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PRESENTATIONS FOR REGULAR UNBRANCHED C_p-COVERINGS OF A KLEIN BOTTLE

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Abstract- In this paper, we obtain presentations for the non-equivalent regular unbranched C_p-coverings of a Klein Bottle by using the Reidemeister-Schreier method, where $C_p =< a \mid a^p = 1 >$ is a cyclic group of order p (p is a prime).

Keywords- Presentation, Regular covering, Schreier transversal.

1. INTRODUCTION

The classification of compact surfaces is well-known [5]: Every compact, connected, orientable surface is homeomorphic to a sphere with g handles attached, where $g \geq 0$ Every compact, connected, non-orientable surface is homeomorphic to a sphere with g cross-cups attached, where $g \geq 1$ Thus, a torus is a sphere with one handle, a projective plane is a sphere with one cross-cup, and a Klein bottle is a sphere with two cross-cups. We call g the genus of the surface. The Euler characteristic κ of a compact, connected surface S is defined by

$$\kappa(S) = \kappa(M) = V - E + F$$

where M is a polygonal subdivision of S with V vertices, E edges, and F faces.

The following relation holds between the genus g and the Euler characteristic κ of a compact, connected surface:

$$g = \begin{cases} 1 - \frac{1}{2}\kappa, & \text{in the orientable case,} \\ 2 - \kappa, & \text{in the non-orientable case.} \end{cases}$$

Let S_1 and S_2 be compact, connected surfaces. Then S_1 and S_2 are homeomorphic if and only if they have the same Euler characteristic and both are orientable or both are non-orientable [5].

Now, if $\sum_g$ is a compact, connected, orientable surface of genus g, then its fundamental group $\prod_g$ is isomorphic to

$$\prod_g = \langle A_1, B_1, \ldots, A_g, B_g \mid [A_1, B_1] \ldots [A_g, B_g] = 1 \rangle,$$

where $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$. If $\sum_g^-$ is a compact, connected, non-orientable surface of genus g, then the fundamental group $\prod_g^-$ of $\sum_g^-$ is generated by the g elements $R_1, R_2, \ldots, R_g$ with a single defining relation

$$R_1^2 R_2^2 \ldots R_g^2 = 1.$$

It is well-known from the coverings of surfaces that the normal subgroups N of the fundamental group of a compact, connected surface are in one-to-one correspondence with the equivalence classes of regular unbranched coverings of the surface with a given finite covering group G. These normal subgroups correspond to the kernels of the epimorphisms from the fundamental group to G, where G is the group of covering transformations.

One can combine character-theoretic techniques and P.Hall's group-theoretic generalization of the Möbius Inversion Formula [1] to obtain the number $\left|\mathrm{Hom}(\prod_g^-, G)\right|$ of homomorphisms from $\prod_g^-$ to G, and then the number $\left|\mathrm{Epi}(\prod_g^-, G)\right|$ of epimorphisms, and hence to evaluate the cardinality $n_g^-(G) = \left|N_g^-(G)\right|$ of the set $N_g^-(G) = \left\{N \lhd \prod_g^- \mid \prod_g^- / N \cong G\right\}$ of the kernels of these epimorphisms. In [3], Jones found formulas for all these numbers, and he also applied these formulas to several finite groups G.

2. COUNTING EPIMORPHISMS AND COVERINGS

In this section, we give a brief summary of the Mobius function and how one can combine the character-theoretic techniques and the Möbius function to obtain the number of equivalence classes of regular unbranched coverings of a compact, connected, non-orientable surfaces with a given finite covering group G.

Let G be a finitely generated group and $S = \left\{H \mid H \leq G, |G:H| < \infty\right\}$. Then the Möbius function $\mu : S \to \mathbf{Z}$ associated with the subgroup-lattice of G is defined recursively by

$$\sum_{H \geq K} \mu(H) = \delta_{K,G} = \begin{cases} 1, & \text{if } K = G, \\ 0, & \text{if } K < G, \end{cases}$$

or equivalently, $\mu(G)=1$, and if $K \neq G$ then $\mu(K)=-\sum_{H>K}\mu(H)$.

Let G be a finite group. Then the number of solutions $r_1, r_2, ..., r_g$ in G of the equation

$$r_1^2 r_2^2 ... r_g^2 = 1 \tag{1}$$

is equal to the number $\left|\mathrm{Hom}(\prod_g^-, G)\right|$ of the homomorphisms from $\prod_g^-$ to G. The following theorem gives a formula for $\left|\mathrm{Hom}(\prod_g^-, G)\right|$.

Theorem 2.1. (Frobenius, Schur): The number $\sigma_g^-(G)=\left|\mathrm{Hom}(\prod_g^-, G)\right|$ of solutions of (1) in a finite group G is given by

$$\sigma_g^-(G)=|G|^{g-1}\sum_{\chi} c_\chi^g \chi(1)^{2-g},$$

where χ ranges over the irreducible complex characters of G and c_χ is the Frobenius-Schur indicator of χ given by

$$c_\chi = |G|^{-1}\sum_{x\in G}\chi(x^2).$$

Proof: See [2, Chapter 5.5].

In [3], Jones used the Möbius function to obtain the number of epimorphisms, and then kernels of epimorphisms from $\prod_g^-$ to G as

$$\Phi_g^-(G)=\sum_{H\leq G}\sigma_g^-(H)\mu(H) \text{ and } n_g^-(G)=\Phi_g^-(G)\Big/\mathrm{Aut}(G),$$

respectively.

When $g=2$ (that is, $\sum_2^-$ is a Klein bottle), and $G=C_p$ (p is a prime), then there are

$$n_2^-(C_p)=\begin{cases}3, & \text{if } p=2,\\ 1, & \text{if } p\neq 2,\end{cases}$$

p-sheeted regular unbranched coverings of the Klein bottle

3. PRESENTATIONS FOR C_p-COVERINGS OF A KLEIN BOTTLE

In this section, we will find generators, and then presentations for C_p-coverings of a Klein bottle by using the Reidemeister-Schreier method [4].

The Reidemeister-Schreier method is an useful technique which will be used to find a presentation for a normal subgroup N, the kernel of an epimorphism $\varphi:\prod_g^- \to G$.

Let G is a finite group and H is a finitely presented group with generators $\{g_i \mid i=1, 2, ..., r\}$ and defining relations $\{R_j \mid j=1,2, ..., s\}$. Let φ be a homomorphism from H onto G, and let H^* denote the set consisting of the generators of H together with their inverses. A Schreier transversal U is defined, in [4] by Johnson. It consists of a set of coset representatives satisfying the following conditions:

a) The identity element $1 \in U$
b) U is closed under right cancellation; i.e. if $x_1 x_2 ... x_n \in U$ then $x_1 x_2 ... x_{n-1} \in U$, where $x_i \in H^*$, $i=1,2,...,n.$

Let $U=\{a_k \mid k=0, 1,..., m\}$ be a Schreier transversal of coset representatives of N in H, and let g be any element of H. If $\varphi(g)=\varphi(a_i)$, that is, if g belongs to the coset $N a_i$, we define a function ψ by putting $\psi(g)=a_i$. Then N is generated by the elements

$$\{a_k g_i \psi(a_k g_i)\}^{-1} \mid i=1, 2, ..., r; \quad k=0, 1, 2, ..., m\}$$

which we call the Schreier generators of N, and the relations

$$\{a_k R_j a_k^{-1}=1 \mid j=1, 2, ..., s; \quad k=0, 1, 2, ..., m\}$$

when expressed in terms of the Schreier generators of N form a complete set of defining relations for N.

Theorem 3.1. Let $G=C_2=<a \mid a^2=1>$ be a cyclic group of order 2, and let $\sum_2^-$ be a Klein bottle. Then there are three 2-sheeted regular unbranched coverings of $\sum_2^-$. Two of them are again Klein bottles, and one of them is a torus. Moreover, there is a 4-sheeted regular unbranched covering of $\sum_2^-$ which is a torus.

Proof: The fundamental group of $\sum_2^-$ is equal to $\prod_2^1 =<R_1, R_2 \mid R_1^2 R_2^2 =>$. to
The solutions r_1, r_2 in C_2 of the equation $r_1^2 r_2^2=1$ with $C_2=<r_1, r_2>$ correspond to

epimorphisms $\prod_2^- \to C_2$. Now $r_1^2 r_2^2 = 1$ is always true since $r_1^2 = r_2^2 = 1$ and $C_2 = < r_1, r_2 >$ is true provided not both r_1 and $r_2 = 1$. Thus there are three possibilities:

(i) $r_1 = 1, \ r_2 = a;$

(ii) $r_1 = a, \ r_2 = 1;$

(iii) $r_1 = a, \ r_2 = a.$

Since $|\mathrm{Aut}(C_2)| = 1$, these three cases give three in-equivalent regular un-branched coverings. Each of them correspond to a normal subgroup $N \lhd \prod_2^-$ of index 2. By the Reidemeister-Schreier method, we find presentations for these normal subgroups as kernels of corresponding epimorphisms.

(i) Let $r_1 = 1, \ \ r_2 = a$, and let $\varphi_1 : \prod_2^- \to C_2; \ R_1 \mapsto 1, R_2 \mapsto a$ be the corresponding epimorphism. We choose as a Schreier transversal for $N_1 = \mathrm{Ker}\,\varphi_1$ the set $U = \{1, R_2\}$. Then the schreier generators for N_1 are

$$x_1 = R_1, \ x_2 = R_2 R_1 R_2^{-1}, \ x_3 = R_2^2,$$

and the corresponding relations are

$$x_1^2 x_3 = 1 \text{ and } x_2^2 x_3 = 1.$$

Eliminating x_3 from these relations, we obtain $x_1^2 x_2^{-2} = 1$. Thus we get a presentation for N_1 as

$$N_1 = \mathrm{Ker}\,\varphi_1 = < X_1, X_2 \mid X_1^2 X_2^2 = 1 >,$$

where $X_1 = x_1 = R_1$ and $X_2 = x_2^{-1} = R_2 R_1^{-1} R_2^{-1}$. Thus the covering $\bar{\sum}$ is a Klein bottle.

(ii) Now $r_1 = a, \ r_2 = 1$, and let $\varphi_2 : \prod_2^- \to C_2; \ R_1 \mapsto a, R_2 \mapsto 1$ be the corresponding epimorphism. If we choose as a Schreier transversal for $N_2 = \mathrm{Ker}\,\varphi_2$, the set $U = \{1, R_1\}$, then we get a presentation for N_2 as

$$N_2 = \mathrm{Ker}\,\varphi_2 = < Y_1, Y_2 \mid Y_1^2 Y_2^2 = 1 >,$$

where $Y_1 = R_2^{-1}$ and $Y_2 = R_1 R_2^{-1} R_1^{-1}$. Thus the covering $\bar{\sum}$ is also a Klein bottle .

Figure 1. (a), (b) illustrate the fundamental regions F of these coverings, respectively. In each case, we have indicated pairs of sides to be identified.

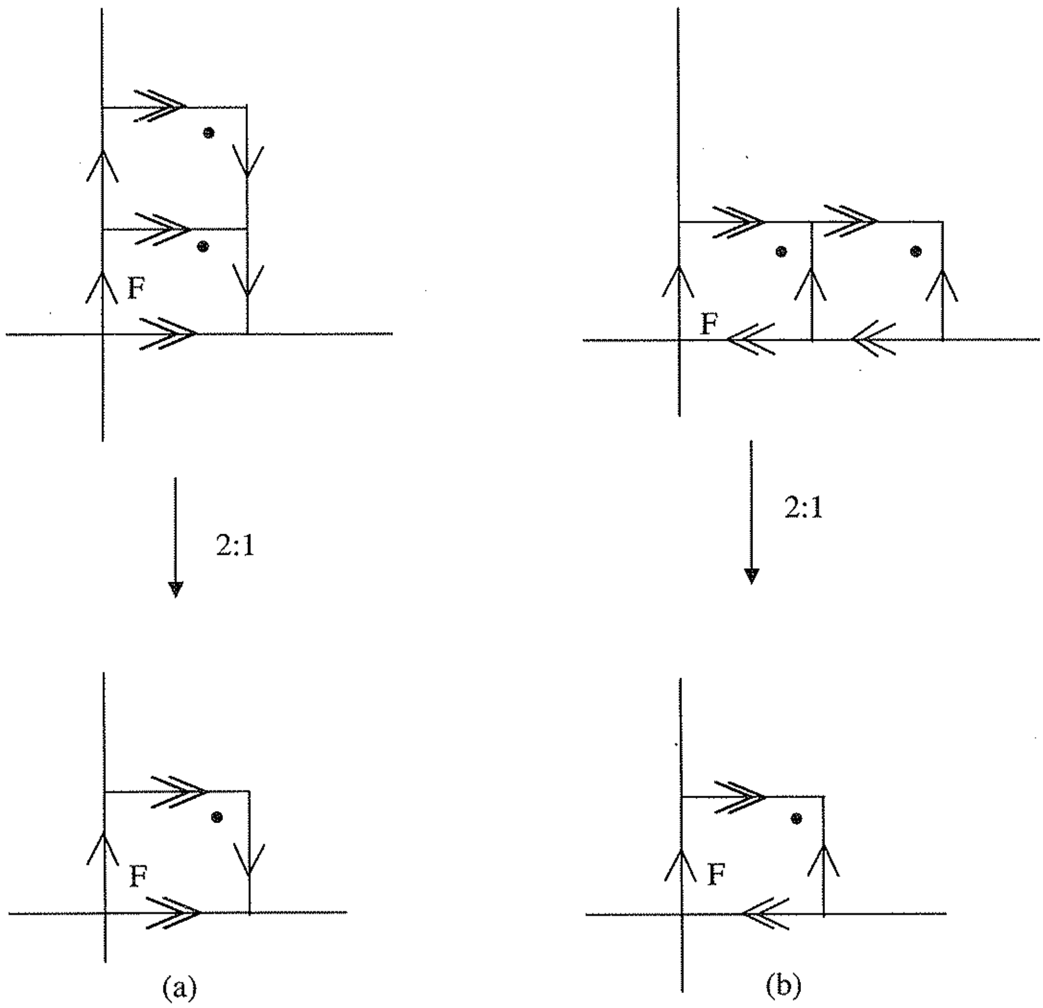

Figure 1. Fundamental Regions F .

(iii) Finally, let $r_1 = a,\ \ r_2 = a$, and let $\varphi_3 : \prod_2^{-} \to C_2 ;\ \ R_1 \mapsto a,\ R_2 \mapsto a$ be the corresponding epimorphism. We may choose as a Schreier transversal for $N_3 = \mathrm{Ker}\,\varphi_3$, the set $U = \{1,\ R_1\}$, then the Schreier generators are

$$S_1 = R_2 R_1^{-1}\ ,\ T_1 = R_1 R_2 \text{ and } S_2 = R_1^2\ ,$$

and relations are

$$S_2 S_1 T_1 = 1 \text{ and } S_2 T_1 S_1 = 1.$$

From these relations, we obtain $S_1^{-1}T_1^{-1}S_1T_1 = 1$, so we get the following presentation for N_3:

$$N_3 = \mathrm{Ker}\,\varphi_3 = < S_1, T_1 \mid S_1^{-1}T_1^{-1}S_1T_1 = 1 > \cong \mathbf{Z}\times\mathbf{Z}.$$

Thus the corresponding covering is a torus (an orientable surface of genus 1). Figure 2 (a) illustrates the fundamental region F of this covering.

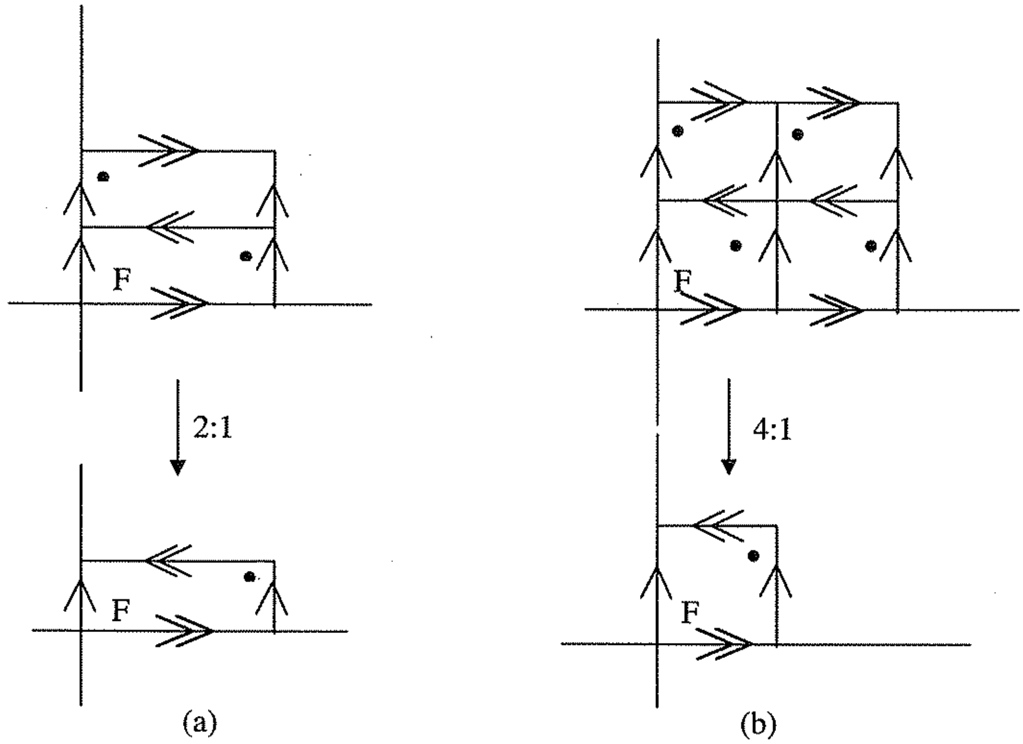

Figure 2. Fundamental Regions F .

Moreover, the intersection of these three normal subgroups is also a normal subgroup N of $\prod_2^-$ with $\prod_2^- / N \cong C_2 \times C_2$ which corresponds to a 4-sheeted regular unbranched covering of the Klein bottle. Since it is also a 2-sheeted regular unbranched covering of the torus, it is also a torus. By the same way, we can find a presentation for N such as

$$N = < K_1, L_1 \mid K_1^{-1}L_1^{-1}K_1L_1 = 1 > \cong \mathbf{Z}\times\mathbf{Z},$$

where $K_1 = X_2X_1^{-1} = R_2R_1^{-1}R_2^{-1}R_1^{-1}$ and $L_1 = X_1X_2 = R_1R_2R_1^{-1}R_2^{-1}$. Thus we have $\prod_2^- / N \cong C_2 \times C_2$ as the group of covering transformations. Figure 2 (b) illustrates the fundamental region F of this covering.

Theorem 3.2. Let $G = C_p = \langle a \mid a^p = 1 \rangle$, ($p$ is an odd prime) be a cyclic group of order p, and let $\sum_2^-$ be a Klein bottle. Then there is only one p-sheeted regular unbranched covering of $\sum_2^-$ which is also a Klein bottle.

Proof: The solutions r_1, r_2 in C_p of the equation $r_1^2 r_2^2 = 1$ with $C_p = \langle r_1, r_2 \rangle$ are $r_1 = a^k$, $r_2 = a^{p-k}$, $1 \le k \le p-1$. Thus there are $p-1$ cases, but since $\left|\mathrm{Aut}(C_p)\right| = p-1$, these $p-1$ cases give $p-1$ equivalent p-sheeted regular unbranched coverings. Therefore we have only one p-sheeted regular unbranched covering.

If we choose one solution, such as $r_1 = a, r_2 = a^{p-1}$, so the corresponding epimorphism

$$\varphi : \prod_2^- \to C_p;\ R_1 \mapsto a,\ R_2 \mapsto a^{p-1},$$

then we may choose as a Schreier transversal for coset representatives for $N = \mathrm{Ker}\,\varphi$ the elements $1, R_1, R_1^2, \ldots, R_1^{p-1}$. Then the Schreier generators for N are $X_1 = R_1^p$, $X_2 = R_2^{1-p}$, and $K_j = R_1^j R_2 R_1^{1-j}$, $j = 1, 2, \ldots, p-1$, thus there are $p+1$ Schreier generators. Conjugating $R_1^2 R_2^2$ by each of p coset representatives of N and expressing the resulting relations in terms of the Schreier generators, we obtain the following relations:

$$R_1^i R_2^2 R_1^{2-i} = K_i\, K_{i-1} = 1, \quad i = 2, 3, \ldots, p-1,$$
$$R_1^p R_2^2 R_1^{2-p} = X_1\, X_2\, K_{p-1} = 1,$$
and
$$R_1^{p+1} R_2^2 R_1^{1-p} = X_1 K_1\, X_2 = 1.$$

From the first $p-1$ relations we get

$$K_1 = K_2^{-1} = K_3 = \ldots = K_{p-2} = K_{p-1}^{-1} = X_1 X_2,$$

putting K_1 in the last relation we obtain

$$X_1^2 X_2^2 = 1.$$

Thus $N = \mathrm{Ker}\,\varphi = \langle X_1, X_2 \mid X_1^2 X_2^2 = 1 \rangle$, where $X_1 = R_1^p$, $X_2 = R_2 R_1^{1-p}$. Thus $\prod_2^- / N \cong C_p$ (as the group of the covering transformations), and the corresponding covering is a Klein bottle. Figure 3 shows the fundamental region F of this covering.

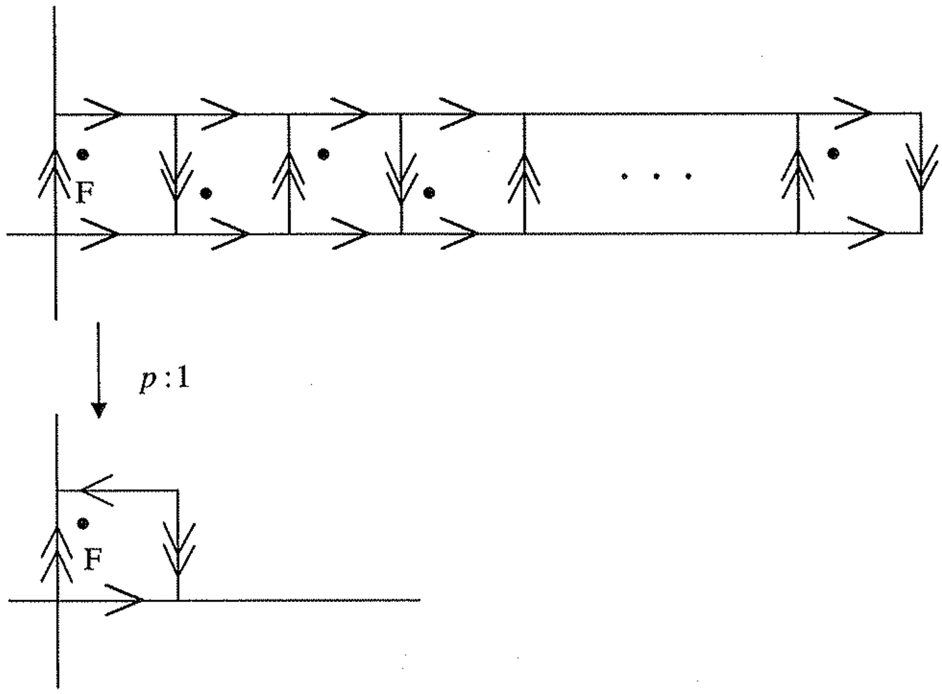

Figure 3. Fundamental Region F of the p-sheeted covering.

REFERENCES

1. P.Hall, The Eulerian function of a group, *Quarterly J. Math. Oxford* **7**, 134-151, 1936.

2. M. Hamermesh, *Group Theory and its Applications to Physical Problems*, Addision-Wesley, Reading, London, 1962.

3. G. A. Jones, Enumerations of Homomorphisms and Surface-Coverings, *Quarterly J. Math. Oxford (2)*, **46**, 485-507, 1995.

4. D. L. Johnson, *Topics in the theory of Group Presentations*, L.M.S. Lecture Note Series, **42**, Cambridge Univ. Press, 1980.

5. W. S. Massey, *A Basic Course in Algebraic Topology*, Springer-Verlag, New York, 1991.