AN EXISTENCE THEOREM FOR GENERAL MIXED MULTIVALUED MILDLY NONLINEAR VARIATIONAL INEQUALITIES WITHOUT CONVEXITY

Salahuddin, Rais Ahmad and S. S. Irfan

Department of Mathematics

Aligarh Muslim University

Aligarh-202002 (India)

salahuddin12@mailcity.com

Abstract- A general mixed multivalued mildly nonlinear variational inequality is considered. We establish an existence result for (GMMNVIP) in H-spaces by using the new result of Chang et al [2].

Keywords- General mixed multivalued mildly nonlinear variational inequality, H-space, Existence theory.

1. INTRODUCTION

In the years 1983-1985, Horvath [3] introduced firstly the concept of H-space. From that time on, Bardaro and Ceppitelli [1], Lee et al [4], Siddiqi et al [5,6] obtained some important results in H-spaces.

Let X and Y be two real Banach spaces and $K \subset X$ be a nonempty closed convex subset of X , $g: K \rightarrow K$ be a continuous mapping, $T, A: K \rightarrow 2^{L(X,Y)}$ be the multivalued mappings, where $L(X,Y)$ is the space of all linear continuous mappings from X into Y .

We consider the following general mixed multivalued mildly nonlinear variational inequality (GMMNVI), recently introduced and studied by Siddiqi et al [7]. Find $u \in K$, $x \in T(u)$ and $y \in A(u)$ such that $g(u) \in K$ and

$$\langle x - y, g(v) - g(u) \rangle + b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } g(v) \in K, \quad (1.1)$$

where $b: K \times K \rightarrow \mathbb{R}$ is a nonlinear form.

2. EXISTENCE THEORY

For the sake of convenience, we first recall some definitions and results.

Definition 2.1 [1]- Let X be a topological space and S be a subset of X . Then S is said to be contractible if there is a point $x \in S$ and a continuous function $g: S \times [0,1] \rightarrow S$, such that $g(x', 0) = x'$ and $g(x', 1) = x$, for all $x' \in S$.

Definition 2.2 [1]- Let X be a topological space and $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X , indexed by the finite subsets of X .

A pair $(X, \{\Gamma_A\})$ is said to be an H-space if $A \subset B$ implies $\Gamma_A \subset \Gamma_B$;

A subset $D \subset X$ is called H-convex if $\Gamma_A \subset D$ holds for every finite subset $A \subset D$;

A subset $D \subset X$ is called weakly H-convex, if for every finite subset $A \subset D$, it results that $\Gamma_A \cap D$ is nonempty and contractible. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space;

A subset $K \subset X$ is called H-compact if there exists a compact, weakly H-convex set $D \subset X$ such that $K \cup A \subset D$ for every finite subset $A \subset X$;

A multivalued function $G: X \rightarrow 2^X$ is called H-KKM mapping if $\Gamma_A \subset \bigcup_{x \in A} G(x)$, for every finite subset $A \subset X$.

Definition 2.3 [8]- Let X and Y be two topological spaces and let $F: X \rightarrow 2^Y$ be a multifunction. F is said to be transfer open valued (resp. transfer closed valued), if for any $x \in X$, $y \in F(x)$ (resp. $y \notin F(x)$) there exist an $x' \in X$ such that

$$y \in \text{int}(F(x')) \text{ (resp. } y \notin \overline{F(x')}).$$

Lemma 2.1 [2]- Let X be a nonempty set, let Y be a topological space and let $G: X \rightarrow 2^Y$ be a multifunction.

(a) G is transfer closed-valued if and only if

$$\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \overline{G(x)}$$

(b) G is transfer open-valued if and only if

$$\bigcup_{x \in X} G(x) = \bigcup_{x \in X} \text{int}(G(x)).$$

Lemma 2.2 [2]- Let $(X, \{\Gamma_A\})$ be an H -space and let $F: X \rightarrow 2^X$ be an H -KKM multifunction such that

(a) F is transfer closed-valued.

(b) There exist a compact subset L of X and an H -compact subset K of X such that, for each weakly H -convex subset D of X with $K \subset D \subset X$, the following holds:

$$\bigcap_{x \in D} \{F(x) \cap D\} \subset L,$$

then

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

Now we state and prove the main result of this paper.

Theorem 2.1 - Let $(X, \{\Gamma_K\})$ be an H -Banach space and Y be an ordered Banach space. Assume that

1° $T, A: X \rightarrow 2^{L(X,Y)}$ be a compact valued, continuous multivalued mappings;

2° $g: X \rightarrow X$ is a continuous mapping;

3° $b: X \times X \rightarrow \mathbb{R}$ is a continuous nonlinear form;

4° the multifunction $u \rightarrow \{v \in X: \exists x \in T(v), y \in A(v) \text{ such that } \langle x - y, g(v) - g(u) \rangle + b(u, g(v)) - b(u, g(u)) \geq 0\}$ is transfer closed-valued;

5° for each $v \in X$, $B_v = \{u \in X: \exists x \in T(v), y \in A(v) \text{ such that } \langle x - y, g(v) - g(u) \rangle + b(u, g(v)) - b(u, g(u)) < 0\}$ is H -convex or empty;

6° there exists a compact set $L \subset X$ and an H -compact set $E \subset X$ such that for each weakly H -convex subset D of X with $E \subset D \subset X$,
 $\{v \in D: \exists x \in T(v), y \in A(v) \text{ such that } \langle x - y, g(v) - g(u) \rangle + b(u, g(v)) - b(u, g(u)) \geq 0, \text{ for all } u \in D\} \subset L.$

Then (GMMMNI) is solvable.

Proof- Let

$$F(u) = \{v \in X: \exists x \in T(v), y \in A(v) \text{ such that}$$

$$\langle x - y, g(v) - g(u) \rangle + b(u, g(v)) - b(u, g(u)) \geq 0\}, u \in X.$$

First we prove that F is an H -KKM mapping. Suppose that F is not an H -KKM mapping. Then there exists a finite subset $K \subset X$ such that $\Gamma_K \not\subset \bigcup_{u \in K} F(u)$. Thus there exists $z \in \Gamma_K$ such that

$$z \notin F(u), \text{ for all } u \in K,$$

that is

$$\langle x - y, g(z) - g(u) \rangle + b(u, g(z)) - b(u, g(u)) < 0, \text{ for all } u \in K, x \in T(z), y \in A(z).$$

By assumption 5°, $K \subset B_z$ and $\Gamma_K \subset B_z$, since B_z is H -convex. Therefore $z \in B_z$, that is there exists $x \in T(z)$, $y \in A(z)$ such that

$\langle x - y, g(z) - g(z) \rangle + b(z, g(z)) - b(z, g(z)) < 0$, which is not possible.

Thus $\Gamma_K \subset \bigcup_{u \in K} F(u)$ for every finite subset K of X , so that F is an H-KKM mapping. By conditions 4° and 6° we know that conditions (a) and (b) in Lemma 2.2 are satisfied.

Therefore by Lemma 2.2, we have

$$\bigcap_{u \in X} F(u) \neq \emptyset,$$

consequently, there exists $u_0 \in X$, $x_0 \in T(u_0)$, $y_0 \in A(u_0)$ such that $g(u_0) \in X$, such that

$$\langle x_0 - y_0, g(v) - g(u_0) \rangle + b(u_0, g(v)) - b(u_0, g(u_0)) \geq 0, \text{ for all } g(v) \in X.$$

This completes the proof.

REFERENCES

1. C. Bardaro and R. Ceppitelli, Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities, J. Math. Anal. Appl. 132, 484 - 490, 1988.
2. S. S. Chang, B. S. Lee, X. Wu, Y. J. Cho and G. M. Lee, On the generalized quasi-variational inequality problems, J. Math. Anal. Appl. 203, 686 - 711, 1996.
3. C. Horvath, Some results on multivalued mappings and inequalities without convexity in 'Nonlinear and Convex Analysis' Lecture Notes in Pure and Appl. Math. Springer Verlag, 1987.
4. G. M. Lee, D. S. Kim and B. S. Lee, Generalized vector variational inequality, Appl. Math. Lett., 9(1), 39 - 42, 1996.
5. A. H. Siddiqi, R. Ahmad and M. F. Khan, Generalized vector variational -like inequalities in H-space, Preprint.
6. A. H. Siddiqi, R. Ahmad and M. F. Khan, Existence of solution for generalized multivalued vector variational inequalities without convexity, Indian J. Pure and Appl. Math. 28(8), 1057-1060, 1997.
7. A. H. Siddiqi, R. Ahmad and M. F. Khan, An iterative scheme for general mixed multivalued mildly nonlinear variational inequalities, Indian J. Pure and Appl. Math. 29(6), 583 - 588, June 1998.
8. G. Q. Tian, Generalization of FKKM theorem and the Ky Fan minimax inequality with applications to maximal elements, price equilibrium and complementarity, J. Math. Anal. Appl. 170, 457- 471, 1992.