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PROJECTION METHODS FOR QUASI-VARIATIONAL INEQUALITIES

Salahuddin
Department of Mathematics
Aligarh Muslim University
Aligarh-202002 (India)

Abstract- In this paper, we consider a new class of variational inequalities which is called the general mixed multivalued nonlinear quasi-variational inequality. We define an iterative algorithm and prove the existence solution for our general mixed multivalued nonlinear quasi-variational inequality by using the updating projection method and convergence criteria for iterative sequences generated by algorithm are also discussed.
Key words - General mixed multivalued nonlinear quasi-variational inequality, Iterative sequences, Algorithm, Strongly monotone mapping, Lipschitz continuous mappings, Projection method.

1. INTRODUCTION

In the last twenty years nonlinear variational inequalities have assumed great importance both from the theoretical and practical points of view due to their applicability in the calculus of variation and different branches of engineering sciences. In recent years, variational inequalities have been extended and generalized in many directions, see [1-9].

In this paper, we define and study a new class of general mixed multivalued nonlinear quasi-variational inequality, which includes a number of known classes of variational inequalities studied previously by many authors in this field. A general and unified iterative algorithm for finding the approximate solutions to this problem is considered by projection method. We prove that the existence of solutions for this problem and convergence of the iterative sequences generated by this algorithm.

Throughout this paper H stands for a real Hilbert space whose norm and inner product are denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. Let K be a nonempty closed convex subset of H. Let A, f, g be the single-valued mappings from H into itself and M, T $:H\to 2^H$ be the multivalued mappings where 2^H denotes the power set of H.

For a nonlinear operator $N(.,.):H\times H\to H$, we consider the problem of finding $u\in K$, $x\in M(u)$, $y\in T(u)$ such that $g(u)\in K$ and

$$\langle A(g(u)), v-g(u)\rangle + \rho\, b(u, v) - \rho b(u, g(u)) \geq \langle A(f(u)), v-g(u)\rangle - \rho \langle N(x, y), v-g(u)\rangle, \text{ for all } v\in K, \quad (1.1)$$

where $\rho > 0$ is a constant and the form $b(.,.):H\times H\to R$ satisfying the following properties:

(1) b(u,v) is linear in first argument,

(2) b(u,v) is bounded, that is, there exists a constant $\upsilon > 0$ such that

$$|b(u,v)| \leq \upsilon \|u\| \; \|v\|, \text{ for all } u,v \in H, \quad (1.2)$$

(3) $b(u,v) - b(u,w) \leq b(u, v-w)$, for all $u,v,w \in H$, (1.3)

(4) $b(u,v)$ is convex in the second argument.

Then problem (1.1) is called general mixed multivalued nonlinear variational inequality problem.

For appropriate and suitable choice of operators g, f, A, T, M and N, a number of known classes of variational inequalities can be obtained as special cases of problem (1.1) studied previously by many authors including Zeng [15], Lions and Stampacchia [11], Cohen [3], Noor [13], Glowinski et al [8] etc.

If the convex set K depends upon the solution, i.e., K is a point-to-set mapping from H into itself, then problem (1.1) becomes the general mixed multivalued nonlinear quasi-variational inequality problem of finding $u \in K$, $x \in M(u)$, $y \in T(u)$ such that $g(u) \in K(u)$ and

$$\langle A(g(u)), v - g(u)\rangle + \rho\, b(u, v) - \rho b(u, g(u)) \geq \langle A(f(u)), v - g(u)\rangle - \rho \langle N(x, y), v - g(u)\rangle, \text{ for all } v \in K(u), \quad (1.4)$$

and a constant $\rho > 0$.

In many important applications, the set K(u) is of the form

$$K(u) = m(u) + K, \quad (1.5)$$

where m is a point-to-point mapping and K is a closed convex set, see [2].

To prove our main result, we need the following lemmas.

Lemma 1.1 [10]- Let $K \subset H$ be a closed convex subset. Then given $z \in H$, we have

$$u = \text{Proj}_K(z) \quad (1.6)$$

if and only if $u \in K$ and

$$\langle u - z, v - u\rangle \geq 0, \text{ for all } v \in K, \quad (1.7)$$

where Proj_K is a projection of H onto K.

Lemma 1.2 [10] - Proj_K is nonexpansive, that is ,

$$\|\text{Proj}_K(u) - \text{Proj}_K(v)\| \leq \|u - v\|, \text{ for all } u,v \in H. \quad (1.8)$$

Lemma 1.3 [12] - If K(u) is of type (1.5), then for each $u,v \in H$

$$\text{Proj}_{K(u)}(v) = m(u) + \text{Proj}_K(v - m(u)).$$

Lemma 1.4 - Let K(u) be of type (1.5). Then $u \in K$, $x \in M(u)$, $y \in T(u)$ is a solution of problem (1.4) if and only if $u \in K$, $x \in M(u)$, $y \in T(u)$ satisfies $g(u) \in K(u)$ and

$$\langle u - \phi(u), v - g(u)\rangle \geq 0, \text{ for all } v \in K(u), \quad (1.9)$$

where $\phi(u): H \to 2^H$ and for some constant $\rho > 0$,

$$\langle \phi(u), v\rangle = \langle u, v\rangle - \rho \langle N(x,y), v\rangle - \rho\, b(u,v) + \langle [Ao(f-g)](u), v\rangle, \quad (1.10)$$

for all $v \in K(u)$ and the operator $Ao(f-g)$ is defined as

$$[Ao(f-g)](u) = A(f(u)) - A(g(u)), \text{ for all } u \in K. \quad (1.11)$$

Proof - Let (u, x, y) be a solution set of problem (1.4). Then we derive $g(u) \in K(u)$ and

$$\langle A(g(u)), v - g(u)\rangle + \rho\, b(u, v) - \rho b(u, g(u)) \geq \langle A(f(u)), v - g(u)\rangle - \rho \langle N(x, y), v - g(u)\rangle, \text{ for all } v \in K(u), \quad (1.12)$$

and a constant $\rho > 0$. Using (1.10) and (1.12), we obtain

$$\langle u - \phi(u), v - g(u)\rangle = \langle u, v - g(u)\rangle - \langle \phi(u), v\rangle + \langle \phi(u), g(u)\rangle$$
$$= \langle u, v\rangle - \langle u, g(u)\rangle - [\langle u, v\rangle - \rho\langle N(x,y), v\rangle - \rho\, b(u,v) + \langle [Ao(f-g)](u), v\rangle]$$
$$+ \langle u, g(u)\rangle - \rho\langle N(x,y), g(u)\rangle - \rho\, b(u,g(u)) + \langle [Ao(f-g)](u), g(u)\rangle$$
$$= \langle [Ao(f-g)](u), g(u) - v\rangle + \rho\, b(u,v) - \rho\, b(u,g(u)) + \rho \langle N(x,y), v - g(u)\rangle \geq 0,$$

for all $v \in K(u)$, which infers that (1.9) holds.

Conversely, let $u \in K$, $x \in M(u)$, $y \in T(u)$ satisfy $g(u) \in K(u)$ and (1.9). Then we have

$$\begin{aligned}\langle u , v - g(u) \rangle &\geq \langle \phi(u), v - g(u) \rangle \\ &= \langle \phi(u), v \rangle - \langle \phi(u), g(u) \rangle \\ &= \langle u , v \rangle - \rho \langle N(x,y) , v \rangle - \rho\, b(u,v) + \langle [Ao(f-g)](u),v\rangle - [\langle u , g(u) \rangle \\ &\quad - \rho \langle N(x,y) , g(u) \rangle - \rho\, b(u,g(u)) + \langle [Ao(f-g)](u) , g(u)\rangle] \\ &= \langle u , v - g(u) \rangle - \rho[\langle N(x,y) , v - g(u)\rangle + b(u,v) - b(u,g(u))] \\ &\quad + \langle [Ao(f-g)](u) , v - g(u)\rangle, \text{ for all } v \in K(u).\end{aligned}$$

It follows that

$$\langle A(g(u)) , v - g(u)\rangle + \rho\, b(u , v) - \rho b(u , g(u)) \geq \langle A(f (u)) , v - g(u)\rangle - \rho \langle N(x , y) , v - g(u)\rangle, \text{ for all } v \in K(u).$$

Thus (u, x, y) is a solution of problem (1.4).

Lemma 1.5 - Let $K(u)$ be defined as (1.5). Then, $u \in K$, $x \in M(u)$, $y \in T(u)$ is a solution of the problem (1.4) if and only if $u \in K$, $x \in M(u)$, $y \in T(u)$ satisfies $g(u) \in K(u)$ and

$$g(u) = m(u) + \mathrm{Proj}_K [g(u) - u + \phi(u) - m(u)], \qquad (1.13)$$

where $m:H \to H$ and $\phi(u)$ is defined as (1.10).

Proof - By Lemma 1.4, $u \in K$, $x \in M(u)$, $y \in T(u)$ is a solution of the problem (1.4) if and only if (1.9) holds. We deduce

$$\langle g(u) - [g(u)-(u - \phi(u))] , v - g(u)\rangle = \langle u - \phi(u) , v - g(u)\rangle \geq 0, \text{ for all } v \in K(u).$$

Hence, by Lemma 1.1, Lemma 1.3 and (1.9) holds if and only if $u \in K$, $x \in M(u)$, $y \in T(u)$ satisfies $g(u) \in K(u)$ and

$$\begin{aligned} g(u) &= \mathrm{Proj}_{K(u)} [g(u) - u + \phi(u)] \\ &= m(u) + \mathrm{Proj}_K [g(u) - u + \phi(u) - m(u)].\end{aligned}$$

We need the following concepts.

Definition 1.1 [14] - For all $u_1, u_2 \in H$, the operator $N(.,.):H\times H \to H$ is said to be strongly monotone and Lipschitz continuous with respect to first argument, if there exist constants $\xi > 0$ and $\eta > 0$ such that

$$\langle N(x_1,.) - N(x_2,.) , u_1 - u_2\rangle \geq \xi \| u_1 - u_2 \|^2, \text{ for all } x_1 \in M(u_1),\ x_2 \in M(u_2)$$

and

$$\| N(u_1,.) - N(u_2,.) \| \leq \eta \| u_1 - u_2 \| .$$

Definition 1.2 [5] - The multivalued operator $M:H \to 2^H$ is said to be H-Lipschitz continuous if there exists a constant $\varepsilon > 0$ such that

$$H(M(u_1) , M(u_2)) \leq \varepsilon \| u_1 - u_2 \| , \text{ for all } u_1, u_2 \in H,$$

where $H(.,.)$ is the Hausdorff metric on Hilbert space.

Lemma 1.6 - Let $N:H\times H \to H$ be strongly monotone and Lipschitz continuous with respect to first argument with constants $\xi > 0$ and $\eta > 0$, respectively; and A, g, $f:H \to H$ be the Lipschitz continuous with constants $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, respectively. Let $N:H\times H \to H$ be a Lipschitz continuous with respect to second argument with constant $\sigma > 0$, Let $M,T:H \to 2^H$ be H-Lipschitz continuous with constants $\varepsilon > 0$ and $\chi > 0$, respectively. Then for any constant $\rho > 0$ there exists $\theta > 0$ such that

$$\| \phi(u_1) - \phi(u_2) \| \leq \theta \| u_1 - u_2 \|, \text{ for all } u_1,u_2 \in H,$$

where $\phi(u)$ is defined as (1.10). It turn out that

$$\theta = (1 - 2 \rho\xi + \rho^2\eta^2 \varepsilon^2)^{1\backslash 2} + \rho\, \upsilon + \rho\, \sigma\, \chi + \alpha (\gamma + \beta). \qquad (1.14)$$

Proof- For all $u_1,u_2\in H$, $x_1\in M(u_1)$, $x_2\in M(u_2)$ $y_1\in T(u_1)$, $y_2\in T(u_2)$, by (1.10) and property (1) of $b(u,v)$, we obtain

$$\begin{aligned}
&|\langle \phi(u_1)-\phi(u_2),v\rangle| \le |\langle u_1-u_2-\rho(N(x_1,y_1)-N(x_2,y_1)),v\rangle| + \rho\ |b(u_1-u_2,v)|\\
&+\rho\,|\langle N(x_2,y_1)-N(x_2,y_2),v\rangle| + |\langle A(f(u_1))-A(f(u_2))-(A(g(u_1))-A(g(u_2))),v\rangle|\\
&\le \| u_1-u_2-\rho(N(x_1,y_1)-N(x_2,y_1))\|\,\|v\| + \rho\,\upsilon\,\|u_1-u_2\|\,\|v\|\\
&+\rho\|N(x_2,y_1)-N(x_2,y_2)\|\ \|v\| + \|A(f(u_1))-A(f(u_2))-(A(g(u_1))-A(g(u_2)))\|\,\|v\|. \qquad (1.15)
\end{aligned}$$

Since operator N is Lipschitz continuous with respect to first argument and M is H-Lipschitz continuous, we have

$$\begin{aligned}
\| N(x_1,y_1)-N(x_2,y_1)\| &\le \eta\,\| x_1-x_2\|\\
&\le \eta\, H(M(u_1),M(u_2))\\
&\le \eta\varepsilon\ \| u_1-u_2\|. \qquad (1.16)
\end{aligned}$$

Further, since operator N is strongly monotone with respect to first argument and (1.16), we have

$$\begin{aligned}
\| u_1-u_2-\rho(N(x_1,y_1)-N(x_2,y_1))\|^2 &= \| u_1-u_2\|^2\\
&-2\rho\,\langle N(x_1,y_1)-N(x_2,y_1),\, u_1-u_2\rangle + \rho^2\,\| N(x_1,y_1)-N(x_2,y_1)\|^2\\
&\le [1-2\,\rho\xi\ +\rho^2\eta^2\,\varepsilon^2]\,\| u_1-u_2\|^2. \qquad (1.17)
\end{aligned}$$

Again, operator N is Lipschitz continuous with respect to second argument and T is H-Lipschitz continuous, we have

$$\begin{aligned}
\| N(x_2,y_1)-N(x_2,y_2)\| &\le \sigma\,\| y_1-y_2\|\\
&\le \sigma\, H(T(u_1),T(u_2))\\
&\le \sigma\,\chi\,\| u_1-u_2\|. \qquad (1.18)
\end{aligned}$$

Since A, g and f are Lipschitz continuous, we have

$$\begin{aligned}
\| A(f(u_1))-A(f(u_2))-(A(g(u_1))-A(g(u_2)))\| &\le \ \| A(f(u_1))-A(f(u_2))\|\\
&+\| A(g(u_1))-A(g(u_2))\|\\
&\le \alpha\,\| f(u_1)-f(u_2)\| + \alpha\ \| g(u_1)-g(u_2)\|\\
&\le \alpha\,\gamma\,\| u_1-u_2\| + \alpha\,\beta\ \| u_1-u_2\|\\
&\le \alpha\,(\gamma+\beta)\,\| u_1-u_2\|. \qquad (1.19)
\end{aligned}$$

Adding (1.15), (1.17), (1.18) and (1.19), we have

$$|\langle \phi(u_1)-\phi(u_2),v\rangle| \le \theta\,\| u_1\ -u_2\|\ \ \| v\|,$$

where θ is given by (1.14), it follows that

$$\begin{aligned}
\| \phi(u_1)-\phi(u_2)\| &\le \sup_{v\in H}\ |\langle \phi(u_1)-\phi(u_2),v\rangle| \backslash \| v\|\\
&\le \theta\,\| u_1\ -u_2\|. \qquad (1.20)
\end{aligned}$$

2. MAIN RESULT

In this section, we define an iterative algorithm for finding the approximate solutions of the general mixed multivalued nonlinear quasi-variational inequality problem (1.4) and prove the approximate solutions converge strongly to the exact solution of the problem (1.4).

Algorithm 2.1- Given $u_0\in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + m(u_n) + \mathrm{Proj}_{K}\,[g(u_n) - u_n + \phi(u_n) - m(u_n)],$$

where

$$\langle \phi(u_n),v\rangle = \langle u_n,v\rangle - \rho\,\langle N(x_n\,,y_n),v\rangle - \rho\, b(u_n,v) + \langle [A\mathrm{o}(f-g)](u_n)\,,v\,\rangle,$$

for all $v\in K$ and some $\rho > 0$, $x_n\in M(u_n)$, $y_n\in T(u_n)$, $n = 0,1,2,\ldots$.

Theorem 2.1- Let $g:H\to H$ be a strongly monotone and Lipschitz continuous with constants $\delta > 0$ and $\beta > 0$, respectively. Let $M,T:H\to 2^H$ be H-Lipschitz continuous with constants $\varepsilon > 0$ and $\chi > 0$, respectively. Let A, f, $m:H\to H$ be Lipschitz continuous with constants $\alpha > 0$, $\gamma > 0$ and $p > 0$, respectively. Let $N:H\times H\to H$ be a strongly monotone with respect to first argument with constant $\xi > 0$ and Lipschitz continuous with respect to first and second argument with constants $\eta > 0$ and $\sigma > 0$, respectively. Let the form $b(u,v)$ satisfy the properties (1)-(4). Assume further that

$$q < 2^{-1} \text{ and } \xi > (\upsilon + \sigma\chi)(1-2q) + [(\eta^2\varepsilon^2 - (\upsilon+\sigma\chi)^2)\, 4q\,(1-q)]^{1/2}$$

where

$$q = (1-2\delta+\beta^2)^{1\backslash 2} + p + \alpha(\gamma+\beta)\, 2^{-1}.$$

Then the problem (1.4) has a solution (u^*, x^*, y^*) and for each constant $\rho > 0$ with

$$|\rho - [\xi + (2q-1)(\upsilon+\sigma\chi)][\eta^2\varepsilon^2 - (\upsilon+\sigma\chi)^2]^{-1}|$$
$$< [(\xi + (2q-1)(\upsilon+\sigma\chi))^2 - 4q(1-q)(\eta^2\varepsilon^2 - (\upsilon+\sigma\chi)^2)]^{1\backslash 2}[\eta^2\varepsilon^2 - (\upsilon+\sigma\chi)^2]^{-1},$$

the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 2.1, converge strongly to u^*, x^* and y^*, respectively.

Proof- By Lemma 1.5, $u\in H$, $x\in M(u)$, $y\in T(u)$ is a solution set of the problem (1.4) if and only if $u\in H$, $x\in M(u)$, $y\in T(u)$ satisfies (1.13) and the mapping $F:H\to 2^H$ is define by

$$F(u) = \cup_{x\in M(u)} \cup_{y\in T(u)} [u - g(u) + m(u) + \text{Proj}_K [g(u) - u + \phi(u) - m(u)]],$$

for all $u\in H$, where $\phi(u)$ is defined as (1.10). For each $u_1, u_2\in H$, we have

$$\| F(u_1) - F(u_2)\| \le \| u_1 - u_2 - (g(u_1) - g(u_2)) + m(u_1) - m(u_2)\|$$
$$+ \| \text{Proj}_K [g(u_1) - u_1 + \phi(u_1) - m(u_1)] - \text{Proj}_K [g(u_2) - u_2 + \phi(u_2) - m(u_2)]\|.$$

By Lemma 1.2, we have

$$\| F(u_1) - F(u_2)\| \le 2\,\| u_1 - u_2 - (g(u_1) - g(u_2))\| + 2\,\| m(u_1) - m(u_2)\|$$
$$+ \| \phi(u_1) - \phi(u_2)\|. \tag{2.1}$$

Since g is strongly monotone and Lipschitz continuous, we have

$$\| u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \le \| u_1 - u_2\|^2 - 2\langle g(u_1) - g(u_2), u_1 - u_2\rangle$$
$$+ \| g(u_1) - g(u_2)\|^2$$
$$\le \| u_1 - u_2\|^2 - 2\delta\,\| u_1 - u_2\|^2 + \beta^2\| u_1 - u_2\|^2$$
$$\le (1 - 2\delta + \beta^2)\| u_1 - u_2\|^2. \tag{2.2}$$

Again using the Lipschitz continuity of m, we have

$$\| m(u_1) - m(u_2)\| \le p\,\| u_1 - u_2\|. \tag{2.3}$$

It follows from (2.1), (2.2), (2.3) and Lemma 1.6, that

$$\| F(u_1) - F(u_2)\| \le 2(1-2\delta+\beta^2)^{1\backslash 2}\| u_1 - u_2\| + 2p\,\| u_1 - u_2\|$$
$$+ \{(1 - 2\rho\xi + \rho^2\eta^2\varepsilon^2)^{1\backslash 2} + \rho\upsilon + \rho\sigma\chi + \alpha(\gamma+\beta)\}\| u_1 - u_2\|$$
$$\le 2\{(1-2\delta+\beta^2)^{1\backslash 2} + p + \alpha(\gamma+\beta)2^{-1}\}\| u_1 - u_2\| + \{(1 - 2\rho\xi + \rho^2\eta^2\varepsilon^2)^{1\backslash 2}$$
$$+ \rho(\upsilon + \sigma\chi)\}\| u_1 - u_2\|$$
$$\le (2q + \theta_1)\| u_1 - u_2\|,$$

where

$$\theta_1 = (1 - 2\rho\xi + \rho^2\eta^2\varepsilon^2)^{1\backslash 2} + \rho(\upsilon + \sigma\chi)$$

and

$$q = (1-2\delta+\beta^2)^{1\backslash 2} + p + \alpha(\gamma+\beta)2^{-1}.$$

Since $q < 2^{-1}$, $\eta\varepsilon > (\upsilon + \sigma\chi)$ and

$$\xi > (\upsilon + \sigma\chi)(1-2q) + [(\eta^2\varepsilon^2 - (\upsilon+\sigma\chi)^2)\, 4q\,(1-q)]^{1\backslash 2}.$$

This implies that for each $\rho > 0$ with
$|\rho - [\xi + (2q-1)(\upsilon + \sigma\chi)][(\eta^2\varepsilon^2 - (\upsilon + \sigma\chi)^2)]^{-1}|$
$< [(\xi + (2q-1)(\upsilon + \sigma\chi))^2 - 4q(1-q)(\eta^2\varepsilon^2 - (\upsilon + \sigma\chi)^2)]^{1\backslash 2}[\eta^2\varepsilon^2 - (\upsilon + \sigma\chi)^2]^{-1}$,
we have

$$2q + \theta_1 = 2q + (1 - 2\rho\xi + \rho^2\eta^2\varepsilon^2)^{1\backslash 2} + \rho(\upsilon + \sigma\chi) < 1.$$

Hence, F is a contraction mapping. Then, it follows that F has a fixed point $u^* \in H$, that is

$$u^* = u^* - g(u^*) + m(u^*) + \mathrm{Proj}_K [g(u^*) - u^* + \phi(u^*) - m(u^*)].$$

Hence by Lemma 1.5, there exist $u^* \in H$, $x^* \in M(u^*)$ and $y^* \in T(u^*)$ such that (u^*, x^*, y^*) is a solution of problem (1.4). Since $u^* \in H$, $x^* \in M(u^*)$ and $y^* \in T(u^*)$ satisfies (1.13), i.e.,

$$g(u^*) = m(u^*) + \mathrm{Proj}_K [g(u^*) - u^* + \phi(u^*) - m(u^*)]. \tag{2.4}$$

By (2.4) and Algorithm 2.1, we obtain

$$\begin{aligned} \| u_{n+1} - u^* \| &\le \| u_n - u^* - (g(u_n) - g(u^*)) \| + \| m(u_n) - m(u^*) \| \\ &+ \| \mathrm{Proj}_K [g(u_n) - u_n + \phi(u_n) - m(u_n)] - \mathrm{Proj}_K [g(u^*) - u^* + \phi(u^*) - m(u^*)] \| \\ &\le 2 \| u_n - u^* - (g(u_n) - g(u^*)) \| + 2 \| m(u_n) - m(u^*) \| \\ &+ \| \phi(u_n) - \phi(u^*) \| \\ &\le (2q + \theta_1) \| u_n - u^* \| \\ &\le (2q + \theta_1)^n \| u_1 - u^* \|. \end{aligned}$$

Noting that $2q + \theta_1 < 1$, we know that $\{u_n\}$ converges to u^*. Now $x_n \in M(u_n)$, $x^* \in M(u^*)$ and M is H-Lipschitz continuous, we have

$$\| x_n - x^* \| \le H(M(u_n), M(u^*)) \le \varepsilon \| u_n - u^* \| \to 0 \text{ as } n \to \infty,$$

i.e., $\{x_n\}$ strongly converges to x^*. Similarly, we can prove that $\{y_n\}$ strongly converges to y^*. This completes the proof.

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