\ \vdots \end{Bmatrix}, \tag{S3}$$

in which $\bar{u} = \frac{1}{T}\int_0^T \langle \hat{u}(t) \rangle dt$ is the time-average of the input. Now, the above system of equations can be exactly integrated, after truncation to a finite-order.

Using an extensive code written in Mathematica, the above system of linear stochastic equations can be treated and integrated as an Itô process, and the results for various orders of truncation between 2 and 6 versus the numerically exact solution are displayed below

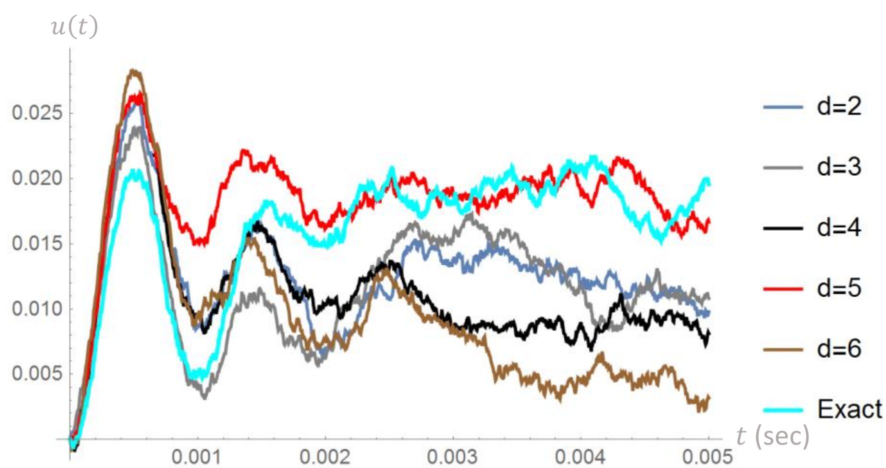

Fig. S1. The stochastic solution function $u(t) = \langle \hat{u}(t) \rangle$ versus time given in various orders of approximation.

It is still not quite clear that the method is convergent to the exact solution, since the Itô integration of a Weiner process every time is carried over a different sequence of random numbers. This difficulty cannot be avoided in principle, since there is no way to reset the numerically random sequence.

Therefore, as a double check, we take the expectation values, which discards the noise term, and transform a mean-field approximation to reach a similar system of differential equations, however, expressed in terms of the expectation value function $\langle\hat{u}(t)\rangle$ and its higher orders. This is equivalent to solving the nonlinear differential equation

$$\tau\frac{d}{dt}\langle\hat{u}(t)\rangle = -\mu\langle\hat{u}(t)\rangle - \kappa\left[e^{\langle\hat{u}(t)\rangle} - 1\right] + v(t), \tag{S4}$$

given the fact that $\langle\hat{n}(t)\rangle = 0$.

Doing this immediately reveals the convergence property of our proposed method, illustrated in the next figure:

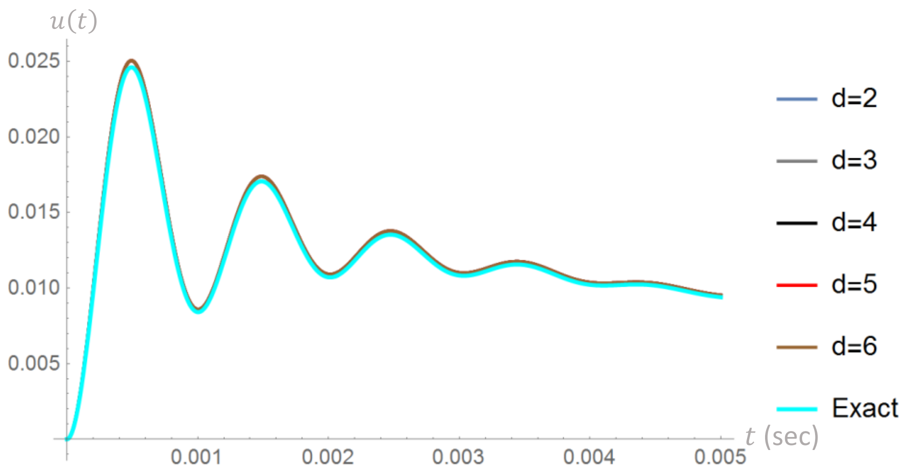

Fig. S2. Expectation value function $\langle\hat{u}(t)\rangle$ versus time given in various orders of approximation. Convergence to the exact solution obtained from numerical solution of (S4) is rapid by increasing order.

As it can be clearly verified, the numerical solutions are so rapidly and accurately converging to the exact solution, that they are practically indistinguishable beyond the two lowest truncation orders.

Mathematica packages developed for this problem can be supplied by direct request from the author.