Mathematical and Computational Applications, Vol. 9, No. 3, pp. 349-358, 2004.
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SOLVABILITY CONDITIONS AND GREEN FUNCTIONAL CONCEPT FOR LOCAL AND NONLOCAL LINEAR PROBLEMS FOR A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

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Abstract- A generally nonlocal problem is investigated for a class of second order differential equations with weak singularities. New adjoint problem and Green functional concepts are introduced for completely nonhomogeneous problem. The solvability conditions of the completely nonhomogeneous problem and corresponding adjoint problem are obtained in an alternative form.

Keywords- Adjoint problem, Green functional, nonlocal problem.

1. INTRODUCTION

Let R be the space of real numbers, let $X=(x_0,x_1)$ be a finite interval, let L_p, $1\le p<\infty$, be the space of the p-integrable functions on X, let L_∞ be the space of the measurable and essentially bounded functions on X. The main aim of this work is to investigate the solvability conditions and the Green function of the differential equation

$$(V_2u)(x)\equiv(g(x)u')'+A_1(x)u'+A_0(x)u=z_2(x),\quad x\in X \tag{1}$$

associated with conditions

$$V_iu\equiv a_iu(x_0)+b_iu_1(x_0)+\int_{x_0}^{x_1}B_i(\xi)u_1'(\xi)d\xi=z_i,\quad i=0,1 \tag{2}$$

in which $u_1(x)=g(x)u'(x)$. Here, $z_2\in L_p$ and $z_i\in R\ (i=0,1)$ are given free terms. We will assume that $g\in L_\infty$ is a given function with $\frac{1}{g}\in L_p$ for a given p with $1\le p\le\infty$ and $A_1\in L_\infty$, $A_0\in L_p$, $B_i\in L_q$, $a_i\in R, b_i\in R$ are given elements, where ${}^1\!/_p+{}^1\!/_q=1$. The conditions (2) can be considered as general form of local and nonlocal conditions for second order equations (see [1]).The initial and classical type boundary conditions and also multipoint and integral type nonlocal conditions (see,e.g., [2-4]) are special cases of (2) for a suitable choise of a_i,b_i and $B_i(\xi)$. The operator V_2 of the equation (1) has generally nonsmooth coefficients and its basic part $(g(x)u')'$ may also have weak singularities, for example, at finite number points (in the closure $\overline{X}$ of X) in which $g(x)$ is continuous and zero. In addition, the problem (1), (2) does not have a traditional form adjoint problem even for simple cases of (1) and (2). Therefore, some unsolvable difficulties arise in application of the classical methods for this problem. For this reason, a new constructive method based on [5,6] is gıven for the investigation of the problem (1),(2).This method uses certain convenient isomorphic decompositions of a Sobolev space with the weight $g(x)$ of the solutions and adjoint space of this space.A concept of the adjoınt problem, called as an "adjoint system" is introduced by these decompositions for the problem (1),(2).An explicit form of this adjoint system is obtained by quite simple calculations in different from the traditional form adjoint problem. Such an adjoint system becomes a system of three integro-algebraic equations with an unknown triple $(f_2(\xi), f_1, f_0)$ of one function $f_2(\xi)$ and two real numbers f_1 and f_0. One of these equations is an integral equation and other two of them are algebraic equations. The role of this system is similar to that of the adjoint operator equation in the general theory of linear equations in the Banach spaces (see,e.g.,[7,8]). The solvability conditions of the completely nonhomogeneous problem and corresponding adjoint system are obtained in an alternative theorem. A concept of the Green functional for completely nonhomogeneous problem is introduced as a solution $(f_2(\xi,x), f_1(x), f_0(x))$ of the adjoint system with a special free term depend on $x\in \overline{X}$ as a parameter. By the definition, the Green functional is a linear bounded functional on the space of the triples $(z_2(x), z_1, z_0)$ representing the free terms of (1), (2). The presented adjoint system and Green functional concepts are more natural than the traditional adjoint problem and Green function concepts, respectively. The presented method is different in principle than the classical methods. Therefore, we have use also some notations which differ from the traditionals.

2. THE SPACE OF THE SOLUTIONS AND ITS ADJOINT SPACE

We can consider the equation (1) as a compact form of a certain system of two first order differential equations with unknowns $u\in W_p^1$ and $u_1\in W_p^1$, where W_p^1 is the space of all $u\in L_p$ with derivative $u'\in L_p$. This shows that it is natural to consider the problem (1), (2) in the Sobolev space $W_{g,p}$ with the weight $g(x)$ of all $u\in L_p$ with $u'\in L_p$ and $(gu')'\in L_p$ and also $\|u\|_{W_{g,p}} = \|u\|_{L_p} + \|u'\|_{L_p} + \|u_1'\|_{L_p}$, $u_1 = gu'$. The operator $V=(V_2, V_1, V_0)$ of the problem (1), (2) is a linear bounded operator from $W_{g,p}$ into the Banach space $E_p = L_p\times R\times R$ of the triples $z=(z_2(x), z_1, z_0)$ with $\|z\|_{E_p} = \|z_2\|_{L_p} + |z_1| + |z_0|$. We can consider the problem (1), (2) as an operator equation

$$Vu = z \tag{3}$$

The following assertions are true for $W_{g,p}$. The trace operators D_0 and D_1 defined by $D_0 u = u(\beta)$ and $D_1 u = u_1(\beta)$ are surjective and bounded from $W_{g,p}$ onto R for an arbitrary $\beta \in \bar{X}$. The operator N defined by $Nu = (u_1'(x), u_1(\beta), u(\beta))$ is a homeomorphism (linear) from $W_{g,p}$ onto E_p and has the bounded inverse. In addition, any function $u \in W_{g,p}$ can be restored as

$$u(x) = u(x_0) + u_1(x_0)v(x, x_0) + \int_{x_0}^{x} u_1'(\xi)v(x, \xi)d\xi, \qquad \left(v(x, \xi) = \int_{\xi}^{x} \frac{ds}{g(s)} \right). \tag{4}$$

The derivative $u'(x)$ of a function $u \in W_{g,p}$ generally does not have a trace at a given point $\beta \in \bar{X}$, while $u_1(x) = g(x)u'(x)$ as an absolutely continuous function on $\bar{X}$ always has a finite trace at an arbitrary point $\beta \in \bar{X}$ for all $u \in W_{g,p}$. If $g(x)$ is piecewise continuous on $\bar{X}$, then the continuity condition of $u_1(x)$ on $\bar{X}$ can be considered as interface conditions at discontinuity points of $g(x)$ (see [2], p.80). The structure of the adjoint space $W_{g,p}^*$ is obtained in the following theorem for the general case of $g(x)$ for the first time.

Theorem 1. If $1 \le p < \infty$, then any linear bounded functional $F \in W_{g,p}^*$ can be represented as

$$F(u) = \int_{x_0}^{x_1} u_1'(x)\varphi_2(x)dx + u_1(x_0)\varphi_1 + u(x_0)\varphi_0, \quad u \in W_{g,p}, \tag{5}$$

by a unique element $\varphi = (\varphi_2(x), \varphi_1, \varphi_0) \in E_q$. Any $F \in W_{g,\infty}^*$ can be represented as

$$F(u) = \int_{x_0}^{x_1} u_1'(x)d\varphi_2 + u_1(x_0)\varphi_1 + u(x_0)\varphi_0, \quad u \in W_{g,\infty}, \tag{6}$$

by a unique element $\varphi = (\varphi_2(e), \varphi_1, \varphi_0) \in \hat{E}_1 = (BA(\Sigma, \mu)) \times R \times R$, where μ is the Lebesque measure on R, Σ is σ-algebra of the μ-measurable subsets $e \subset X$ and $BA(\Sigma, \mu)$ is the space of all bounded additive functions $\varphi_2(e)$ defined on Σ with $\varphi_2(e) = 0$ when $\mu(e) = 0$ (see [7], p.192). The inverse is also valid, that is, if $\varphi \in E_q$ then the functional (5) is bounded on $W_{g,p}$ with $1 \le p < \infty$, and if $\varphi \in \hat{E}_1$ then the functional (6) is bounded on $W_{g,\infty}$.

Proof. The adjoint operator N^* of N with $\beta = x_0$ is a homeomorphism from E_p^* onto $W_{g,p}^*$. Therefore, for any linear bounded functional $F \in W_{g,p}^*$ there exists one and only one $\varphi \in E_p^*$ such that $F = N^*\varphi$ or $F(u) = \varphi(Nu)$ for all $u \in W_{g,p}$. Any $\varphi \in E_p^*$ is a triple $\varphi = (\varphi_2, \varphi_1, \varphi_0) \in L_p^* \times R^* \times R^*$ of the linear bounded functionals φ_2, φ_1 and φ_0 defined on L_p, R and R, respectively. That is,

$$F(u) = \varphi_2(u_1') + \varphi_1(u_1(x_0)) + \varphi_0(u(x_0)), \quad u \in W_{g,p}.$$

In other hand, $R^* = R$, $L_p^* = L_q$ for $1 \le p < \infty$ and $L_\infty^* = BA(\Sigma, \mu)$ in the sense of a isomorphism (see [7], p.191). This shows that any $F \in W_{g,p}^*$ can be represented as (5) when $1 \le p < \infty$ and as (6) when $p = \infty$. The inverse of these follows from (5) and (6).

3. ADJOINT SYSTEM AND SOLVABILITY CONDITIONS

Now we use Theorem 1 to find an explicit form of the adjoint operator V^* of V. We consider any element $f = (f_2(x), f_1, f_0) \in E_q$ as a linear bounded functional defined on E_p. Then, by the definition and (4), we have

$$f(Vu) = \int_{x_0}^{x_1} f_2(x) \left\{ u_1'(x) + \frac{A_1(x)}{g(x)}[\, u_1(x_0) + \int_{x_0}^{x} u_1'(\xi)d\xi \,] \right.$$
$$\left. + A_0(x)[\, u(x_0) + u_1(x_0)v(x, x_0) + \int_{x_0}^{x} u_1'(\xi)v(x,\xi)d\xi] \right\} dx$$
$$+ \sum_{i=0}^{1} f_i \, [\, a_i u(x_0) + b_i u_1(x_0) + \int_{x_0}^{x_1} B_i(\xi) u_1'(\xi)d\xi \,].$$

Thus, the following identity is true

$$\begin{aligned} f(Vu) &\equiv \int_{x_0}^{x_1} f_2(x)(V_2u)(x)dx + f_1(V_1u) + f_0(V_0u) \\ &= \int_{x_0}^{x_1} (w_2 f)(\xi)u_1'(\xi)d\xi + (w_1 f)u_1(x_0) + (w_0 f)u(x_0) \qquad (7) \\ &\equiv (wf)(u), \quad \forall f \in E_q, \quad \forall u \in W_{g,p}, \end{aligned}$$

where

$$(w_2 f)(\xi) \equiv f_2(\xi) + \int_{\xi}^{x_1} f_2(s)\,[\,\frac{A_1(s)}{g(s)} + A_0(s)v(s,\xi)\,]\,ds + f_0 B_0(\xi) + f_1\; B_1(\xi) = \varphi_2(\xi), \quad \xi \in X;$$

$$w_1 f \equiv \int_{x_0}^{x_1} f_2(s)\,[\,\frac{A_1(s)}{g(s)} + A_0(s)\, v(s,x_0)\,]\,ds + f_0 b_0 + f_1 b_1 = \varphi_1; \tag{8}$$

$$w_0 f \equiv \int_{x_0}^{x_1} f_2(s) A_0(s) ds + f_0 a_0 + f_1 a_1 = \varphi_0,$$

and $\varphi = (\varphi_2(\xi), \varphi_1, \varphi_0) \in E_q$ is a given element. The operator $w = (w_2, w_1, w_0)$ is bounded from the space E_q of the triples $f = (f_2(\xi), f_1, f_0)$ into itself. The identity (7) and Theorem 1 show that $V^* = w$ for $1 \le p < \infty$ and $w^* = VS$ for $1 < p \le \infty$, where S is the inverse of N with $\beta = x_0$, that is, $S = N^{-1}$. Therefore, the equation

$$wf = \varphi \tag{9}$$

with an unknown $f \in E_q$ and a given $\varphi \in E_q$ can be considered as an adjoint equation of (3). The explicit form of (9) is the system of the integro-algebraic equations (8). We call (8) as the adjoint system of (3). This system include one integral and two algebraic equations. The conditions imposed on the coefficients $A_i(x)$ and $B_i(x)$ show that the operator $Q = w - I_q$ is completely continuous from E_q into itself, where I_q is the identity operator on E_q and $1 < p < \infty$. Therefore, (9) is a canonical Fredholm's equation. Furthermore, S becomes a right regularizer for (3) (see [6] and [8], p.52). Thus, the following theorem is proved.

Theorem 2. Assume that $1 < p < \infty$. Then, Vu=0 has either only the zero solution or a finite number linearly independent solutions in $W_{g,p}$:

(i) if Vu=0 has only the zero solution in $W_{g,p}$, then also wf=0 has only the zero solution in E_q. In this case, the operators $V : W_{g,p} \to E_p$ and $w : E_q \to E_q$ are homeomorphisms,

(ii) if *Vu=0* has m linearly independent solutions $u_1, ..., u_m$ in $W_{g,p}$, then also *wf=0* has m linearly independent solutions $f^{(i)} = (f_2^{(i)}(x), f_1^{(i)}, f_0^{(i)})$, $i = 1, ..., m,$ in E_q. In this case, the equations (3) and (9) have the solutions $u \in W_{g,p}$ and $f \in E_q$, for given $z \in E_p$ and $\varphi \in E_q$, if and only if the following conditions are satisfied, respectively

$$\int_{x_0}^{x_1} f_2^{(i)}(\xi) z_2(\xi) d\xi + f_1^{(i)} z_1 + f_0^{(i)} z_0 = 0, \quad i = 1, ..., m \tag{S}$$

and

$$\int_{x_0}^{x_1} \varphi_2(\xi)u'_{1,i}(\xi)d\xi + \varphi_1 u_{1,i}(x_0) + \varphi_0 u_i(x_0) = 0, \quad i = 1,\dots,m \qquad (S^\bullet)$$

where $u_{1,i}(x) = g(x)u'_i(x)$.

The conditions (S) and $(S^\bullet)$ are more natural than the well known conditions given by traditional form adjoint problems (see, e.g., [2,8]). The conditions (S) and $(S^\bullet)$ for the case $g(x) \equiv 1$ was obtained by [1] for the first time.

4. GREEN FUNCTIONAL CONCEPT

Let $C(\overline{X})$ be the space of the continuous functions on $\overline{X}$.The operator I of the imbedding of the functions $u \in W_{g,p}$ into $C(\overline{X})$ is bounded. That is,

$$|u(x)| \le \alpha_0 \|u\|_{W_{g,p}}, u \in W_{g,p}, \quad \alpha_0\text{=const.>0}$$

for all $x \in \overline{X}$ .In other words ,the functional $\theta(x)$ defined on $W_{g,p}$ by $\theta(x)(u) = u(x)$ is bounded for all $x \in \overline{X}$. Therefore, $\theta(x) \in W_{g,p}$ for all $x \in \overline{X}$. Then, we can consider the equation

$$V^\bullet f = \theta(x) \qquad (10)$$

in which $f \in E_p^\bullet$ is an unknown functional and $x \in \overline{X}$ is a parameter. By the definition (see [6,9]), the fundamental solution of the operator V is a functional $f = f(x) \in E_p^\bullet$ with the parameter $x \in \overline{X}$ for which (10) is true for all $x \in \overline{X}$. We can also consider the equation (10) as an equation in the following functional form

$$(wf)(u) = u(x), \forall u \in W_{g,p} \quad . \qquad (11)$$

It is clear that, the equation (11) is equivalent to the following system

$$\begin{aligned} &(w_2 f)(\xi) = H(x-\xi)v(x,\xi), \quad \xi \in X; \\ &w_1 f = v(x, x_0); \\ &w_0 f = 1, \end{aligned} \qquad (12)$$

in which $f = (f_2(\xi), f_1, f_0) \in E_q$ is an unknown triple, $x \in \overline{X}$ is a parameter and $H(\xi)$ is the Heaviside function on R.Thus, we can introduce the following concept.

Definition 1. Assume that $f(x) = (f_2(\xi, x), f_1(x), f_0(x)) \in E_q$ is a triple with the parameter $x \in \overline{X}$. If *f(x)* is a solution of (12) for all $x \in \overline{X}$, then this triple is called as a Green functional of V.

Theorem 3. Assume that V has at least one Green functional. Then any solution $u \in W_{g,p}$ of (1), (2) can be represented as

$$u(x) = \int_{x_0}^{x_1} f_2(\xi, x) z_2(\xi) d\xi + f_1(x) z_1 + f_0(x) z_0 . \tag{13}$$

Furthermore, $Vu=0$ has only the zero solution.

Proof. From (7) follows the following identity

$$\int_{x_0}^{x_1} f_2(\xi, x) z_2(\xi) d\xi + f_1(x) z_1 + f_0(x) z_0 = \int_{x_0}^{x_1} H(x-\xi) v(x,\xi) u_1'(\xi) d\xi + u_1(x_0) v(x, x_0) + u(x_0).$$

The right hand side of this identity is equal to $u(x)$ by (4). Therefore, (13) is true. The triviality of the solution of $Vu=0$ follows from (13). The theorem is proven.

The necesarry condition for the existence a Green functional is the existence of the left inverse operator V_l^{-1} of V (Theorem 3). The sufficent condition for the existence of a Green functional is given by the following theorem.

Theorem 4. Assume that V has the apriori estimation

$$\| u \|_{W_{g,p}} \le c_0 \| Vu \|_{E_p} , \quad u \in W_{g,p} , \tag{14}$$

where c_0 is a positive constant independent on u. Then there exists at least one Green functional of V, where $1 \le p \le \infty$.

Proof. The operator V has a bounded left inverse V_l^{-1} by (14). Then the image $Im\ w$ of w becomes equal to E_q, that is, $Im\ w = E_q$ (see Theorems 2 and 2^* from [7], p. 357).Therefore, the system (12) as a special case of (9) has at least one solution $f(x) = (f_2(\xi, x), f_1(x), f_0(x)) \in E_q$ for all $x \in \bar{X}$. The theorem is proven.

Remark I. If w has the apriori estimation

$$\| f \|_{E_q} \le c_1 \| wf \|_{E_q} , \quad f \in E_q , \quad c_1 = const > 0 , \tag{15}$$

then (1), (2) always has at least one solution $u \in W_{g,p}$. If both (14) and (15) hold, then V and w become homeomorphisms and there exists a unique Green functional. The estimations such as (14) and (15) can be obtained by Bellman-Gronwall type inequalities from (1), (2) and (8), respectively. In particular, (14) and (15) hold if $\Delta = a_1 b_0 - a_0 b_1 \neq 0$ and $\Delta_1 = \| B_0 \|_{L_q} + \| B_1 \|_{L_q}$ is sufficiently small.

Theorem 5. Assume that $1 < p < \infty$. Then the Green functional is unique if it exists. There exists a Green functional if and only if *Vu=0* has only the solution *u=0.*

Proof. If there exists at least one Green functional, then *Vu=0* has only the solution *u*=0(Theorem 3) . In this case $w : E_q \to E_q$ becomes a homeomorphism (Theorem 2). Therefore, the Green functional as a solution of (12) is unique. The second part of the theorem follows from Theorem 2.

5. CONCLUSION AND APPLICATIONS

The traditional adjoint problem is defined by the classical Green's formula of the integration by parts (see, e.g., [2], p.168). Therefore, it is meaningful only for some classes of local problems with sufficiently smooth coefficients. The presented adjoint system is defined by (7) which is an integration by parts formula having more natural form than the Green's formula. Therefore, the presented method can be used for very general classes of local and nonlocal problems, in particular, for integro-differential problems with generally nonsmooth coefficients satisfying some conditions such as *p*-integrablity and boundedness. As following from [5,6], this method can also be extended for first or higher order equations or systems similar to (1). The constructivity of the presented method plays a very important role in generalizations and applications. The expression (13) shows that the superposition principle for the equation (1) is given by the first element $f_2(\xi, x)$ of the Green functional. Therefore, the function $f_2(\xi, x)$ is a generalized version of the classical Green function.The latter two elements $f_1(x)$ and $f_0(x)$ of the Green functional correspond to the unit effects of the right hand sides of the conditions. If $\Delta \neq 0$, then the function $f_2(\xi, x)$ becomes a solution of an independent Fredholm's equation of the second kind which can be obtained from $(12)_1$. We consider, for example, the boundary conditions

$$V_1 u \equiv u(x_1) = z_1, \quad V_0 u \equiv u(x_0) = z_0. \tag{16}$$

In this case,(12)can be written as

$$(w_2 f) \equiv f_2(\xi) + \int_{\xi}^{x_1} f_2(s) \left[\frac{A_1(s)}{g(s)} + A_0(s) v(s, \xi) \right] ds + f_1 v(x_1, \xi) = H(x - \xi) v(x, \xi), \quad \xi \in X;$$

$$w_1 f \equiv \int_{x_0}^{x_1} f_2(s) \left[\frac{A_1(s)}{g(s)} + A_0(s) v(s, x_0) \right] ds + f_1 v(x_1, x_0) = v(x, x_0), \tag{17}$$

$$w_0 f \equiv \int_{x_0}^{x_1} f_2(s)A_0(s)ds + f_1 + f_0 = 1$$

which can be reduced to the integral equation

$$f_2(\xi)+\int_{\xi}^{x_1} f_2(s)\left[\frac{A_1(s)}{g(s)}+A_0(s)v(s,\xi)\right]ds-\frac{1}{\alpha}v(x_1,\xi)\int_{x_0}^{x_1} f_2(s)\left[\frac{A_1(s)}{g(s)}+A_0(s)v(s,x_0)\right]ds$$
$$=H(x-\xi)v(x,\xi)-\frac{1}{\alpha}v(x_1,\xi)v(x,x_0), \quad \xi\in X; \quad \alpha=v(x_1,x_0). \tag{18}$$

Thus, the "Green function" $f_2(\xi,x)$ can be obtained as a solution of the integral equation (18). This result is obtained by this study for the first time. It can also be easily proven that the classical Green function of (1), (16) with $z_1 = z_0 = 0$ is only a special case of the solution $f_2(\xi,x)$ of (18). The integral equation (18) can also be used for approximately calculations of the Green functional and solution.

Example. We consider the equation

$$(V_2 u)(x) \equiv (g(x)u')' = -z_2(x), \quad x\in(0,1) \tag{19}$$

describing the steady heat conduction in a rod. Here, $g(x)$ is the variable thermal conductivity of the rod and $z_2(x)$ is the density of the distributed body sources of heat. Assume that the heat flow at the left end and the amount of total body heat are prescribed as constants z_1 and z_0, respectively. That is, we have the conditions (see also [10]):

$$\begin{aligned} V_1 u &\equiv u_1(0) = -z_1, \\ V_0 u &\equiv \int_0^1 u(s)ds = z_0. \end{aligned} \tag{20}$$

The problem (19), (20) does not have a traditional form adjoint problem, while this problem always has an adjoint system in the sense of this study. The identity (7) corresponding to this case shows that the adjoint operator w and the Green functional are defined by

$$(w_2 f)(\xi) \equiv f_2(\xi) + f_0 \int_\xi^1 v(s,\xi)\,ds = H(x-\xi)\,v(x,\xi), \quad \xi \in (0,1);$$

$$w_1 f \equiv f_1 + f_0 \int_0^1 v(s,0)\,ds = v(x,0), \tag{21}$$

$$w_0 f \equiv f_0 = 1.$$

Thus, the Green functional is given as

$$\begin{aligned} f_2(\xi,x) &= H(x-\xi)\,v(x,\xi) - \int_\xi^1 v(s,\xi)\,ds, \\ f_1(x) &= v(x,0) - \int_0^1 v(s,0)\,ds, \quad f_0(x) = 1, \end{aligned} \tag{22}$$

and the unique solution $u \in W_{g,p}$ of (19), (20) is

$$u(x) = z_0 - z_1\left[v(x,0) - \int_0^1 v(s,0)\,ds \right] - \int_0^x z_2(\xi)\,v(x,\xi)\,d\xi + \int_0^1 z_2(\xi)\left(\int_\xi^1 v(s,\xi)\,ds \right) d\xi \ . \tag{23}$$

If, for example, $\Phi(x,u) \in C(\overline{X} \times R)$ is a given function and $z_2(x) = \Phi(x,u(x))$, then the equation (19) becomes a nonlinear equation. In this case, the problem (19), (20) is reduced to the nonlinear integral equation given by (23).

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