Communication

# Statistical Mixture of Kaleidoscope States Interacting with a Two-Level Atom: Entropy and Purification 

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#### Abstract

We investigate some of the fundamental features of the interaction of a mixture of coherent states, namely, a Kaleidoscope states mixture, with two-level atoms in the Jaynes-Cummings model framework. We begin our analysis by calculating the von Neumann entropy of the field, which is determined with the help of the virtual atom method. The oscillations appearing in the entropy indicate a state of purity greater than the initial state, i.e., a purification of the initial state due to a transfer of coherence from the atom to the field. In this oscillatory region, we obtain a negative Wigner function that hints at a (noisy) multiple Schrödinger cat.


Keywords: Kaleidoscope-states; mixed-states; virtual-atom

## 1. Introduction

The relation between classical and quantum phenomena has attracted attention over the years, becoming a very discussed problem in quantum mechanics. The main problem is that what causes, in the macroscopic world, the quantum interference of superposition states and the $n$-qubit entanglement states may not be observed [1-3].

Schrödinger cat [4-6] states, or superpositions of coherent states, have attracted the attention of researchers due to their fundamental features. Of particular interest is the case of the superposition of two (or more) coherent states [7,8], where, because of quantum interference, their properties are very different from the properties of the constituent coherent states, as well as from the incoherent superposition or statistical mixtures of such states. For example, the superposition exhibits sub-Poissonian photon statistics, higherorder squeezing, and oscillations in the photon number distribution [8]. These properties clearly differentiate the state of the superposition and statistical mixture of two coherent states [7]. Because superpositions of macroscopically distinguishable states (or Schrödinger cat-like states) may be produced by using coherent states, the problem is important for the quantum theory of measurement [9]. Several schemes have already been proposed to produce a superposition of coherent states, for instance the non-linear interaction of the field in a coherent state with a Kerr-like medium can produce their superposition [4]. Another possible way would be through the interaction between quantized fields, initially prepared in coherent states, with two-level atoms [7,8] or ion laser interactions [9].

On the other hand, entropy [9-11] is one of the main tools for measuring entanglement [2,12]. Precisely, one of the main tasks in the present manuscript is to calculate the entropy of the field for a mixture of Kaleidoscope states or a statistical mixture of coherent states, in the Jaynes-Cummings framework, which we will do with the aid of the ArakiLieb inequality [13]. In principle, it seems impossible to use the Araki-Lieb inequality to calculate the field entropy because the field is in a statistical mixture of states [14]. However, via purification of the mixed density matrix of the quantized field [15], we will be able to
use that inequality in order to calculate the field von Neumann entropy, either for the atom or the field.

In the next section, we define the Kaleidoscope states that we will consider as the initial state in their interaction with a two-level atom. Section 3 deals precisely with this interaction and, there, we calculate the field entropies and their Wigner functions [16], where we show that the statistical mixture of coherent states, i.e., the Kaleidoscope states mixture, goes to a minimum at half the revival time, independently of its number of components. Finally, in Section 4, we summarize our conclusions.

## 2. Kaleidoscope States

Kaleidoscope states are a particular superposition of $n$ coherent states, and are defined in reference [17], as:

$$
\left(\begin{array}{c}
\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle  \tag{1}\\
\sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle \\
\sqrt{\lambda_{3 \alpha}}\left|\psi_{3 \alpha}\right\rangle \\
\vdots \\
\sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle
\end{array}\right)=\frac{1}{n}\left(\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega^{*} & \omega^{* 2} & \cdots & \omega^{*(n-1)} \\
1 & \omega^{* 2} & \omega^{* 4} & \cdots & \omega^{* 2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{*(n-1)} & \omega^{* 2(n-1)} & \cdots & \omega^{*(n-1)(n-1)}
\end{array}\right)\left(\begin{array}{c}
|\alpha\rangle \\
|\omega \alpha\rangle \\
\left|\omega^{2} \alpha\right\rangle \\
\vdots \\
\left|\omega^{(n-1)} \alpha\right\rangle
\end{array}\right)
$$

where $\sqrt{\lambda_{k \alpha}}$ are normalization constants and $\omega=\exp \left(i \frac{2 \pi}{n}\right)$ with $n, k$ integers which satisfy $1 \leq k \leq n$.

We can observe that the Vandermonde matrix that transforms the vectors in Equation (1) is proportional to the so-called discrete Fourier transform, also known as the quantum Fourier transform [18].

On the other hand, an initial statistical mixture of $n$ coherent states may be written as

$$
\begin{equation*}
\hat{\rho}_{F}(0)=\frac{1}{n} \sum_{k=1}^{n}\left|\omega^{(k-1)} \alpha\right\rangle\left\langle\omega^{(k-1)} \alpha\right| . \tag{2}
\end{equation*}
$$

This density matrix may be diagonalized by the virtual atom method [15], where the important issue is to establish a connection between this density and the virtual $(V)$ density operator $\hat{\rho}_{V}=\left|\psi_{V}\right\rangle\left\langle\psi_{V}\right|$, which is calculated from the pure state

$$
\begin{equation*}
\left|\psi_{V}\right\rangle=\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle\left|A_{1}\right\rangle+\sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle\left|A_{2}\right\rangle+\sqrt{\lambda_{3 \alpha}}\left|\psi_{3 \alpha}\right\rangle\left|A_{3}\right\rangle+\cdots+\sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle\left|A_{n}\right\rangle, \tag{3}
\end{equation*}
$$

where $\left\{\left|A_{k}\right\rangle\right\}$ is the virtual atom basis and $\left|\psi_{k \alpha}\right\rangle$ are the Kaleidoscope states, whenever $k=1, . ., n$. After tracing the density operator $\hat{\rho}_{V}=\left|\psi_{V}\right\rangle\left\langle\psi_{V}\right|$, over the virtual atom states $\left\{\left|A_{k}\right\rangle\right\}$, we obtain the block diagonal density matrix

$$
\begin{equation*}
\hat{\rho}_{F}(0)=\sum_{k=1}^{n} \lambda_{k \alpha}\left|\psi_{k \alpha}\right\rangle\left\langle\psi_{k \alpha}\right| \tag{4}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{k \alpha}=\frac{\mathrm{e}^{-|\alpha|^{2}}}{n} \sum_{m=1}^{n} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right) \tag{5}
\end{equation*}
$$

Similarly, by tracing the density $\hat{\rho}_{V}=\left|\psi_{V}\right\rangle\left\langle\psi_{V}\right|$ over the field basis, we obtain the virtual atom (VA) density matrix

$$
\hat{\rho}_{V A}=\frac{1}{n}\left(\begin{array}{ccccc}
1 & \langle\alpha \mid \omega \alpha\rangle^{*} & \left\langle\alpha \mid \omega^{2} \alpha\right\rangle^{*} & \cdots & \left\langle\alpha \mid \omega^{(n-1)} \alpha\right\rangle^{*}  \tag{6}\\
\langle\alpha \mid \omega \alpha\rangle & 1 & \langle\alpha \mid \omega \alpha\rangle^{*} & \cdots & \left\langle\alpha \mid \omega^{(n-2)} \alpha\right\rangle^{*} \\
\left\langle\alpha \mid \omega^{2} \alpha\right\rangle & \langle\alpha \mid \omega \alpha\rangle & 1 & \cdots & \left\langle\alpha \mid \omega^{(n-3)} \alpha\right\rangle^{*} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left\langle\alpha \mid \omega^{(n-1)} \alpha\right\rangle & \left\langle\alpha \mid \omega^{(n-2)} \alpha\right\rangle & \left\langle\alpha \mid \omega^{(n-3)} \alpha\right\rangle & \cdots & 1
\end{array}\right),
$$

and, taking into account that $\left\langle\alpha \mid \omega^{k} \alpha\right\rangle=\left\langle\alpha \mid \omega^{(n-k)} \alpha\right\rangle^{*}$ and $\left\langle\alpha \mid \omega^{k} \alpha\right\rangle=\exp \left(-|\alpha|^{2}\right) \exp \left(\omega^{k}|\alpha|^{2}\right)$, the Equation (6) may be rewritten as:

$$
\hat{\rho}_{V A}=\frac{\exp \left(-|\alpha|^{2}\right)}{n}\left(\begin{array}{ccccc}
\exp \left(|\alpha|^{2}\right) & \exp \left(\omega^{*}|\alpha|^{2}\right) & \exp \left(\omega^{* 2}|\alpha|^{2}\right) & \cdots & \exp \left(\omega^{*(n-1)}|\alpha|^{2}\right)  \tag{7}\\
\exp \left(\omega^{*(n-1)}|\alpha|^{2}\right) & \exp \left(|\alpha|^{2}\right) & \exp \left(\omega^{*}|\alpha|^{2}\right) & \cdots & \exp \left(\omega^{*(n-2)}|\alpha|^{2}\right) \\
\exp \left(\omega^{*(n-2)}|\alpha|^{2}\right) & \exp \left(\omega^{*(n-1)}|\alpha|^{2}\right) & \exp \left(|\alpha|^{2}\right) & \cdots & \exp \left(\omega^{\left.*(n-3)|\alpha|^{2}\right)}\right. \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\exp \left(\omega^{*}|\alpha|^{2}\right) & \exp \left(\omega^{* 2}|\alpha|^{2}\right) & \exp \left(\omega^{* 3}|\alpha|^{2}\right) & \cdots & \exp \left(|\alpha|^{2}\right)
\end{array}\right)
$$

where

$$
\left|A_{k}\right\rangle \rightarrow \frac{1}{\sqrt{n}}\left(\begin{array}{c}
1  \tag{8}\\
\omega^{(k-1)} \\
\omega^{2(k-1)} \\
\omega^{3(k-1)} \\
\vdots \\
\omega^{(n-1)(k-1)}
\end{array}\right)
$$

are eigenvectors of $\hat{\rho}_{V A}$, whose eigenvalues are given by (5). For details of the above results, we refer to the Appendixes A and B.

## 3. Interaction of Kaleidoscope States with a Two-Level Atom

The interaction between a quantized field and a two-level atom (under rotating wave approximation) is given by the Jaynes-Cummings interaction Hamiltonian [19] (for simplicity we have set $\hbar=1$ ),

$$
\begin{equation*}
\hat{H}_{I}=\lambda\left(\hat{a}^{\dagger} \sigma_{-}+\hat{a} \sigma_{+}\right), \tag{9}
\end{equation*}
$$

where we have considered on-resonance conditions (equal field and atomic transition frequencies). In the above equation, $\lambda$ is the coupling constant, $\hat{a}$ and $\hat{a}^{\dagger}$ and the annihilation and creation operators, respectively, and $\sigma_{+}$and $\sigma_{-}$are the raising and lowering Pauli operators, respectively. The evolution operator, $\hat{U}=\exp \left(-i \hat{H}_{I} t\right)$, in the $2 \times 2$ basis is given by

$$
\hat{U}=\left(\begin{array}{cc}
\cos \left(\lambda t \sqrt{\hat{a} \hat{a}^{\dagger}}\right) & -i \hat{V} \sin \left(\lambda t \sqrt{\hat{a}^{\dagger} \hat{a}}\right)  \tag{10}\\
-i \hat{V}^{\dagger} \sin \left(\lambda t \sqrt{\hat{a} \hat{a}^{\dagger}}\right) & \cos \left(\lambda t \sqrt{\hat{a}^{\dagger} \hat{a}}\right)
\end{array}\right)
$$

with $\hat{V}$ and $\hat{V}^{\dagger}$ the London phase operators [20]. For details of the above and next results, we refer to the appendices.

If we consider the atom initially prepared in the state $|e\rangle$ and the field in a statistical mixture of $n$ coherent states, i.e., the system is initially prepared in $\hat{\rho}(0)=\hat{\rho}_{F}(0)|e\rangle\langle e|$, with $\hat{\rho}_{F}(0)$ defined by (2), then the time evolved density matrix is given by

$$
\hat{\rho}=\left(\begin{array}{cc}
\sum_{k=1}^{n}\left|\mathrm{C}_{k}\right\rangle\left\langle\mathrm{C}_{k}\right| & \left|\mathrm{C}_{n}\right\rangle\left\langle\mathrm{S}_{1}\right|+\sum_{k=2}^{n}\left|\mathrm{C}_{k-1}\right\rangle\left\langle\mathrm{S}_{k}\right|  \tag{11}\\
\left|\mathrm{S}_{1}\right\rangle\left\langle\mathrm{C}_{n}\right|+\sum_{k=2}^{n}\left|\mathrm{~S}_{k}\right\rangle\left\langle\mathrm{C}_{k-1}\right| & \sum_{k=1}^{n}\left|\mathrm{~S}_{k}\right\rangle\left\langle\mathrm{S}_{k}\right|
\end{array}\right),
$$

with

$$
\begin{align*}
& \left|C_{k}\right\rangle=\mathrm{e}^{-\frac{|\alpha|^{2}}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m+k}}{\sqrt{(n m+k)!}} \cos (\lambda t \sqrt{n m+k+1})|n m+k\rangle, \quad(k=1,2, \cdots n-1),  \tag{12}\\
& \left|\mathrm{C}_{n}\right\rangle=\mathrm{e}^{-\frac{|\alpha|^{2}}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m}}{\sqrt{(n m)!}} \cos (\lambda t \sqrt{n m+1})|n m\rangle, \tag{13}
\end{align*}
$$

$$
\begin{equation*}
\left|\mathrm{S}_{k}\right\rangle=-i \mathrm{e}^{-\frac{|\alpha|^{2}}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m+k-1}}{\sqrt{(n m+k-1)!}} \sin (\lambda t \sqrt{n m+k})|n m+k\rangle, \quad(k=1,2, \cdots n) . \tag{14}
\end{equation*}
$$

In the study of the interaction between two subsystems, namely atom (A) and field ( F ), the subsystem entropies $S_{A}$ and $S_{F}$, and the total entropy $S_{A F}$ play an essential role and they obey the Araki-Lieb inequality [13],

$$
\begin{equation*}
\left|S_{A}-S_{F}\right| \leq S_{A F} \leq S_{A}+S_{F} \tag{15}
\end{equation*}
$$

Particularly, when the states of the subsystems are prepared in pure states, we have that $S_{A F}=0$, and the entropies of the subsystems will be equal $S_{A}=S_{F}$. This situation is precisely our case because the states of the field (3) and atom are pure states, and we can apply the virtual atom method [15] in order to find the entropies.

Therefore, tracing over the field states, we obtain the reduced block diagonal virtual atom density matrix

$$
\hat{\rho}_{V A}=\left(\begin{array}{ccccc}
\left\langle C_{1} \mid C_{1}\right\rangle & \left\langle C_{1} \mid S_{1}\right\rangle^{*} & \cdots & 0 & 0  \tag{16}\\
\left\langle C_{1} \mid S_{1}\right\rangle & \left\langle S_{1} \mid S_{1}\right\rangle & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \left\langle C_{n} \mid C_{n}\right\rangle & \left\langle C_{n} \mid S_{n}\right\rangle^{*} \\
0 & 0 & \cdots & \left\langle C_{n} \mid S_{n}\right\rangle & \left\langle S_{n} \mid S_{n}\right\rangle
\end{array}\right)
$$

whose eigenvalues are:

$$
\begin{equation*}
\lambda_{k}^{ \pm}=\frac{1}{2}\left(\left\langle C_{k} \mid C_{k}\right\rangle+\left\langle S_{k} \mid S_{k}\right\rangle\right) \pm \frac{1}{2} \sqrt{\left(\left\langle C_{k} \mid C_{k}\right\rangle-\left\langle S_{k} \mid S_{k}\right\rangle\right)^{2}+4\left|\left\langle C_{k} \mid S_{k}\right\rangle\right|^{2}}, \tag{17}
\end{equation*}
$$

for $k=1,2,3,4, \ldots, n$, such that the field entropy can be written in the form

$$
\begin{equation*}
S_{F}=-\sum_{k=1}^{n} \lambda_{k}^{+} \ln \lambda_{k}^{+}-\sum_{k=1}^{n} \lambda_{k}^{-} \ln \lambda_{k}^{-} . \tag{18}
\end{equation*}
$$

Finally, tracing the density (11) over the atomic states, we obtain the reduced density matrix for the field as

$$
\begin{equation*}
\hat{\rho}_{F}=\sum_{k=1}^{n}\left|\mathrm{C}_{k}\right\rangle\left\langle\mathrm{C}_{k}\right|+\sum_{k=1}^{n}\left|\mathrm{~S}_{k}\right\rangle\left\langle\mathrm{S}_{k}\right| . \tag{19}
\end{equation*}
$$

As two coherent states are sufficiently apart when $\alpha \approx 2$, they may be considered orthogonal. They may be considered orthogonal as $\langle\alpha \mid-\alpha\rangle=\exp \left(-2|\alpha|^{2}\right)$ and, as $\alpha$ becomes larger, the exponential approaches zero. On the other hand, the Kaleidoscopestates are orthogonal for any $\alpha>0$, and $\hat{\rho}_{V A}$ will be a block diagonalizable density as was shown in Equation (16).

The field entropy is plotted as a function of the scaled time $\lambda t$ in Figure 1, for Kaleidoscope-states with different values of statistical mixture of coherent states, namely, $n=1,2,4,8$ and 16 , and $\alpha=6.0$. Their initial values are calculated as $\ln (n)$ and are indicated in the figure by dotted lines. It may be seen that the entropies have similar behavior: each one possesses a global minimum of about $\lambda t \approx 19$; for all values of $n \geq 2$ the field entropy is below the dotted line corresponding to their initial values, making it clear that a purification process takes place. This may be explained as follows: the photon distribution for a mixture of coherent states is a Poissonian distribution, just as the one for a coherent state. This means that the collapses and revivals occur exactly at the same time for each of the components, namely $t_{R} / 2=\pi|\alpha| / \lambda$, with $t_{R}$ the so-called revival time. This phenomenon happens because each of the components of the mixture does not interfere with any other components as the state considered is not a superposition of states but
rather a statistical mixture. Once the interaction begins, each component is divided into two counter-rotating components (in phase space, see Figure 2) that produce the purest state precisely at half the revival time [21]. In this region and in each case, the field becomes purer than its initial state and oscillations appear for $n \geq 2$. Such purification occurs because of a transfer of coherence from the atom to the field [22].


Figure 1. The evolution of the field entropy of Kaleidoscope-States as a function of the scaled time $\lambda t$ and different values of statistical mixture of coherent states $n=1,2,4,8$ and 16 , with $\alpha=6.0$. Their initial values are calculated as $\ln (n)$ and are indicated in the figure by dotted lines.

Here, some questions arise: what will the form of the above-mentioned purer states for different values of a statistical mixture of coherent states $n$ about $\lambda t \approx 19$ be? What will the form of the state where the field entropy reaches its maximum value for different values of a statistical mixture of coherent states $n$ be? In order to try to answer the above questions, we calculated the field Wigner function for the reduced density matrix field defined in Equation (19). The Wigner function may be written as [16]

$$
\begin{align*}
W_{F}(\beta) & =\frac{2}{\pi} \operatorname{Tr}_{F}\left\{\hat{\rho}_{F} \hat{D}(\beta)(-1)^{\hat{n}} \hat{D}^{\dagger}(\beta)\right\}, \\
& =\frac{2}{\pi} \sum_{k=1}^{n}\left\langle C_{k}\right| \hat{\rho}_{F} \hat{D}(\beta)(-1)^{\hat{n}} \hat{D}^{\dagger}(\beta)\left|C_{k}\right\rangle+\frac{2}{\pi} \sum_{k=1}^{n}\left\langle\mathrm{~S}_{k}\right| \hat{\rho}_{F} \hat{D}(\beta)(-1)^{\hat{n}} \hat{D}^{\dagger}(\beta)\left|\mathrm{S}_{k}\right\rangle, \tag{20}
\end{align*}
$$

where $\hat{D}(\beta)=\exp \left(\beta \hat{a}^{\dagger}-\beta^{*} \hat{a}\right)$ is the displacement operator.
In Figure $2 \mathrm{a}-\mathrm{c}$, we show the field Wigner function at $\lambda t=0$, and we clearly see the two, four, and eight peaks corresponding to each coherent state for $n=2,4$ and 8 , respectively. Additionally, in Figure 2d-f, we show the field Wigner function corresponding to the Kaleidoscope-State with $n=2,4$ and 8 respectively when the field would become a purer state at time $\lambda t \approx 19.15$. These Wigner functions resemble a Schrödinger cat state of 2,4 and 8 components, where we note the characteristic interference structure. We clearly see the formation of quantum interferences halfway between the $n$ humps. As we mentioned above, once the interaction starts, each of the components is divided into two counter-rotating terms and, at half the revival time, the humps recombine with the traveling contributions from other component states; that, however, is not a coherent process, as each one of the components interferes only with itself. At that time, the Wigner function becomes negative, which is a clear signature of a non-classical state. This fact, plus the purification of the field shown in Figure 1, hints that multiple Schrödinger cats are being generated. The frequency of the interference structure increases as the separation distance $\alpha$ increases [23]. For example, setting $\alpha=4$ and $n=2$, the entropy has a similar behavior as in Figure 1 but, as we can see in Figure 3a, now its minimum is around $\lambda t \approx 12.5$, and its corresponding field Wigner function has an interference structure with a lower frequency, as is shown in Figure 4c. Finally, when $\lambda t \approx 19$, the initial mixture of Kaleidoscope States (as we showed
in Figure $2 \mathrm{a}-\mathrm{c}$ ) gains purity as was suggested by the entropy behavior, and the negativity of the field Wigner function are an indicator of the non-classical properties of the state [23] at $\lambda t \approx 19.15$.


Figure 2. Wigner function for Kaleidoscope-State with $\alpha=6.0$ for several values of the time $\lambda t$ and $n$. For $\lambda t=0$ : (a) $n=2$, (b) $n=4$ and (c) $n=8$; for $\lambda t \approx 19.15$ : (d) $n=2$, (e) $n=4$ and (f) $n=8$.

On the other hand, for the case of the maximum value of the field entropy $\lambda t \approx 2.0$ as it is shown in Figure 3a, we see that each peak of the initial coherent state in Figure 4a, split into two counter-rotating terms, where each one of the components interferes only with itself and spread on the circle of radius $\alpha=4$, and the Wigner function, may have negative values, as is shown in Figure 4b, which is an indicator of the non-classical properties of the state [23].

For the sake of completeness, the atom entropy is plotted in Figure 3b as a function of the scaled time $\lambda t$ with $n=2$ and $\alpha=4.0$. This entropy quickly saturates to the value of $\ln (2)$, and it may be seen that the entropy has similar behavior for other values of $n$ and $\alpha$.

The above results can be compared with previous ones in Figure 1 for $n=16$ and $\alpha=6$. In that case, it was shown that the field entropy has its minimum around $\lambda t \approx 19.15$. In Figure 4 d , we show the field Wigner function at $\lambda t=0$, and we clearly see the sixteen peaks corresponding to each of $n=16$ coherent states. For the case of the maximum value of the field entropy $(\lambda t \approx 1.15)$, we see that each peak of the initial coherent state splits into two counter-rotating terms, where each component interferes only with itself, and it the formation of the quantum interference structure on the circle of radius $\alpha=6$ may also be seen. In this case, the Wigner function has no negative values. Finally, when the time goes to $\lambda t \approx 19.15$, the peaks interfere and the formation of the quantum interference structures may be seen halfway between the 16 humps, which is a clear signature of a non-classical state.


Figure 3. The evolution of the: (a) field entropy and (b) and atom entropy; for a Kaleidoscope-State as a function of the scaled time $\lambda t$ with $n=2$ and $\alpha=4.0$. The initial value of field entropy is calculated as $\ln (2)$ and is indicated in the figure by a dotted line.


Figure 4. Cont.


Figure 4. Field Wigner function for Kaleidoscope-State for several values of the time $\lambda t$ and $n$. For $n=2$ (a) $\lambda t=0$, (b) $\lambda t \approx 2.0$ and (c) $\lambda t \approx 12.5$. And for $n=16$ (d) $\lambda t=0$, (e) $\lambda t \approx 1.15$ and (f) $\lambda t \approx 19.15$.

## 4. Conclusions

We have shown that the Jaynes-Cummings interaction with an initial Kaleidoscope mixture of coherent states may be modeled by the virtual Hamiltonian method by extending the atomic Hilbert space such that a virtual pure state may be associated as an initial wavefunction. In particular, we have seen that the purification procedure takes us from a mixed field density matrix to a pure wave function that involves a virtual $2 n$-level atom, as can be seen in the $2 n$ terms in Equation (19). Finally, we should mention that the effects presented in the field entropy for the initial field state given by a statistical mixture of the constituent states are reflected in the appearance of oscillations that give rise to negative Wigner functions resembling multiple Schrödinger's cats.

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## Appendix A. Initial Field Density

Following the virtual atom method to the density matrix (2) we propose:

$$
\begin{equation*}
\left|\psi_{V}\right\rangle=\frac{|\alpha\rangle}{\sqrt{n}}\left|a_{1}\right\rangle+\frac{|\omega \alpha\rangle}{\sqrt{n}}\left|a_{2}\right\rangle+\frac{\left|\omega^{2} \alpha\right\rangle}{\sqrt{n}}\left|a_{3}\right\rangle+\cdots+\frac{\left|\omega^{(n-1)} \alpha\right\rangle}{\sqrt{n}}\left|a_{n}\right\rangle \tag{A1}
\end{equation*}
$$

where for simplicity we represented the ket in the Hilbert space by the standard basis of $n$ dimensional Euclidean space as:

$$
\left|a_{1}\right\rangle=\left(\begin{array}{c}
1  \tag{A2}\\
0 \\
0 \\
\vdots \\
0
\end{array}\right),\left|a_{2}\right\rangle=\left(\begin{array}{c}
0 \\
1 \\
0 \\
\vdots \\
0
\end{array}\right),\left|a_{3}\right\rangle=\left(\begin{array}{c}
0 \\
0 \\
1 \\
\vdots \\
0
\end{array}\right), \cdots,\left|a_{n}\right\rangle=\left(\begin{array}{c}
0 \\
0 \\
0 \\
\vdots \\
1
\end{array}\right),
$$

(we do here as it is done when dealing with Pauli-spin operators: one may go to the $2 \times 2$ matrix form or work with the well known $\sigma$ operators). From Equation (1) and considering orthogonal properties of Discrete Fourier Transform (DFT), one has

$$
\left(\begin{array}{c}
|\alpha\rangle / \sqrt{n}  \tag{A3}\\
|\omega \alpha\rangle / \sqrt{n} \\
\left|\omega^{2} \alpha\right\rangle / \sqrt{n} \\
\vdots \\
\left|\omega^{(n-1)} \alpha\right\rangle / \sqrt{n}
\end{array}\right)=\frac{1}{\sqrt{n}}\left(\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^{2} & \cdots & \omega^{(n-1)} \\
1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{array}\right)\left(\begin{array}{c}
\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle \\
\sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle \\
\sqrt{\lambda_{3 \alpha}}\left|\psi_{3 \alpha}\right\rangle \\
\vdots \\
\sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle
\end{array}\right)
$$

Thus, according to the above formula (A3), we can rewrite Equation (A1) as:

$$
\begin{align*}
\left|\psi_{V}\right\rangle & =\frac{1}{\sqrt{n}}\left(\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle+\sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle+\cdots+\sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle\right)\left|a_{1}\right\rangle \\
& +\frac{1}{\sqrt{n}}\left(\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle+\omega \sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle+\cdots+\omega^{(n-1)} \sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle\right)\left|a_{2}\right\rangle \\
& +\frac{1}{\sqrt{n}}\left(\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle+\omega^{2} \sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle+\cdots+\omega^{2(n-1)} \sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle\right)\left|a_{3}\right\rangle \\
& \vdots \\
& +\frac{1}{\sqrt{n}}\left(\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle+\omega^{(n-1)} \sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle+\cdots+\omega^{(n-1)(n-1)} \sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle\right)\left|a_{n}\right\rangle \\
& =\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle \frac{1}{\sqrt{n}}\left(\left|a_{1}\right\rangle+\left|a_{2}\right\rangle+\left|a_{3}\right\rangle+\cdots+\left|a_{n}\right\rangle\right) \\
& +\sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle \frac{1}{\sqrt{n}}\left(\left|a_{1}\right\rangle+\omega\left|a_{2}\right\rangle+\omega^{2}\left|a_{3}\right\rangle+\cdots+\omega^{(n-1)}\left|a_{n}\right\rangle\right) \\
& \vdots \\
& +\sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle \frac{1}{\sqrt{n}}\left(\left|a_{1}\right\rangle+\omega^{(n-1)}\left|a_{2}\right\rangle+\omega^{2(n-1)}\left|a_{3}\right\rangle+\cdots+\omega^{(n-1)(n-1)}\left|a_{n}\right\rangle\right) \\
& =\sqrt{\lambda_{1 \alpha}}\left|\psi_{1 \alpha}\right\rangle\left|A_{1}\right\rangle+\sqrt{\lambda_{2 \alpha}}\left|\psi_{2 \alpha}\right\rangle\left|A_{2}\right\rangle+\cdots+\sqrt{\lambda_{n \alpha}}\left|\psi_{n \alpha}\right\rangle\left|A_{n}\right\rangle \tag{A4}
\end{align*}
$$

where

$$
\left(\begin{array}{c}
\left|A_{1}\right\rangle  \tag{A5}\\
\left|A_{2}\right\rangle \\
\left|A_{3}\right\rangle \\
\vdots \\
\left|A_{n}\right\rangle
\end{array}\right)=\frac{1}{\sqrt{n}}\left(\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^{2} & \cdots & \omega^{(n-1)} \\
1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{array}\right)\left(\begin{array}{c}
\left|a_{1}\right\rangle \\
\left|a_{2}\right\rangle \\
\left|a_{3}\right\rangle \\
\vdots \\
\left|a_{n}\right\rangle
\end{array}\right)
$$

with

$$
\left|A_{k}\right\rangle=\left(\begin{array}{c}
1  \tag{A6}\\
\omega^{(k-1)} \\
\omega^{2(k-1)} \\
\vdots \\
\omega^{(n-1)(k-1)}
\end{array}\right)=\left(\begin{array}{c}
\left\langle a_{1} \mid A_{k}\right\rangle \\
\left\langle a_{2} \mid A_{k}\right\rangle \\
\left\langle a_{3} \mid A_{k}\right\rangle \\
\vdots \\
\left\langle a_{n} \mid A_{k}\right\rangle
\end{array}\right)
$$

then these new virtual atom states $\left\{\left|A_{i}\right\rangle_{i}\right\}$ are orthogonal kets because the discrete Fourier transform (DFT) is an orthogonal matrix too.

Using the above results it can be shown that the density matrix associated with $\left|\psi_{V}\right\rangle$ defined by Equation (A1) will be

$$
\begin{equation*}
\hat{\rho}_{V}=\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n}\left|\omega^{(i-1)} \alpha\right\rangle\left|a_{i}\right\rangle\left\langle a_{j}\right|\left\langle\omega^{(j-1)} \alpha\right|=\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\lambda_{i \alpha} \lambda_{j \alpha}^{*}}\left|\psi_{i \alpha}\right\rangle\left|A_{i}\right\rangle\left\langle A_{j}\right|\left\langle\psi_{j \alpha}\right|, \tag{A7}
\end{equation*}
$$

where we have used relation (A4). Taking the trace of the density matrix (A7) over the field states we get the virtual atom density operator

$$
\begin{equation*}
\hat{\rho}_{V A}=\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n}\left\langle\omega^{(j-1)} \alpha \mid \omega^{(i-1)} \alpha\right\rangle\left|a_{i}\right\rangle\left\langle a_{j}\right|=\sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\lambda_{i \alpha} \lambda_{j \alpha}^{*}}\left\langle\psi_{j \alpha} \mid \psi_{i \alpha}\right\rangle\left|A_{i}\right\rangle\left\langle A_{j}\right| . \tag{A8}
\end{equation*}
$$

But in reference [17], it was remarked that Kaleidoscope states are orthogonal, then

$$
\begin{equation*}
\hat{\rho}_{V A}=\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n}\left\langle\omega^{(j-1)} \alpha \mid \omega^{(i-1)} \alpha\right\rangle\left|a_{i}\right\rangle\left\langle a_{j}\right|=\sum_{i=1}^{n} \lambda_{i \alpha}\left|A_{i}\right\rangle\left\langle A_{i}\right| . \tag{A9}
\end{equation*}
$$

We will now do an analysis of the eigenvalue problem for $\hat{\rho}_{V A}$, which we may start writing

$$
\begin{align*}
\hat{\rho}_{V A}\left|A_{k}\right\rangle & =\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n}\left\langle\omega^{(j-1)} \alpha \mid \omega^{(i-1)} \alpha\right\rangle\left|a_{i}\right\rangle\left\langle a_{j} \mid A_{k}\right\rangle=\lambda_{k \alpha}\left|A_{k}\right\rangle, \\
& =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \exp \left(\omega^{*(j-i)}|\alpha|^{2}\right) \frac{1}{\sqrt{n}} \omega^{(j-1)(k-1)}\left|a_{i}\right\rangle=\lambda_{k \alpha}\left|A_{k}\right\rangle,  \tag{A10}\\
& =\left(\begin{array}{c}
b_{1} \\
b_{2} \\
b_{3} \\
\vdots \\
b_{n} .
\end{array}\right),
\end{align*}
$$

where

$$
\begin{align*}
b_{l} & =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n} \sum_{j=1}^{n} \exp \left(\omega^{*(j-l)}|\alpha|^{2}\right) \frac{1}{\sqrt{n}} \omega^{(j-1)(k-1)} \\
& =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n} \frac{1}{\sqrt{n}} \omega^{(l-1)(k-1)} \sum_{m=1-l}^{n-l} \omega^{m(k-1)} \exp \left(\omega^{* m}|\alpha|^{2}\right)  \tag{A11}\\
& =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=l}^{n+l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right)
\end{align*}
$$

with $1 \leq l \leq n$. When $l=1$, we have

$$
\begin{equation*}
b_{1}=\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{1} \mid A_{k}\right\rangle \sum_{m=1}^{n} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right)=\lambda_{k \alpha}\left\langle a_{1} \mid A_{k}\right\rangle \tag{A12}
\end{equation*}
$$

as $\left\langle a_{1} \mid A_{k}\right\rangle \neq 0$, then we have eigenvalue

$$
\begin{equation*}
\lambda_{k \alpha}=\frac{\mathrm{e}^{-|\alpha|^{2}}}{n} \sum_{m=1}^{n} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right) . \tag{A13}
\end{equation*}
$$

In a similar fashion, for $l>1$ we have

$$
\begin{align*}
b_{l} & =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=l}^{n+l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right), \\
& =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{n+l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right)-\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right), \\
& =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{n} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right)-\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right) \\
& +\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=n+1}^{n+l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right), \\
& =\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{n} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right)-\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{l-1} \omega^{* m(k-1)} \exp \left(\omega^{m}|\alpha|^{2}\right) \\
& +\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{l-1} \omega^{*(m+n)(k-1)} \exp \left(\omega^{m+n}|\alpha|^{2}\right) . \tag{A14}
\end{align*}
$$

Because $\omega^{n}=\left[\exp \left(i \frac{2 \pi}{n}\right)\right]^{n}=1$, we can cancel the last two sums in Equation (A14), and we have that

$$
\begin{equation*}
b_{l}=\frac{\mathrm{e}^{-|\alpha|^{2}}}{n}\left\langle a_{l} \mid A_{k}\right\rangle \sum_{m=1}^{n} \omega^{* m(k-l)} \exp \left(\omega^{m}|\alpha|^{2}\right)=\lambda_{k \alpha}\left\langle a_{l} \mid A_{k}\right\rangle \tag{A15}
\end{equation*}
$$

as $\left\langle a_{l} \mid A_{k}\right\rangle \neq 0$, then we have eigenvalue

$$
\begin{equation*}
\lambda_{k \alpha}=\frac{\mathrm{e}^{-|\alpha|^{2}}}{n} \sum_{m=1}^{n} \omega^{* m(k-l)} \exp \left(\omega^{m}|\alpha|^{2}\right) \tag{A16}
\end{equation*}
$$

With the help of Equations (A12) and (A15), the eigenvalue Equation (A10) can be written as,

$$
\hat{\rho}_{V A}\left|A_{k}\right\rangle=\lambda_{k \alpha}\left(\begin{array}{c}
\left\langle a_{1} \mid A_{k}\right\rangle  \tag{A17}\\
\left\langle a_{1} \mid A_{k}\right\rangle \\
\left\langle a_{n} \mid A_{k}\right\rangle \\
\vdots \\
\left\langle a_{n} \mid A_{k}\right\rangle
\end{array}\right)=\lambda_{k \alpha}\left|A_{k}\right\rangle
$$

Now, taking the trace of the density matrix (A7) over field states we get the field density:

$$
\begin{equation*}
\hat{\rho}_{F}=\frac{1}{n} \sum_{i=1}^{n}\left|\omega^{(i-1)} \alpha\right\rangle\left\langle\omega^{(i-1)} \alpha\right|=\sum_{i=1}^{n} \lambda_{i \alpha}\left|\psi_{i \alpha}\right\rangle\left\langle\psi_{i \alpha}\right| . \tag{A18}
\end{equation*}
$$

## Appendix B. Jaynes-Cummings Dynamics

The Jaynes-Cummings time evolution operator is given by

$$
\hat{U}(t)=\left(\begin{array}{cc}
\cos \left(\lambda t \sqrt{\hat{a} \hat{a}^{\dagger}}\right) & -i \hat{V} \sin \left(\lambda t \sqrt{\hat{a}^{\dagger} \hat{a}}\right)  \tag{A19}\\
-i \hat{V}^{\dagger} \sin \left(\lambda t \sqrt{\hat{a} \hat{a}^{\dagger}}\right) & \cos \left(\lambda t \sqrt{\hat{a}^{\dagger} \hat{a}}\right)
\end{array}\right),
$$

where $\hat{V}$ and $\hat{V}^{\dagger}$ are the London phase operators [20]. If we consider the initial density matrix $\hat{\rho}(0)=\hat{\rho}_{F}(0)|e\rangle\langle e|$, where $\hat{\rho}_{F}(0)$ was defined in Equation (4), we obtain:

$$
\begin{align*}
\hat{\rho}(t) & =\hat{U}(t) \hat{\rho}(0) \hat{U}^{\dagger}(t)=\hat{U}(t)\left(\begin{array}{cc}
\hat{\rho}_{F}(0) & 0 \\
0 & 0
\end{array}\right) \hat{U}^{\dagger}(t), \\
& =\left(\begin{array}{cc}
\sum_{k=1}^{n}\left|C_{k}\right\rangle\left\langle C_{k}\right| & \left|C_{n}\right\rangle\left\langle S_{1}\right|+\sum_{k=2}^{n}\left|C_{k-1}\right\rangle\left\langle S_{k}\right| \\
\left|S_{1}\right\rangle\left\langle C_{n}\right|+\sum_{k=2}^{n}\left|S_{k}\right\rangle\left\langle C_{k-1}\right| & \sum_{k=1}^{n}\left|S_{k}\right\rangle\left\langle S_{k}\right|
\end{array}\right), \tag{A20}
\end{align*}
$$

where,

$$
\begin{align*}
& \left|C_{k}\right\rangle=\sqrt{\lambda_{(k+1) \alpha}} \cos \left(\lambda t \sqrt{\hat{a} \hat{a}^{\dagger}}\right)\left|\psi_{(k+1) \alpha}\right\rangle, \quad(k=1,2,3, \ldots, n-1), \\
& \left|C_{n}\right\rangle=\sqrt{\lambda_{1 \alpha}} \cos \left(\lambda t \sqrt{\hat{a} \hat{a}^{+}}\right)\left|\psi_{1 \alpha}\right\rangle, \\
& \left|S_{k}\right\rangle=-i \sqrt{\lambda_{k \alpha}} \hat{V}^{\dagger} \sin \left(\lambda t \sqrt{\hat{a} \hat{a}^{\dagger}}\right)\left|\psi_{k \alpha}\right\rangle, \quad(k=1,2,3, \ldots, n) . \tag{A21}
\end{align*}
$$

Nevertheless, Kaleidoscope states can be rewritten as:

$$
\begin{align*}
\left|\psi_{k \alpha}\right\rangle & =\frac{\mathrm{e}^{-|\alpha|^{2} / 2}}{\sqrt{\lambda_{k \alpha}}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m+k-1}}{(n m+k-1)!} \hat{a}^{\dagger(n m+k-1)}|0\rangle, \\
& =\frac{\mathrm{e}^{-|\alpha|^{2} / 2}}{\sqrt{\lambda_{k \alpha}}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m+k-1}}{\sqrt{(n m+k-1)!}}|n m+k-1\rangle . \tag{A22}
\end{align*}
$$

With the above relation, we can write

$$
\begin{align*}
& \left|C_{k}\right\rangle=\mathrm{e}^{-\frac{|\alpha|^{2}}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m+k}}{\sqrt{(n m+k)!}} \cos (\lambda t \sqrt{n m+k+1})|n m+k\rangle, \quad(k=1,2,3, \ldots, n-1), \\
& \left|C_{n}\right\rangle=\mathrm{e}^{-\frac{|\alpha|^{2}}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m}}{\sqrt{(n m)!}} \cos (\lambda t \sqrt{n m+1})|n m\rangle \\
& \left|S_{k}\right\rangle=-i \mathrm{e}^{-\frac{|\alpha|^{2}}{2}} \sum_{m=0}^{+\infty} \frac{\alpha^{n m+k-1}}{\sqrt{(n m+k-1)!}} \sin (\lambda t \sqrt{n m+k})|n m+k\rangle, \quad(k=1,2,3, \ldots, n) . \tag{A23}
\end{align*}
$$

From the time evolution density matrix (A20), we can obtain the reduced block diagonal atom density matrix by tracing over the field states, so that

$$
\hat{\rho}_{A}=\left(\begin{array}{ccccc}
\left\langle C_{1} \mid C_{1}\right\rangle & \left\langle C_{1} \mid S_{1}\right\rangle^{*} & \cdots & 0 & 0  \tag{A24}\\
\left\langle C_{1} \mid S_{1}\right\rangle & \left\langle S_{1} \mid S_{1}\right\rangle & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \left\langle C_{n} \mid C_{n}\right\rangle & \left\langle C_{n} \mid S_{n}\right\rangle^{*} \\
0 & 0 & \cdots & \left\langle C_{n} \mid S_{n}\right\rangle & \left\langle S_{n} \mid S_{n}\right\rangle
\end{array}\right)
$$

whose characteristic polynomial is:

$$
\operatorname{det}\left(\hat{\rho}_{A}-\lambda \hat{I}\right)=\prod_{k=1}^{n} \operatorname{det}\left(\begin{array}{cc}
\left\langle C_{k} \mid C_{k}\right\rangle-\lambda & \left\langle C_{k} \mid S_{k}\right\rangle^{*}  \tag{A25}\\
\left\langle S_{k} \mid C_{k}\right\rangle & \left\langle S_{k} \mid S_{k}\right\rangle-\lambda
\end{array}\right)=0
$$

and we get the eigenvalues

$$
\begin{equation*}
\lambda_{k}^{ \pm}=\frac{1}{2}\left(\left\langle C_{k} \mid C_{k}\right\rangle+\left\langle S_{k} \mid S_{k}\right\rangle\right) \pm \frac{1}{2} \sqrt{\left(\left\langle C_{k} \mid C_{k}\right\rangle-\left\langle S_{k} \mid S_{k}\right\rangle\right)^{2}+4\left|\left\langle C_{k} \mid S_{k}\right\rangle\right|^{2}} \tag{A26}
\end{equation*}
$$

for $k=1,2,3, \ldots, n$. By recalling the properties of the trace of the block diagonal matrix we calculate the atomic von Neumann entropy defined as

$$
\begin{align*}
S_{A} & =-\operatorname{Tr}\left\{\hat{\rho}_{A} \ln \hat{\rho}_{A}\right\}, \\
& =-\left(\sum_{k=1}^{n}\left\langle\mathrm{C}_{k} \mid \mathrm{C}_{k}\right\rangle\right) \ln \left(\sum_{k=1}^{n}\left\langle\mathrm{C}_{k} \mid \mathrm{C}_{k}\right\rangle\right)-\left(\sum_{k=1}^{n}\left\langle\mathrm{~S}_{k} \mid \mathrm{S}_{k}\right\rangle\right) \ln \left(\sum_{k=1}^{n}\left\langle\mathrm{~S}_{k} \mid \mathrm{S}_{k}\right\rangle\right),  \tag{A27}\\
& =-\sum_{k=1}^{n} \lambda_{k}^{+} \ln \lambda_{k}^{+}-\sum_{k=1}^{n} \lambda_{k}^{-} \ln \lambda_{k}^{-} .
\end{align*}
$$

Finally, by tracing the density (A20) over the atomic states, we obtain the reduced density matrix for the field as

$$
\begin{equation*}
\hat{\rho}_{F}=\sum_{k=1}^{n}\left|\mathrm{C}_{k}\right\rangle\left\langle\mathrm{C}_{k}\right|+\sum_{k=1}^{n}\left|\mathrm{~S}_{k}\right\rangle\left\langle\mathrm{S}_{k}\right| . \tag{A28}
\end{equation*}
$$

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