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GREEN'S FUNCTION FOR PARALLEL PLANES AND AN OPEN RECTANGULAR CHANNEL-FLOW.

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Abstract- Here we use a more convenient technique to generate a faster convergent Green's function needed for solving Laplace's equation in two cases: the first domain is bounded by two parallel planes; and the second is an infinite open rectangular prism. Green's function usually is expressed as a series of images which is slowly convergent, and that is why we transform it into an integral representation which is rapidly convergent and stable. Many examples are herein given and discussed for the numerical applications of the above two cases; and then we make a comparison between our calculations and some others.

1. INTRODUCTION

Various authors are interested in Green's function, which has numerous applications in different fields of engineering and science. Green's function from one dimension up to three dimension have been discussed and deduced throughout [1, 2, 3, 4] by means of different techniques, e.g., eigenfunction expansion, series of images, integral representation, etc. Many implementations of Green's function are found throughout the solution of boundary value problems (heat equation, wave equation, Laplace equation,). In this paper we deduce a good suitable convergent formula of Green's function which is needed in solving Laplace equation for two domains, namely, one between two parallel walls and the second in an infinite open rectangular prism [5,7]; both sketched in Fig (1).

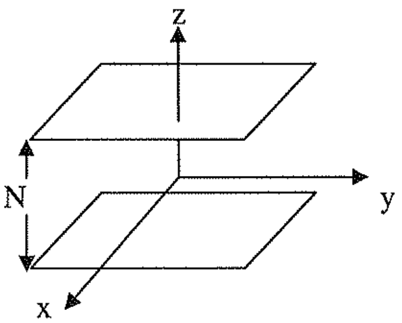

Fig (1-1) Parallel plates

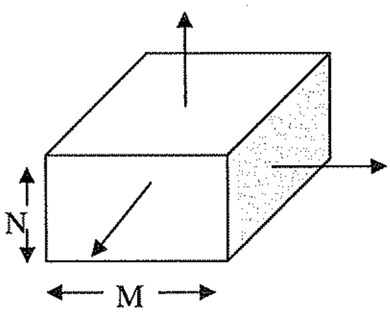

Fig (1-2) Rectangular channel

Here G_1 is the Green's function, which satisfies the periodic boundary conditions on Parallel planes $z=\pm N/2$, and with the potential of a source located half way between the parallel planes. The subscript of G indicates that G_1 is periodic in one direction. Likewise G_2 is the potential of a source located at the center of the rectangular prism defined by the two pairs of parallel planes $y=\pm M/2$ and $z=\pm N/2$; and in this case G_2 satisfies the periodic boundary conditions in two directions. The Green's function G_2 can be constructed from the free-space singularity $1/r$ such that $r=\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{1/2}$ where (x, y, z) is the Cartesian coordinate system and (ξ,η,ζ) is the location of source point inside the prism. However we must note that Green's function in terms of sums of images usually contain series, which converge very slowly and so are unsuitable for numerical work.

2. FORMULATION OF THE PROBLEM

Green's functions, which satisfy homogenous Neumann or Dirichlet boundary conditions on $z=\pm N/2$ can be constructed from G_1 by the method of images, where G_1 with V_1 satisfy the Laplace equation

$$\nabla^2(G_1+V_1)=0, \tag{1}$$

such that $\nabla^2=[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}]$ and V_1 is the free-space Green's function that satisfies

$$\nabla^2 V_1=-4\pi\delta(x)\delta(y)\sum_{k=-\infty}^{\infty}\delta(z-kN), \tag{2}$$

where δ is the Dirac delta function. Similarly for G_2, we obtain that

$$\nabla^2(G_2+V_2)=0, \tag{3}$$

$$\nabla^2 V_2=-4\pi\,\delta(x)\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(y-jM)\delta(z-kN). \tag{4}$$

The resulting infinite series in (2) and (4), as we mentioned above, are not suitable for computation but we take them as the starting point for our discussion. Equivalently we can define G_1 and G_2 respectively the solutions of Poisson equations

$$\nabla^2 G_1=-4\pi\delta(x)\delta(y)\sum_{k=-\infty}^{\infty}\delta(z-kN), \tag{5}$$

$$\nabla^2 G_2=-4\pi\delta(x)\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(y-jM)\delta(z-kN). \tag{6}$$

By using the method of images [1-5, 7], the solutions of (5) and (6) are then given respectively by

$$G_1(R_1,z;N)=(R_1^2+z^2)^{-1/2}+\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\left\{\left[R_1^2+(z-kN)^2\right]^{-1/2}-|kN|^{-1}\right\}, \tag{7}$$

$$G_2(x,y,z;M,N)=R_2^{-1}+\sum_{j,k}\left\{\left[x^2+(y-jM)^2+(z-kN)^2\right]^{-1/2}-\left[(jM)^2+(kN)^2\right]^{-1/2}\right\}, \tag{8}$$

where $R_1=\sqrt{x^2+y^2}$, $R_2=\sqrt{x^2+y^2+z^2}$, and in (8) the summation is take over all combinations of positive and negative integers j, k except $j = k = 0$. In the infinite series in (7) and (8), a constant, $|kN|^{-1}$ and $\left[(jM)^2+(kN)^2\right]^{-1/2}$, respectively, is subtract from each term to make them converge. In the next section a rectangular channel was taken to be of width M. Without loss of generality its height N can be assumed to be equal to unity, with unbounded length. And also our discussion focuses on derivation of alternative forms of Green's function, which converge more rapidly than those, given by (7) and (8). The first form of Green's function is the improvement eigenfunctions expansion and the second one is the integral representations for the two cases under consideration.

3. FORMS OF THE SOLUTION

The classical forms of Green's function (7) and (8) are the eigenfunction expansions and methods of deriving them are found throughout [1-4]. In the next part of this section we deduce their integral representations by aid of [8].

3.1 Eigenfunction expansions of G_1 and G_2

An alternative representation of G_1 as an eigenfunction expansion given throughout [5, 7] and repeated here for convenience as

$$G_1(R_1,z)=-2(C+\ln\frac{R_1}{2})+4\sum_{k=1}^{\infty}K_0(2\pi kR_1)\cos(2\pi kz), \tag{9}$$

where C is the Euler constant and K_0 is the modified Bessel function of second kind of order zero. For G_2 we start by taking the Fourier transform of equation (6) with respect to x, thus obtaining

$$\left[-(2\pi A)^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\tilde{G}_2(A,y,z)=-4\pi\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(y-jM)\delta(z-kN), \tag{10}$$

where

$$\left.\begin{aligned} G_2(x,y,z)&=\int_{-\infty}^{\infty}e^{i2\pi Ax}\tilde{G}_2(A,y,z)\,dA\\ \tilde{G}_2(A,y,z)&=\int_{-\infty}^{\infty}e^{-i2\pi Ax}G_2(x,y,z)\,dx\end{aligned}\right\}. \tag{11}$$

For the right hand side of equation (10) we use the Poisson summation formula [1, 8,10], which can be formally write as

$$\left.\begin{aligned}&\sum_{j=-\infty}^{\infty}\delta(u+2\pi j)=\frac{1}{2\pi}\sum_{\mu=-\infty}^{\infty}F(\mu),\\&F(\mu)=\int_{-\infty}^{\infty}e^{-i2\pi\mu j}\,\delta(u+2\pi j)\,dj,\end{aligned}\right\}. \tag{12}$$

Then equation (10) takes the form

$$\left[-(2\pi A)^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\tilde{G}_2(A,y,z)=-\frac{4\pi}{M}\left[1+2\sum_{j=1}^{\infty}\cos(\frac{2\pi jy}{M})\right]\left[1+2\sum_{k=1}^{\infty}\cos(2\pi kz)\right]. \tag{13}$$

A solution can be given by the formula

$$\tilde{G}_2(A,y,z)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\tilde{a}_{jk}(A)\cos(\frac{2\pi jy}{M})\cos(2\pi kz) \tag{14}$$

where the unknown coefficients $\tilde{a}_{jk}$ can be obtained by substituting from (14) into (13) and collecting the coefficients of like terms giving

$$\tilde{a}_{jk}(A)=\frac{a_j a_k}{\pi M}\left[A^2+(j/M)^2+k^2\right]^{-1},\qquad a_l=\begin{cases}1 & if\ l=0\\ 2 & if\ l\geq 1\end{cases}. \tag{15}$$

Therefore to obtain the eigenfunction expansion of G_2, we take the inverse Fourier transform of (14) with respect to A, where we use Cauchy integral formula and gamma function properties in our calculations

$$G_2(x,y,z;M)=\alpha-\frac{2\pi}{M}|x|+\sum_{\substack{j,k=0\\(j+k)>0}}^{\infty}a_j a_k\frac{e^{-2\pi\sqrt{(j/M)^2+k^2}|x|}}{M\sqrt{(j/M)^2+k^2}}\cos(\frac{2\pi jy}{M})\cos(2\pi kz), \tag{16}$$

where α is an additional constant to make equation (16) compatible with (8); and as in [5]

$$\alpha=2\left[\ln(\frac{4\pi}{M})-C\right]-8\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}K_0(2\pi jkM). \tag{17}$$

During our calculations when we use equation (16); α can be calculated once and for all for a given M value.

3.2 Integral representation

A more suitable formula for Green's function, for the above mentioned two cases, can be obtained by deriving the integral representations of G_1 and G_2; respectively. This method represents transforming Green's function into a more convergent and suitable form for calculation.

For G_1 we start from the eigenfunction expansion (9) of G_1 and the integral form of the modified Bessel function

$$K_0(kx) = \int_1^{\infty} \frac{e^{-kxu}}{\sqrt{u^2-1}}\,du \; . \tag{18}$$

Therefore to obtain the integral representation of G_1, we begin with the summed geometric progression

$$\sum_{k=1}^{\infty} e^{i2\pi kz} e^{-2\pi kR_1 u} = \frac{e^{i2\pi z}}{e^{2\pi R_1 u} - e^{i2\pi z}} \, . \tag{19}$$

Multiplying both sides, of the preceding equation, by $\frac{1}{\sqrt{u^2-1}}$ and integrating with respect to u from 1 to ∞, we obtain that

$$\sum_{k=1}^{\infty} K_0(2\pi kR_1)\cos(2\pi kz) = \mathrm{Re}\left[\int_1^{\infty} \frac{e^{i2\pi z}}{\sqrt{u^2-1}\left(e^{2\pi R_1 u} - e^{i2\pi z}\right)}\,du\right]. \tag{20}$$

The above integral has an integrable singularity at u=1, and to avoid this singularity we can generate an alternative form by making the substitution $t = \left(u^2-1\right)^{1/2}$. Therefore the final expression of G_1 takes the form

$$G_1(R_1, z) = -2\left(C + \ln(R_1/2)\right) + 2\int_0^{\infty} \frac{\left(\cos(2\pi z) - e^{-2\pi R_1 \beta}\right)}{\beta\left(\cosh(2\pi R_1 \beta) - \cos(2\pi z)\right)}\,dt \, , \tag{21}$$

Where $\beta = (t^2+1)^{1/2}$ and C=. 577215665. We must note that, the integrand in the left-hand-side of above equation tends to zero as t tends to infinity, and also absolutely convergent for any real values of R_1 and z. Moreover, solving $\cosh(2\pi R_1 \beta) - \cos(2\pi z) = 0$ gives four imaginary roots only; therefore the integral has no poles on the real t-axis. Hence there is no difficulty in evaluating the integral and consequently Green's function using equation (21).

For G_2, its integral representation can be deduced using that of G_1; since G_2 can be viewed as the superposition of an infinite number of G_1 functions rowed in the z-direction. Therefore the two-equation (7) and (8) give

$$G_2(x, y, z; M) = G_1(R_1, z) + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \left[G_1\left(\sqrt{x^2 + (y - jM)^2}, z\right) - G_1\left(\left|jM\right|, 0\right)\right] \tag{22}$$

Here we also must note that, a finite constant $G_1\left(\left|jM, 0\right|\right)$, which independent of any variable, subtracted from each term in infinite series to secure convergence. The infinite summation in the right hand side of (22), after substitution from (21) into (22) for the two occurrences of G_1, becomes

$$-2\sum_{j=1}^{\infty}\left[\ln\sqrt{x^2+(y+jM)^2}+\ln\sqrt{x^2+(y-jM)^2}-2\ln(jM)\right]$$
$$-8\sum_{j=1}^{\infty}\int_0^{\infty}\frac{dt}{\beta\left[e^{2\pi jM\beta}-1\right]}+2\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\int_0^{\infty}\frac{\cos(2\pi z)-e^{-2\pi\sqrt{x^2+(y-jM)^2}\beta}}{\beta\left[\cosh(2\pi\sqrt{x^2+(y-jM)^2}\beta)-\cos(2\pi z)\right]}dt. \tag{23}$$

For the first infinite summation in above equation, we can replace it in (23) by the infinite product

$$-2\ln\left|\prod_{j=1}^{\infty}\left(1-(\frac{Z}{jM})^2\right)\right|, \tag{24}$$

where we use the complex variable $Z=y+ix$, and finally using the sine identity

$$\sin(\theta)=\theta\prod_{j=1}^{\infty}\left\{1-(\frac{\theta}{j\pi})^2\right\}. \tag{25}$$

For the second infinite summation in (23), by taking $\beta_2=2\pi M\beta$ such that M must be greater than zero and the different real values of β starting from one, therefore

$$\sum_{j=1}^{\infty}\frac{1}{e^{\beta_2 j}-1}\cong\sum_{j=1}^{\infty}\frac{1}{e^{\beta_2 j}}, \quad \forall\ \beta_2>>1\ . \tag{26}$$

The left-hand-side of above equation is a geometric series and its sum equal to $1/(e^{\beta_2}-1)$. Therefore the second infinite summation in (23) can be replaced by the very well known approximate formula:

$$\sum_{j=1}^{\infty}\frac{1}{e^{2\pi jM\beta}-1}\cong\frac{1}{e^{2\pi M\beta}-1}\quad . \tag{27}$$

And from our numerical experiment we have found that the two sides of above equation are identical up to 12th decimal places when j sufficiently tends to infinity ($j\geq 10$). Finally we found that the last infinite summation in (23) can be rearranged and can be expressed by the formula

$$\sum_{j=1}^{\infty}\left[\frac{\sinh(\beta_{+j})}{\cosh(\beta_{+j})-\cos(2\pi z)}+\frac{\sinh(\beta_{-j})}{\cosh(\beta_{-j})-\cos(2\pi z)}-2\right], \tag{28}$$

Where $\beta_{\pm j}=2\pi\sqrt{x^2+(y\pm jM)^2}\ \beta$. Therefore equation (22) can be rearranged as

$$G_2(x,y,z;M)=G_1(R_1,z)-2\ln\left|\frac{\sin(\pi Z/M)}{\pi Z/M}\right|-8\int_0^{\infty}\frac{dt}{e^{2\pi M\beta}-1}$$
$$+2\sum_{j=1}^{\infty}\int_0^{\infty}\left\{\beta^{-1}\left[\frac{\sinh(\beta_{+j})}{\cosh(\beta_{+j})-\cos(2\pi z)}+\frac{\sinh(\beta_{-j})}{\cosh(\beta_{-j})-\cos(2\pi z)}-2\right]\right\}dt \tag{29}$$

where G_1 is computed using (21). The above two formulae (21) and (29) for G_1 and G_2 respectively are rapidly convergent and more suitable for numerical work.

4. APPLICATIONS AND DISCUSSION

We shall introduce here a comparison between our integral representation technique which was described in the preceding section for G_1 and G_2, and those expressions given by (9) and (16); see [5, 7]. During our calculations, we have used four Fortran subroutines. The first two are developed to evaluate Green's function G_1 for two infinite parallel plates $(N=1)$ using eigenfunction expansion (9) and integral representation (21) respectively. We have used the Fortran 77 program for the special function K_0 from the Numerical Recipes Fortran package, which was found among the version 4.0 of Fortran Power Station. The other two Fortran programs are used to evaluate G_2 for an infinite rectangular prism, its high N equal unity, by mean of equation (16) and (29) respectively. As an illustrative example, our calculations for that case have been performed with the width of the prism M equal 0.5. The two subroutines for equation (21) and (29) depend on the numerical integration subroutine that evaluates the integral terms. Since Green's function is difficult to compute in the neighborhood of the singularity $(x=y=z=0)$, then for the integrals we have used a double-precision version of Quadpack routine [9], which are designed especially for computing the singular integrals by adaptive algorithms.

A comparison for the solution of a variety of problems is conveniently made to test the difference between the various techniques of the solution. The maximum absolute error for both single and double precisions between the eigenfunction expansions and integral representations, for G_1 and G_2, is shown in Table (1) for the illustrative examples given below. To test the result of each of the equation (9), (16), (21), and (29), we list the computed values of G_1 and G_2 over the selected regions

Table (1). Max. Abs. errors for the illustrative examples.

Method	Single precision	Double precision
Eqs. (9) and (21)	6.41001×10^{-5}	3.752081×10^{-9}
Eqs. (16) and (29)	1.28798×10^{-4}	9.893472×10^{-8}

in Table (2) and (3) respectively. Each table contains the two calculated values of Green's function computed by the above two methods, eigenfunction expansion and integral representation. We note that, the infinity value of the index j, in any sum of the form $\sum_{j=0}^{\infty}$, is restricted to be less than or equal 300, to achieve seven significant digits convergence. It is also noteworthy that the values of integration when the upper limit of integration are equal or greater than 20 converge. It must be noted that the eigenfunction expansions (9) and (16) are better than that integral representations (21) and (29) for the points which are close to the singularity, and for the complementary domain, the transformed equations (21) and (29) converge faster than that eigenfunction expansions (9) and (16).

Table (2). The computed values of $G_1 - (R_1^2 + z^2)^{-1/2}$ for the two parallel plates using the two equations (9) and (21), respectively.

R_1	z	$G_1 - (R_1^2 + z^2)^{-1/2}$ Using Eigenfunction expansion [Eq. (9)]	$G_1 - (R_1^2 + z^2)^{-1/2}$ Using integral representation[Eq. (21)]
0.0	0.0	0.0000000	0.0000000
0.1		-0.0118826	-0.0119467
0.2		-0.0468435	-0.0468776
0.5		-0.2600045	-0.0260006
1.0		-0.7644666	-0.7644657
2.0		-1.6544260	-1.6544260
5.0		-3.1870130	-3.1870130
0.0	0.125	0.0380492	0.0380784
0.1		0.0251386	0.0251434
0.2		-0.0125631	-0.0125526
0.5		-0.2387620	-0.2387616
1.0		-0.7578230	-0.7578223
2.0		-1.6534540	-1.6534540
5.0		-3.1869500	-3.1869500
0.0	0.250	0.1588782	0.1588825
0.1		0.1425015	0.1424786
0.2		0.0952126	0.0952133
0.5		-0.1743575	-0.1743584
1.0		-0.7382844	-0.7382844
2.0		-1.6505700	-1.6505700
5.0		-3.1867630	-3.1867630
0.0	0.375	0.3855276	0.3856086
0.1		0.3612491	0.3612455
0.2		0.2925355	0.2925493
0.5		-0.0652182	-0.0652185
1.0		-0.7070581	-0.7070587
2.0		-1.6458710	-1.6458710
5.0		-3.1864530	-3.1864530
0.0	0.500	0.7725786	0.7725892
0.1		0.7296228	0.7296171
0.2		0.6131608	0.6131377
0.5		0.0894464	0.0894500
1.0		-6.6622475	-6.6622561
2.0		-1.6395080	-1.6395080
5.0		-3.1860200	-3.1860200

Table (3). The computed values of $G_2 - (x^2 + y^2 + z^2)^{-1/2}$ for the rectangular prism, with M =0.5, using equations (16) and (29), respectively.

x	y	z	$G_2 - (x^2 + y^2 + z^2)^{-1/2}$ Using Eigenfunction expansion [Eq. (16)]	$G_2 - (x^2 + y^2 + z^2)^{-1/2}$ Using integral representation[Eq. (28)]
0.0	0.00	0.00	0.0000000	0.0000000
0.1			-0.1593445	-0.1594733
0.2			-0.6091952	-0.6096951
0.5			-3.0565170	-3.0567363
1.0			-8.5162640	-8.5164374
5.0			-57.9892300	-57.9896154
0.0	0.00	0.10	-0.0277193	-0.0277276
0.1			-0.1838328	-0.1839275
0.2			-0.6259532	-0.6249441
0.5			-3.0543370	-3.0540594
1.0			-8.5127330	-8.5125151
5.0			-57.9891900	-57.9893754
0.0	0.00	0.20	-0.0825307	-0.0824592
0.1			-0.2313914	-0.2315063
0.2			-0.6561516	-0.6559029
0.5			-3.0428260	-3.0426311
1.0			-8.5020210	-8.5024401
5.0			-57.9890700	-57.9893556
0.0	0.00	0.50	0.2847325	0.2846821
0.1			0.1274853	0.1271886
0.2			-0.3206990	-0.3209362
0.5			-2.8230850	-2.8229236
1.0			-8.4256360	-8.4250198
5.0			-57.9882400	-57.9887022
0.0	0.05	0.10	0.0160345	0.0168786
0.1			-0.1441912	-0.1446965
0.2			-0.5957882	-0.5958997
0.5			-3.0462070	-3.0455433
1.0			-8.5115060	-8.5115803
5.0			-57.9891900	-57.9892285
0.0	0.10	0.20	0.0507860	0.0507992
0.1			-0.1098112	-0.1098691
0.2			-0.5614933	-0.5615846
0.5			-3.0146480	-3.0146561
1.0			-8.4973450	-8.4973558
5.0			-57.9890300	-57.9890412
0.0	0.25	0.50	0.4665427	0.4665428
0.1			0.3020893	0.3020899
0.2			-0.1654895	-0.1654897
0.5			-2.7440410	-2.7440411
1.0			-8.4040890	-8.4040946
5.0			-57.9880000	-57.9880324

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