THERMOELASTIC PROBLEM FOR AN INFINITE PLATE WITH A CURVILINEAR HOLE HAVING FINITE POLES

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Abstract-In the present paper, Muskhelishvili's complex variable method of solving two-dimensional elasticity problems has been applied to derive exact expressions for Goursat's functions for the boundary value problems of the infinite plate weakened by a hole having many poles and arbitrary shape which is conformally mapped on the domain outside a unit circle by means of general rational mapping function. Also the components of stress are obtained, when a stationary heat is flowing uniformly in the perpendicular direction of the hole. Some applications are considered. The interesting cases when the shape of the hole takes different shapes are included as special cases.

Keywords-Fundamental problem of elasticity, infinite plate with hole, complex variable method, rational mapping, flowing heat.

1. INTRODUCTION AND BASIC EQUATIONS

Problems dealing with isotropic homogeneous perforated infinite plates have been investigated by several authors [1-10]. Some of them [1,2,4] used Laurent's theorem to express each complex potential as a power series, others [5,6,7] used a complex variable method of Cauchy type integrals.

Consider a thin infinite plate of thickness h with a curvilinear hole C where the origin lies inside the hole conformally mapped on the domain outside a unit circle γ by means of a rational mapping function $z = cw(\zeta)$, subject to the condition $z'(\zeta)$ does not vanish or become infinite outside a unit circle γ , $\zeta = e^{i\psi}$ ($0 \leq \psi \leq 2\pi$). Here, we take the x -axis to be the horizontal axis which is perpendicular to the y -axis. A heat flow given by $\Theta = qy$ is flowing uniformly in the direction of the negative y -axis, where the increasing temperature Θ is assumed to be constant across the thickness of the plate i.e $\Theta = \Theta(x,y)$, and q is the constant temperature gradient. The uniform flow of heat is distributed by the presence of an insulated curvilinear hole C , and the heat equation satisfies the relation

$$\nabla^2 \Theta = 0 \quad , \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial \Theta}{\partial n} = 0 \quad , \quad \text{on the boundary } r = r_0 \quad (1)$$

Neglecting the variation of the strain and the stress with respect to the thickness of the plate, the thermoelastic potential Φ satisfies the formula (see [3])

$$\nabla^2 \Phi = (1 + \nu) \alpha \Theta \quad , \quad (2)$$

where α is a scalar representing the coefficient of thermal expansion and ν is Poisson's ratio. Assume the faces of the plate are free of applied loads.

It is known that [3] the first and second fundamental problems in the plane theory of thermoelasticity are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$. These functions must satisfy the boundary conditions

$$K\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi_1(t)} = f(t), \quad (3)$$

where for the first fundamental problem $K = -1$, $f(t)$ is a given function of stresses; while for the second fundamental problem $K = k > 1$, $f(t)$ is a given function of the displacement, k is called the thermal conductivity of the material and t denotes the affix of a point on the boundary.

Formula (3) for the first and second fundamental problems, respectively, take the following form

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} = \frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y} + \frac{1}{2G} \int_0^s [iX(s) - Y(s)] ds + C, \quad (4)$$

$$k\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)} = u + iv - \frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y} \quad (5)$$

where the applied stresses $X(s)$ and $Y(s)$ are prescribed on the boundary of the plane, s is the length measured from arbitrary point, u, v are the displacement components, G is the shear modulus, and Φ represents the thermoelastic potential function. Also, here the applied stresses $X(s)$ and $Y(s)$ must satisfy the following (see[3])

$$X(s) = \sigma_{xx} \frac{dy}{ds} - \sigma_{xy} \frac{dx}{ds}, \quad Y(s) = \sigma_{yx} \frac{dy}{ds} - \sigma_{yy} \frac{dx}{ds}$$

where σ_{xx} , σ_{yy} and σ_{xy} are the stress components given by (see[3])

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4G[\phi'(z) + \overline{\phi'(z)} - \lambda\Theta] \\ \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= 2G\left[\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} + 2i\frac{\partial^2 \Phi}{\partial x \partial y}\right] - 4G[z\phi''(z) + \overline{\psi'(z)}] \end{aligned} \quad (6)$$

where $\lambda = \frac{1+\nu}{2}\alpha$ is the coefficient of heat transfer.

A very powerful method for solving the thermoelastic problems makes use of conformal mapping to reduce the problem for any given region whose boundary C satisfies certain regularity conditions, to a corresponding problem for a region with unit circle hole. Through the rational mapping function $z = cw(\zeta)$ where $c > 0$ and $w(\zeta)$ does not vanish or become infinite for $|\zeta| > 1$, the infinite region outside a closed contour may be conformally mapped outside the unit circle γ .

The two complex potential analytic functions $\phi_1(z)$ and $\psi_1(z)$ are written in the form (see[3])

$$\phi_1(z) = -\frac{S_x + iS_y}{2\pi(1+k)} \ln \zeta + c\Gamma\zeta + \phi(\zeta) \quad (7)$$

$$\psi_1(z) = \frac{k(S_x - iS_y)}{2\pi(1+k)} \ln \zeta + c\Gamma^*\zeta + \psi(\zeta) \quad (8)$$

where S_x, S_y are the components of the resultant vector of all external forces acting on the boundary, Γ, Γ^* represent the stresses at infinity, generally complex; $\phi(\xi), \psi(\xi)$ are single valued analytic function within the region outside the unit circle γ and $\phi(\infty) = \psi(\infty) = 0$, which means that, $\phi(z)$ and $\psi(z)$ are homomorphic functions at infinity.

It will be assumed that $\Gamma = \bar{\Gamma}$ and $S_x = S_y = 0$ for the first fundamental problem.

The rational mapping $z = cw(\zeta)$ maps the boundary C of the given region occupied by the middle plane of the plate in the z -plane onto the unit circle γ in the ζ -plane. Curvilinear coordinates (r, ψ) are thus introduced into the z -plane which are the maps of the polar coordinates in the ζ -plane as given by $\zeta = re^{i\psi}$.

Substituting $w(\zeta)$ into (3), we have

$$K\phi_1(cw(\zeta)) - \frac{w(\zeta)}{w'(\zeta)} \overline{\phi_1'(cw(\zeta))} - \overline{\psi_1(cw(\zeta))} = f(cw(\zeta)), \quad \phi_1' = \frac{d\phi}{d\zeta} \quad (9)$$

Muskhelishvili [4] used the transformation $z = cw(\zeta) = c(\zeta + m\zeta^{-1})$ in (9) when $K = -1$, for the first fundamental problem, and $K = \chi = \frac{\lambda+3\mu}{\lambda+\mu} > 1$ for the second

fundamental problem, λ, μ are called the Lamé's constants, while χ is called Muskhelishvili's constant, for solving the problem of stretching of an infinite plate weakened by an elliptic hole. This transformation conformally maps the infinite domain bounded internally by an ellipse onto the domain outside the unit circle $|\zeta| = 1$ in the ζ -plane. Also the application of the Hilbert problem for a stretched infinite plate weakened by a circular cut is discussed in [5]. The following rational mapping functions

$$z = c \frac{\zeta + m_1 \zeta^{-1}}{1 - n_1 \zeta^{-1}} \quad (c > 0, |n_1| < 1, \text{ see [6]}) \quad (10)$$

$$z = c \frac{\zeta + m_1 \zeta^{-1} + m_2 \zeta^{-2}}{1 - n_1 \zeta^{-1}} \quad (c > 0, |n_1| < 1, \text{ see [7,10]}) \quad (11)$$

$$z = c \frac{\zeta + m_1 \zeta^{-\ell}}{1 - n_1 \zeta^{-\ell}} \quad (c > 0, |n| < 1; \ell = 1, \dots, p, \text{ see [[8]])} \quad (12)$$

$$z = c \frac{\zeta + \sum_{i=1}^3 m_i \zeta^{-i}}{1 - n_1 \zeta^{-1}} \quad (c > 0, |n_1| < 1, \text{ see [9]}) \quad (13)$$

where $c > 0$; m 's and n 's are real parameters restricted such that $z^{-1}(\zeta)$ does not vanish or become infinite outside γ , are used by El-Sirafy and Abdou [6], Abdou [7], Abdou and Khar-Eldin [8] and Abdou and Badr [9] respectively in (9), to solve the first and second fundamental problems of the infinite plate with a curvilinear hole C in the same previous domain. Abdou and Salama [10] used the rational mapping (11) in Eq. (9) to obtain the stress components for the first and second boundary value problem in the thermoelastic infinite plate weakened by a curvilinear hole C .

In this paper, the complex variable method has been applied to obtain the two complex analytic potential functions (Goursat's functions) $\phi(\zeta), \psi(\zeta)$ and the three stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ for the first and second boundary value problems in

thermoelasticity of the same previous domain, for an infinite plate weakened by a curvilinear hole C conformally mapped outside a unit circle γ by the rational mapping function,

$$z = c w(\zeta) = c \frac{\zeta + \sum_{j=1}^p m_j \zeta^{-j}}{\prod_{j=1}^p (1 - n_j \zeta^{-1})} \quad (c > 0, |n_j| < 1) \quad (14)$$

when a heat $\Theta = qy$ is flowing uniformly in the negative direction of y -axis. The increasing temperature Θ is assumed to be constant across the thickness the plate. Here, in (14) m 's and n 's are real parameters restricted such that $w(\zeta)$ does not vanish or become infinite outside the unit circle γ . The interesting cases when the shape of the hole is an ellipse, hypotrochoidal, a crescent or a cut having the shape of a circular arc are included as special ones. Holes corresponding to certain combinations of the parameters m 's and n 's are sketched (see Fig. 1-6). Some applications of the first and second fundamental problems of the infinite plate with a curvilinear hole having several poles are investigated.

2. METHOD OF SOLUTION

In view of the definition $z = x + iy = \rho e^{i\theta}$, and the rational mapping function of Eq. (13), the solution of the boundary value problem of Eq. (1) is given by

$$\Theta = q \left[\operatorname{Im} z + \frac{r_0^2 \sin^2 \theta}{\operatorname{Im} z} \right] \quad (15)$$

In determining the thermoelastic potential of Eq. (2), the uniform heat flow may be disregarded. So the formula (2) takes the form

$$\nabla^2 \Phi = q(1+\nu) \alpha \frac{r_0^2 \sin^2 \theta}{\operatorname{Im} z} \quad (16)$$

Using the definition of $\nabla^2 \Phi$ in polar coordinates and solving (16) in this domain, we have

$$\Phi(z, \bar{z}) = \frac{q(1+\nu)r_0^2}{4} \operatorname{Im} z \cdot \ell n z \bar{z} \quad (17)$$

Hence the values of Θ and Φ are completely determined.

The expression $w(\zeta^{-1})/w'(\zeta)$ can be written in the form

$$\frac{w(\zeta^{-1})}{w'(\zeta)} = \alpha(\zeta^{-1}) + \beta(\zeta), \quad (18)$$

where

$$\alpha(\zeta) = \sum_{\ell=1}^p \frac{h_{\ell}}{\zeta - n_{\ell}},$$

$$h_\ell = \frac{\left[n_\ell^{P+1} + \sum_{j=1}^P m_j n_\ell^{P-j} \right] \left(\prod_{j=1}^P (1 - n_j n_\ell) \right) \left(\prod_{\substack{j=1 \\ \ell \neq j}}^P (n_\ell - n_j) \right)^{-1}}{\left[(1+\ell) + \sum_{j=1}^P (P-j) m_j n_\ell^{1+j} - \sum_{j=1}^P \frac{1}{1-n_j n_\ell} - \left(\sum_{j=1}^P m_j n_\ell^{1+j} \right) \sum_{j=1}^P \frac{1}{1-n_j n_\ell} \right]}, \quad (19)$$

and $\beta(\zeta)$ is a regular function for $|\zeta| > 1$.

Using (7), (8) and (18), the boundary condition (9) may be written in the form

$$K\phi(\sigma) - \alpha(\sigma)\overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = f_*(\sigma), \quad (20)$$

where $\sigma = e^{i\theta}$ denotes the values of ζ on the boundary of the unit circle γ , while

$$\psi_*(\zeta) = \psi(\zeta) + \beta(\zeta)\phi'(\zeta),$$

$$f_*(\zeta) = F(\zeta) - ck_1 \Gamma \zeta + c\overline{\Gamma^*} \zeta^{-1} + N(\zeta)(\alpha(\zeta) + \overline{\beta(\zeta)}),$$

$$N(\zeta) = c\overline{\Gamma} - \frac{S_x - iS_y}{2\pi(1+k)} \zeta,$$

and

$$F(\zeta) = f(t) = f(cw(\zeta)) \quad (21)$$

Assume that the derivatives of $F(\sigma)$ must satisfy the Hölder condition.

Our aim is to determine the functions $\phi(\zeta)$ and $\psi(\zeta)$, for the various fundamental problems, from (20). For this multiplying both sides of (20) by $\frac{1}{2\pi i} \frac{d\sigma}{\sigma - \zeta}$ then

integrating the result around the unit circle γ and evaluating the integrals thus formulated by residue theorems, one has

$$K\phi(\zeta) + \frac{1}{2\pi i} \int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)} d\sigma}{\sigma - \zeta} = c\overline{\Gamma^*} \zeta^{-1} - A(\zeta) + \sum_{j=1}^P \frac{h_j N(n_j)}{\zeta - n_j} \quad (22)$$

where

$$A(\zeta) = -\frac{1}{2\pi i} \sum_{v=0}^{\infty} \frac{1}{\zeta^{1+v}} \int \sigma^v F(\sigma) d\sigma \quad (|\zeta| > 1). \quad (23)$$

Using (2.4) we have

$$\frac{1}{2\pi i} \int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)} d\sigma}{\sigma - \zeta} = c \sum_{j=1}^P \frac{h_j b_j}{n_j - \zeta}, \quad (24)$$

where b_j 's are complex constant to be determined. Hence

$$-K\phi(\zeta) = A(\zeta) - c\overline{\Gamma^*} \zeta^{-1} + \sum_{j=1}^P \frac{h_j}{n_j - \zeta} (cb_j + N(n_j)). \quad (25)$$

Differentiating (25) with respect to ζ , and using the result of $\overline{\phi'(\sigma)}$ in (24), we obtain

$$c K b_j + c n_j^2 \Gamma^* + d_j h_j (c \bar{b}_j + \overline{N(n_j)}) = -\overline{A'(n_j)},$$

where

$$d_{j,\ell} = n_j^2 (1 - n_j n_\ell)^{-2} \quad (\ell = 1, \dots, P).$$

Hence

$$b_j = \frac{K E_j - h_j d_{j,\ell} \bar{E}_j}{c(K^2 - h_j^2 d_{j,\ell}^2)},$$

where

$$E_j = -\overline{A'(n_j)} - c \Gamma^* n_j^2 - h_j d_{j,k} \overline{N(n_j)} \quad (26)$$

Also, from (20), $\psi(\zeta)$ can be determined in the form

$$\psi(\zeta) = \frac{c k_1 \bar{\Gamma}}{\zeta} - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi_*(\zeta) + \sum_{j=1}^P \frac{h_j \zeta}{1 - n_j \zeta} \phi_*(n_j^{-1}) + B(\zeta) - B \quad (27)$$

where

$$\phi_*(\zeta) = \phi'(\zeta) + \overline{N(\zeta)},$$

$$B(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{\overline{F(\sigma)}}{\sigma - \zeta} d\sigma,$$

and

$$B = \frac{1}{2\pi i} \int_{\gamma} \overline{F(\sigma)} \frac{d\sigma}{\sigma}. \quad (28)$$

Using (15), (17), (25) and (27) in (6), after some derivatives and algebraic relations, we have

$$\sigma_{xx} = 2G[-\gamma(z^2 + 4z\bar{z} + \bar{z}^2) \operatorname{Im} z + \operatorname{Re}(2\phi'(\zeta) - M(\zeta, \bar{\zeta}))]$$

$$\sigma_{yy} = 2G[\gamma(z^2 + \bar{z}^2) \operatorname{Im} z + \operatorname{Re}(2\phi'(\zeta) + M(\zeta, \bar{\zeta}))]$$

and

$$\sigma_{xy} = 4G[\gamma(z\bar{z} - 2(\operatorname{Im} z)^2) \operatorname{Re} z + \operatorname{Im} M(\zeta, \bar{\zeta})], \quad (29)$$

where

$$\gamma = \frac{(1+\nu)r_0^2}{4z^2\bar{z}^2},$$

$$\begin{aligned} M(\zeta, \bar{\zeta}) = & c w(\zeta) \overline{\phi''(\zeta)} - c K \Gamma \zeta^2 - \frac{d}{d\zeta} \left[\frac{w(\zeta)}{w'(\zeta)} \overline{\phi_1(\zeta)} \right] + \\ & + \sum_{j=1}^P \frac{h_j \zeta^2}{(\zeta - n_j)^2} \phi_*(n_j) + \overline{B'(\zeta)}. \end{aligned} \quad (30)$$

3. SPECIAL CASES

(I) For $n_j = 0$, $0 \leq m_j \leq 1$, $j = 1, 2, \dots, P$, we get the rational mapping function

$$z = c(\zeta + m_1 \zeta^{-1} + m_2 \zeta^{-2} + \dots + m_P \zeta^{-P}). \quad (31)$$

The physical interest of the mapping (3.1) comes from the following

1. A circle of radius c : $m_j = 0$, $j = 1, 2, \dots, P$.
2. An ellipse $m_j = 0$, $j \geq 2$.
3. A square with rounded corners with diagonals parallel to the x - and y -axis $m_1 = m_j = 0$, $j \geq 3$, $m_2 = \text{about } 0.1$, the same square with its sides parallel to the axis $m_1 = m_j = 0$, $j \geq 3$, $m_2 = \text{about } -0.1$.
4. An ovaloid $m_j = 0$, $j \geq 3$, $m_1 = \text{about } 0.3$, $m_2 = \text{about } -0.05$.
5. A triangle $m_j = 0$, $j \neq 3$.

More information and applications on technology for the special cases of the mapping (31) are found in [1,2,4,5].

(II) For $m_j = 0$, $j \geq 2$, $m_1 = -1$ the boundary C degenerated into a circular cut with P poles (see Fig. 1, $P = 1$). And for m_1 takes values near -1 , $P = 1$, the edge of the hole resemble the shape of a crescent.

Many interesting cases may be derived according to some methodology (see Figs. 1-6).

4. EXAMPLES

4.1. Curvilinear hole for an infinite plate subjected to a uniform tensile stress and flowing heat

For $K = -1$, $\Gamma = \frac{P}{4}$, $\Gamma^* = -\frac{P}{2} e^{-2i\theta}$, $0 \leq \theta \leq 2\pi$, $S_x = S_y = f = 0$ and $\Theta = qy$ we

have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P , making an angle θ with the x -axis, and a flowing heat in the negative direction of y -axis. The plate weakened by a curvilinear hole C having finite poles which is free from stress and the Goursat's functions (25)-(27) take the form

$$\phi(\zeta) = \frac{cP}{2} \left[\exp(2i\theta) \zeta^{-1} + \sum_{j=1}^P \frac{h_j Q_j}{n_j - \zeta} \right], \quad (32)$$

$$\psi(\zeta) = -\frac{cP}{4} \zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi_*(\zeta) + \sum_{j=1}^P \frac{h_j \phi_*(n_j^{-1}) \zeta}{1 - n_j \zeta}, \quad (33)$$

where

$$Q_j = \left[\frac{\frac{1}{2} - n_j^2 \cos 2\theta}{1 - h_j d_j} + \frac{n_j^2 \sin 2\theta}{1 + h_j d_j} \right],$$

and

$$\phi_*(\zeta) = \phi'(\zeta) + \frac{cP}{4}. \quad (34)$$

The stress components can be obtained, directly from Eq. (29), with the aid of (32)-(34).

4.2. Curvilinear hole having finite poles the edge of which is subject to a uniform pressure in the present of flowing heat

For $K = -1$, $S_x = S_y = \Gamma = \Gamma^* = 0$, $\Theta = qy$ and $f(t) = Pt$, where P is a real constant. The formulae (25) – (27) become

$$\phi(\zeta) = \sum_{k,j=1}^P \frac{(n_k^{P+1} + m_j n_k^{P-j})(1 - h_j d_{j,k})}{(n_k - \zeta)(1 + h_j d_{j,k})} P, \quad (35)$$

and

$$\psi(\zeta) = -\frac{w(\zeta^{-1})}{w'(\zeta)}\phi'(\zeta) - cP \sum_{j=1}^P (n_j + \zeta^{-1}) + \sum_{j=1}^P \frac{h_j \zeta}{1 - n_j \zeta} \phi'(n_j^{-1}). \quad (36)$$

Hence (35) – (36) give the solution of the first fundamental problem when the edge of the hole is subject to uniform pressure P . Putting in (35) and (36) $-iT$ instead of P , we have the first fundamental problem when the edge of the hole is subject to a uniform tangential stress T . Using (29), the stress components, in this case, can be directly obtained.

4.3. Uni-directional tension of an infinite plate with a rigid curvilinear centre and a heat flowing in the negative direction of y-axis

For $K = k$, $\Gamma = \frac{P}{4}$, $\Gamma^* = -\frac{P}{2}e^{-2i\theta}$, $S_x = S_y = 0$, $f(t) = 2iPet$ and $\Theta = qy$, we have the two complex functions

$$-k\phi(\zeta) = \frac{cP}{2}e^{2i\theta}\zeta^{-1} + 2ic\mu\varepsilon \sum_{j,\ell=1}^P \frac{k(n_\ell^{P+1} + m_j n_\ell^{P-j})}{(n_\ell - \zeta)(k + h_j d_{j,\ell})} + \frac{cP}{2} \sum_{j,\ell=1}^P \frac{h_j Q_{j,\ell}^{(2)}}{n_j - \zeta}, \quad (37)$$

$$\begin{aligned} \psi(\zeta) = & 2ic\mu\varepsilon \sum_{j=1}^P n_j + c\left(-\frac{kP}{4} + 2\mu\varepsilon i\right)\zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \\ & + \sum_{j=1}^P \frac{h_j \zeta}{1 - n_j \zeta} \phi_*(n_j^{-1}), \end{aligned} \quad (38)$$

where

$$Q_{j,\ell}^{(2)} = \frac{(k + 2n_j^2 \cos 2\theta)}{2(k + h_j d_{j,\ell})} - i \frac{n_j^2 \sin 2\theta}{(k - h_j d_{j,\ell})},$$

and

$$\phi_*(\zeta) = \phi'(\zeta) + \frac{cP}{4}. \quad (39)$$

Therefore, we have the case of uni-directional tension of an infinite plate with a rigid curvilinear centre. The constant ε can be determined from the condition that the

resultant moment of the forces acting on the curvilinear centre from the surrounding material must vanish i.e.

$$M = \operatorname{Re} \left\{ \int \left[\psi(\zeta) - \frac{cP}{2} e^{-2i\theta} \zeta \right] w'(\zeta) d\zeta \right\} = 0 \quad (40)$$

Hence, we have

$$\varepsilon = \frac{P(1+k) \left(\sum_{j=1}^P n_j + N \right) \sin 2\theta}{4\mu [1 + \sum_j n_j + L]},$$

where

$$N = \sum_{j,\ell=1}^P \frac{h_j n_j^2 n_\ell^2}{(1 - n_j n_\ell)^2 (k - h_j d_{j,\ell})},$$

$$L = \sum_{j,\ell=1}^P \frac{n_k^{P+3} + m_j n_\ell^{P-j+2}}{(1 - n_j n_\ell)^2 (k - h_j d_{j,\ell})}. \quad (41)$$

Case 1: Bi-axial tension with $K = k$, $S_x = S_y = 0$, $\Gamma = \bar{\Gamma} = \frac{P}{2}$, $\Gamma^* = 0$ and $f(t) = 2\mu g(t)$,

$\Theta = qy$ under the same condition of example (34), one obviously will have $\varepsilon = 0$ and the two complex functions are

$$\phi(\zeta) = \frac{cP}{2} \sum_{j,\ell=1}^P \frac{h_j}{(\zeta - n_j)(k + h_j d_{j,\ell})} \quad (42)$$

$$\psi(\zeta) = \frac{ckP}{2} \zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi_*(\zeta) + \sum_{j=1}^P \frac{h_j \zeta}{1 - n_j \zeta} \phi_*(n_j^{-1}), \quad (43)$$

where

$$\phi_*(\zeta) = \phi' + \frac{cP}{2}. \quad (44)$$

Case 2: When the curvilinear centre is not allowed to rotate:

Under the condition of the preceding example (34) the rigid curvilinear kernel is restrained in its original position by a couple which is not sufficient to rotate the kernel, then $\varepsilon = 0$.

Hence the two complex functions are

$$-k\phi(\zeta) = \frac{cP}{2} e^{2i\theta} \zeta^{-1} + \frac{cP}{2} \sum_{j,\ell=1}^P \frac{h_j Q_{j,\ell}^{(2)}}{n_j - \zeta}, \quad (45)$$

and

$$\psi(\zeta) = \frac{ckP}{4} \zeta^{-1} - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi_*(\zeta) + \sum_{j=1}^P \frac{h_j \zeta}{1 - n_j \zeta} \phi_*(n_j^{-1}), \quad (46)$$

where $Q_{j,\ell}^{(2)}$ and $\phi_*(\zeta)$ are given by (4.8).

The resultant moment is given by

$$M = \frac{c P \pi (1+k)}{k} \left[\sum_{j=1}^P n_j + \sum_{j,\ell=1}^P \frac{h_j n_j^2 n_\ell}{(1-n_j n_\ell)^2 (k-h_j d_{j,\ell})} \right] \sin 2\theta. \quad (47)$$

Case 3: When a couple with a given moment acts on the curvilinear hole.

We assume that the stresses vanish at infinity. Under the same conditions of example (34), the complex functions take the following form

$$\phi(\zeta) = 2ic\mu\varepsilon \sum_{j,\ell=1}^P \frac{(n_\ell^{P+1} + m_j n_\ell^{P-j})}{(\zeta - n_\ell)(k + h_j d_{j,\ell})}, \quad (48)$$

$$\psi(\zeta) = 2ic\mu\varepsilon \left(\sum_{j=1}^P n_j + \zeta^{-1} \right) - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi' + \sum_{j=1}^P \frac{h_j \zeta}{1-n_j \zeta} \phi'(n_j^{-1}), \quad (49)$$

where

$$\varepsilon = \frac{Mk}{4\pi c\mu \left[1 + \sum_{j=1}^P n_j + L \right]}, \quad (50)$$

and L is given by (41). The components of stress can be obtained, in each case, by using (29)-(30).

4.4. The force acts on the centre of the curvilinear hole:

In this case, it will be assumed that the stresses vanish at infinity. It is easily seen that the kernel does not rotate. In general, the kernel remains in its original position.

Hence, one assumes $\Gamma = \Gamma^* = f(t) = 0$, $K = k$ and $\Theta = qy$, the Goursat's functions are

$$\begin{aligned} \phi(\zeta) = \frac{c}{2\pi k(1+k)} \sum_{j,\ell=1}^P \frac{ch_j}{\zeta - n_j} & \left[\frac{kh_j d_{j,\ell}(S_x + iS_y)}{c(k^2 - h_j d_{j,\ell})} - \right. \\ & \left. - \left\{ 1 + \frac{h_j^2 d_{j,\ell}^2}{c(k^2 - h_j^2 d_{j,\ell}^2)} \right\} (S_x - iS_y) \right], \end{aligned} \quad (51)$$

$$\psi(\zeta) = \sum_{j=1}^P \frac{h_j \zeta}{1-n_j \zeta} \phi_*(n_j^{-1}) - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi_*(\zeta), \quad (52)$$

where

$$\phi_*(\zeta) = \phi'(\zeta) - \frac{S_x + iS_y}{2\pi(1+k)\zeta}. \quad (53)$$

Therefore, we have the solution of the second fundamental problem in the case, when a force (S_x, S_y) acts on the centre of the curvilinear kernel and when a heat is flowing in the negative direction of y -axis.

5. CONCLUSION

From the above results and discussions, the following may be concluded:

- 1) In the theory of two dimensional linear elasticity one of the most useful techniques for the solution of boundary value problems for awkwardly shaped region is to transformation the region into one simpler shape.
- 2) The mapping function (14) maps the curvilinear hole C in the z -plane onto the domain outside a unit circle ζ -plane under the condition $w'(\zeta)$ does not vanish or become infinite outside γ .
- 3) The physical interest of the mapping (14) comes from its strong special cases which discussed here, moreover many new cases can be obtained according to the technology of the work, where the fundamental problems of the infinite plate with a curvilinear hole having finite poles are not discussed before.
- 4) The complex variable method (Cauchy method) is considered as one of the best method for solving the integro-differential equations (22) taken on closed contour γ , and obtained the two complex potential functions $\phi(z)$ and $\psi(z)$ directly.
- 5) Here, we assumed the conformal mapping of Eq. (14), which has n_ℓ singular points, $\ell = 1, 2, \dots, p$ i.e we can say that $z \rightarrow \infty$ at $\zeta = n_\ell$, $\ell = 1, 2, \dots, p$. Also at infinity we can say that the function $\Theta_{\text{inf}}(z, \bar{z})$ is equivalent to the term $\Theta_{\text{inf}} = qy$, and the thermoelasticity potential $\Phi = (z, \bar{z})$ is equivalent, at infinity, to the value

$$\Phi_{\text{inf}}(z, \bar{z}) = \frac{(1+\nu)\alpha q}{8} z \bar{z} \text{Im} z$$

So, we can say that the stress components, at infinity, are relative to the following equations

$$\sigma_{xx} + \sigma_{yy} + 2(1+\alpha)\Theta_{\text{inf}} = 8GA$$

$$\sigma_{yy} - \sigma_{xx} + 4G \left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \bar{z}^2} \right] \Phi_{\text{inf}}(z, \bar{z}) = 8GB$$

$$\sigma_{xy} + 4iG \left[\frac{\partial^2}{\partial \bar{z}^2} - \frac{\partial^2}{\partial z^2} \right] \Phi_{\text{inf}}(z, \bar{z}) = 4GC$$

where the real constants A , B and C are related to the stress at infinity.

- 6) The complex function $\phi(z)$ is considered the solution of the integral equation with Cauchy kernel

$$\phi'(\zeta) + \frac{1}{2\pi i} \int_{\gamma} K(\zeta, \sigma) \overline{\phi'(\sigma)} d\sigma = A'(\zeta) - \lambda w'(\zeta)$$

where

$$\lambda = \frac{\overline{\phi'(0)}}{w'(0)}, \quad K(\zeta, \sigma) = \frac{w(\sigma) - w(\zeta) - (\sigma - \zeta) w'(\zeta)}{w'(\sigma)(\sigma - \zeta)^2},$$

and

$w(\zeta)$ is given by (14).

- 7) This paper can be considered as a generalization of the work of the infinite plate with a curvilinear hole under certain conditions presented in [1-10]
Future work: The influence of a small hole having finite poles of rigid inclusion on the transverse flexure of thin plates will be discussed.

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