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PARAMETRIZATION OF TORSION UNITS IN $U_1(ZS_3)$

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Abstract-In this study, a representation of the group ring ZS_3 is obtained by using a faithful irreducible representation of S_3 of degree 2. In ZS_3, all torsion units are expressed in terms of parameters by means of this representation. It is shown that any torsion unit in $U_1(ZS_3)$ can be expressed in terms of two parameters.
Key words- Parametrization, torsion units

1.INTRODUCTION

Let G be a finite group and $U_1(ZG)$ be the unit group of integral group ring ZG with augmentation 1. In mid-sixties, Zassenhaus made seemingly very strong conjectures. The first one is :

$$\gamma \in U_1(ZG) \text{ and } |\gamma| < \infty \implies \text{for some } g \in G \;\; \gamma = \alpha g \alpha^{-1},\; \exists \alpha \in QG, \qquad \textbf{(ZC1)}$$

where QG is a rational group algebra.

This conjecture has been proved for metacyclic groups $<a>$ $<x>$ with $(|a|, |x|)=1$ by Milies-Sehgal[1]. Here, all metacyclic groups can be considered as a split extention of two cyclic groups. On the other hand, for nilpotent class-two groups the conjecture has been proved by Sehgal[2]. The special case $|G|$=pq of the next result goes back to Bhandari-Luthar[3].

Theorem 1.1 (Milies–Sehgal [1]). Let G be a split extension of a cyclic p-group and a cyclic p′-group with faithful action. Let $u \in U_1(ZG)$ be an element of finite order. Then, there exist an $\alpha \in QG$, $g \in G$ such that $u = \alpha^{-1} g \alpha$.

So far, it has not been given a concrete construction about torsion units satisfying the first conjecture. In this study, we deal with the unit group of the integral group ring of symmetric group S_3. The unit group of ZS_3 was first characterized by Hughes and Pearson[5] and another description was given by Allen and Hobby[6]. The last description was given by Jespers and Parmenter[7]. But none of them characterized the torsion units explicitely. We have investigated the construction of the torsion units in the integral group ring of the group S_3 in terms of parameters.

Now, let us write the representation of ZS_3 obtained by using faithful irreducible representation of $S_3 =< a, b : a^3 = b^2 = 1,\; bab^{-1} = a^{-1} >$ (James and Liebeck [4]).

$$\rho : S_3 \to GL(2,Z)$$
$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$
$$b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By extending the representation ρ of S_3 linearly over Z, we can get a representation $\overline{\rho}$ from ZS_3 to Z^{2x2}, a matrix ring of integers as follows :

$$\overline{\rho} : ZS_3 \longrightarrow Z^{2x2}$$
$$\sum \gamma_g g \mapsto \sum \gamma_g \rho(g).$$

For an arbitrary $\gamma = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \beta_0 b + \beta_1 ba + \beta_2 ba^2 \in ZS_3$, the image of γ is

$$\overline{\rho}(\gamma) = \begin{bmatrix} \alpha_0 - \alpha_2 - \beta_1 + \beta_2 & \alpha_1 - \alpha_2 + \beta_0 - \beta_1 \\ -\alpha_1 + \alpha_2 + \beta_0 - \beta_2 & \alpha_0 - \alpha_1 + \beta_1 - \beta_2 \end{bmatrix}.$$

2. SOME BASIC RESULTS

Now, let us state some results which will be used to characterize torsion units in the unit group of the integral group ring of S_3 .

Lemma 2.1 Let G be a metacyclic group and $N = G'$, its commutator subgroup. Consider the following natural ring homomorphism :

$$\varphi : ZG \longrightarrow Z(G/N)$$
$$\sum \gamma_g g \mapsto \sum \gamma_g (gN).$$

If $\gamma \in U_1(ZG)$ is a torsion unit then $\varphi(\gamma) = gN$, for some $g \in G$.

Proof. Let $\gamma \in U_1(ZG)$ be a torsion unit. From Theorem1.1 γ is a rational conjugate to $g \in G$. That is, $\gamma = \alpha g \alpha^{-1}$ for some $\alpha \in QG$ and $g \in G$. Since G/N is abelian and φ is the natural ring homomorphism, we have

$$\varphi(\gamma) = \varphi(\alpha g \alpha^{-1}) = \varphi(\alpha)\varphi(g)\varphi(\alpha)^{-1} = \varphi(g) = gN .$$

Lemma 2.2 Let $\overline{\rho}$ be the representation of ZG obtained by extending linearly, by a representation ρ of finite degree of a group G. If $\gamma \in U_1(ZG)$ is a rational conjugate to $g \in G$ then $|\overline{\rho}(\gamma)| = |\rho(g)|$.

Proof. Clear.

Preposition 2.3 (Berman-Higman[2]) Let G be a finite group. Suppose $\gamma \in U(ZG)$ is a torsion unit, namely, $\gamma^n = 1$ for some natural number n. If $\gamma_e \neq 0$ then $\gamma = \pm 1$.

Corollary 2.4 $\gamma \in U_1(ZG)$ is a torsion unit and $\gamma \neq 1$ then $\gamma_e = 0$.

Corollary 2.5 If $\gamma \in U_1(ZG)$ is a rational conjugate to $g \in Z(G)$, then $\gamma = g$. That is, any torsion unit which is rational conjugate to an element in the center of a group is trivial.

3. PARAMETRIZATION OF TORSION UNITS

$S_3 = \{1\} \cup \{a, a^2\} \cup \{b, ba, ba^2\}$ has three conjugacy classes. By Theorem1.1, we can say that any torsion unit must be conjugate to one of the conjugacy classes above. As a result, a torsion unit of order 3 is conjugate to a and a torsion unit of order 2 is conjugate to b. Now let us characterize these two types of torsion units.

Teorem 3.1 Let $\gamma_3 \in U_1(ZS_3)$ be a torsion unit of order 3. γ_3 can be expressed in terms of two parameters as follows

$$\gamma_3 = [n-3k+2+\frac{3k^2-3k+1}{n}]a+[1-n+3k-2-\frac{3k^2-3k+1}{n}]a^2+[n-4k+2+\frac{3k^2-3k+1}{n}]b$$
$$+[2k-1-n]ba+[1-2k+\frac{3k^2-3k+1}{n}]ba^2,$$

where $k \in Z$ is an arbitrary parameter and n is a divisor of $3k^2$-$3k$+1.

Proof. Let $\gamma_3 \in U_1(ZS_3)$ be a torsion unit of order 3. Since S_3 is a split extesion of <a> and <b> with faithful action, by Theorem1.1, γ_3 is a rational conjugate to $a \in S_3$. In this case, by Lemma 2.1, we have

$$\varphi(\gamma_3) = aN = \begin{cases} \alpha_0 + \alpha_1 + \alpha_2 = 1 \\ \beta_0 + \beta_1 + \beta_2 = 0. \end{cases}$$

By proposition 2.3 and Corollary 2.4, we can write all coefficients in terms of the three parameters α_1, β_0 and β_1 as follows :

$$\alpha_0 = 0,\ \alpha_2 = 1-\alpha_1,\ \beta_2 = -\beta_0 - \beta_1 .$$

Now, if we consider the representation $\bar{\rho}$ then, we get

$$\bar{\rho}(\gamma_3) = \begin{bmatrix} -1+\alpha_1-\beta_0-2\beta_1 & -1+2\alpha_1+\beta_0-\beta_1 \\ 1-2\alpha_1+2\beta_0+\beta_1 & -\alpha_1+\beta_0+2\beta_1 \end{bmatrix}.$$

By Lemma 2.2, we have

$$|\bar{\rho}(\gamma_3)| = 1 \ \Rightarrow\ \beta_0^2 + \beta_0\beta_1 + \beta_1^2 = \alpha_1^2 - \alpha_1 .$$

If we use the substitution $\alpha_1 - \beta_0 = k \in Z$, then

$$(\beta_0+k)^2 - (\beta_0+k) = \beta_0^2 + \beta_0\beta_1 + \beta_1^2 \Rightarrow (2k-1-\beta_1)\beta_0 = \beta_1^2 - k^2 + k$$
$$\Rightarrow n\beta_0 = (2k-1-n)^2 - k^2 + k,\ \ (n \in Z \text{ ve } \beta_1 = 2k-1-n)$$
$$\Rightarrow n\beta_0 = n(n-4k+2) + 3k^2 - 3k + 1$$
$$\Rightarrow n \mid (3k^2 - 3k + 1).$$

Hence, we can write free coefficients in terms of the parameters of n, k as follows:

$$\beta_0 = n - 4k + 2 + \frac{3k^2 - 3k + 1}{n},\ \beta_1 = 2k - 1 - n \text{ and } \alpha_1 = n - 3k + 2 + \frac{3k^2 - 3k + 1}{n}.$$

Theorem 3.2 Let $\gamma_2 \in U_1(ZS_3)$ be a torsion unit of order 2.Then, γ_2 can be expressed in terms of two parameters as follows:

$$\gamma_2 = [n-3k-1+\frac{3k^2+2k}{n}](a-a^2)+[n-4k-1+\frac{3k^2+2k}{n}]b+[2k+1-n]ba+[2k+1-\frac{3k^2+2k}{n}]ba^2,$$

where $k\in Z$ an arbitrary parameter and n is a divisor of $3k^2+2k$.

Poof. Let $\gamma_2 \in U_1(ZS_3)$ be a torsion unit of order 2. By Theorem1.1, γ_2 is a rational conjugate to $b\in S_3$. In this case, by Lemma 2.1, we have

$$\varphi(\gamma_2) = bN \Rightarrow \left.\begin{matrix} \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ \beta_0 + \beta_1 + \beta_2 = 1 \end{matrix}\right\} \Rightarrow \alpha_0 = 0,\ \alpha_2 = -\alpha_1,\ \beta_2 = 1 - \beta_0 - \beta_1.$$

Rewriting the representation $\overline{\rho}$ in terms of the parameters α_1, β_0 and β_1, we have

$$\overline{\rho}(\gamma_2) = \begin{bmatrix} 1-\alpha_1-\beta_0-2\beta_1 & 2\alpha_1+\beta_0-\beta_1 \\ -1-2\alpha_1-2\beta_0+\beta_1 & -1-\alpha_1+\beta_0+2\beta_2 \end{bmatrix}.$$

On the other hand, by Lemma 2.2

$$|\overline{\rho}(\gamma_2)| = -1 \Rightarrow \beta_0^2 + \beta_0\beta_1 + \beta_1^2 - \beta_0 - \beta_1 = \alpha_1^2.$$

For $\alpha_1 - \beta_0 = k \in Z$, we get

$$(\beta_0+k)^2 = \beta_0^2+\beta_0\beta_1+\beta_1^2-\beta_0-\beta_1 \Rightarrow (2k+1-\beta_1)\beta_0 = \beta_1^2-\beta_1-k^2$$

$$\Rightarrow n\beta_0 = (2k+1-n)^2-(2k+1-n)-k^2,\ (n\in Z \text{ ve} \beta_1 = 2k+1-n)$$

$$\Rightarrow n\beta_0 = n(n-4k-1)+3k^2+2k$$

$$\Rightarrow n \mid (3k^2+2k).$$

So, $\beta_0 = n-4k-1+\dfrac{3k^2+2k}{n}$, $\beta_1 = 2k+1-n$, $\alpha_1 = n-3k-1+\dfrac{3k^2+2k}{n}$.

Corollary 3.3 Any torsion unit in $U_1(ZS_3)$ can be exspressed in terms of two parameters; one of them is free and the other one depends on free parameter.

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