FLOW OF A THIRD GRADE FLUID BETWEEN CONCENTRIC CIRCULAR CYLINDERS

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Abstract- The steady-state flow of a third grade fluid between concentric circular cylinders with heat transfer is examined. The temperature of pipes is assumed to be higher than the temperature of the fluid. Reynold's model viscosity case is treated. Approximate analytical solutions are presented using perturbation technique.

Key words: annuli flow, third grade fluids, perturbation analysis

1. INTRODUCTION

Third grade fluids are used as a non-Newtonian fluid model in many papers. Some of the works are as follows: Massoudi and Christie [1] used finite-difference method and investigated the solutions for the flow of third grade fluid in a pipe. Fosdick and Rajagopal [2] made a complete thermodynamical analysis of third-grade fluids and showed the restrictions on the stress constitutive equation. Rajagopal [3] investigated some stability characteristics of third grade fluids and showed that they exhibit features different from that of Newtonian and second grade fluids. The flow of third grade fluids between heated parallel plates was treated by Szeri and Rajagopal [4] and the flow past an infinite porous flat plate with suction was examined by Rajagopal *et al.* [5]. Baldoni *et al.* [6] examined the helical flow of a third grade fluid between eccentric cylinders. Secondary flow of a third grade fluid between two eccentrically placed cylinder was investigated by Mollica and Rajagopal [7]. Pakdemirli [8] derived the boundary layer equations for third grade fluids using a general coordinate system. The coordinate system makes the equations independent of body shape immersed into flow. Conventional and multiple deck boundary layers were treated extensively for second and third grade fluids by the same author [9]. Yürüsoy and Pakdemirli [10] found approximate analytical solutions for the flow of a third grade fluid in a pipe. The solutions agree with numerical solutions in reference [1]. Lubrication of a slider bearing with a special third grade fluid was considered by Yürüsoy and Pakdemirli [11].

A general mathematical problem arises in fluids of grade n . Since the equations are of higher order than the corresponding Newtonian flow, additional boundary conditions are needed. The issue was discussed in reference [5]. Later a detailed discussion was presented by Rajagopal [12] and relevant references were given in that study.

In this work, the steady state flow of a third grade fluid between concentric circular cylinders with heat transfer is investigated. The temperature of the pipes is assumed to be higher than the temperature of the fluid. The analysis is based on perturbation technique. Reynold's model viscosity case is treated. Velocity and temperature profile will be calculated approximately. Numerical plots of velocity and temperature profiles will be given.

2. EQUATIONS OF MOTIONS

The non-dimensional form of the equations of motion of a third grade fluid in a pipe with heat transfer was derived by Massoudi and Christie [1]

$$\frac{d\mu}{dr} \frac{dv}{dr} + \frac{\mu}{r} \left(\frac{dv}{dr} + r \frac{d^2v}{dr^2} \right) + \frac{\Lambda}{r} \left(\frac{dv}{dr} \right)^2 \left(\frac{dv}{dr} + 3r \frac{d^2v}{dr^2} \right) = C \quad (1)$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r} \frac{d\theta}{dr} + \Gamma \left(\frac{dv}{dr} \right)^2 \left[\mu + \Lambda \left(\frac{dv}{dr} \right)^2 \right] = 0 \quad (2)$$

$$v(r_i) = v(r_o) = \theta(r_i) = \theta(r_o) = 0 \quad (3)$$

where r is the dimensionless radius ($r_i < r < r_o$), r_i is dimensionless radius of inner cylinder, r_o is dimensionless radius of outer cylinder, v is the dimensionless velocity, θ is the dimensionless temperature and μ is the dimensionless viscosity. These terms are related to the dimensional ones as follows (see Ref. [7]).

$$R^* = \frac{\bar{r}_o - \bar{r}_i}{2}, R = \frac{\bar{r}_i}{\bar{r}_o}, r_i = \frac{R}{1-R}, r_o = \frac{1}{1-R}, r = \frac{\bar{r}}{R^*}, v = \frac{\bar{v}}{V_0}, \theta = \frac{\bar{\theta} - \theta_w}{\bar{\theta}_m - \theta_w}, \mu = \frac{\bar{\mu}}{\mu_0} \quad (4)$$

where, V_0 is a reference velocity, μ_0 is a reference viscosity and $\bar{\theta}_m$ is the bulk mean fluid temperature.

The dimensionless parameters involved in equations (1) and (2) are

$$C = \frac{C_1 R^{*2}}{\mu_* V_0}, \quad C_1 = \frac{\partial p}{\partial z}, \quad \Gamma = \frac{\mu_* V_0^2}{k(\bar{\theta}_m - \theta_w)}, \quad \Lambda = \frac{\beta_3 V_0^2}{\mu_* R^{*2}} \quad (5)$$

where C_1 is the pressure drop in the axial direction, Γ is related to the Prandtl number and Eckert number, Λ is the dimensionless non-Newtonian viscosity, $\mu_* = \mu_0 e^{\theta_w}$ and k is the thermal conductivity. Solutions will be presented for three different viscosity models.

3. REYNOLD'S MODEL

One assumes expansions for velocity and temperature as follows

$$v = v_0 + \epsilon v_1 \quad (6)$$

$$\theta = \theta_0 + \epsilon \theta_1 \quad (7)$$

where ϵ is a small parameter. Viscosity depends on temperature in an exponential manner [1]

$$\mu = e^{-M\theta} \quad (8)$$

Inspecting the original equations, to retain the same terms at the first order as in the constant viscosity case, in addition to Λ , one has to select M also small

$$\Lambda = \epsilon \lambda, \quad M = \epsilon m \quad (9)$$

Viscosity and its derivative now reads approximately as follows

$$\mu \cong 1 - \epsilon m \theta$$

$$\frac{d\mu}{dr} \cong -\epsilon m \theta' \quad (10)$$

Substituting all into the original equations of motion and separating at each order of ϵ , one has

Order 1

$$r v_0'' + v_0' = Cr \quad (11)$$

$$\theta_0'' + \frac{1}{r} \theta_0' + \Gamma v_0'^2 = 0 \quad (12)$$

$$v_0(r_i) = v_0(r_o) = \theta_0(r_i) = \theta_0(r_o) = 0 \quad (13)$$

Order ϵ

$$r v_1'' + v_1' - m \theta_0 v_0' - r m \theta_0 v_0'' - r m \theta_0' v_0' = -\lambda v_0'^2 (v_0' + 3r v_0'') \quad (14)$$

$$\theta_1'' + \frac{1}{r} \theta_1' + \Gamma (v_0')^2 (-m \theta_0 + \lambda v_0'^2) + 2 \Gamma v_0' v_1' = 0 \quad (15)$$

$$v_1(r_i) = v_1(r_o) = \theta_1(r_i) = \theta_1(r_o) = 0 \quad (16)$$

For the first order, solutions are

$$v_0 = \frac{Cr^2}{4} + c_1 \ln(r) + c_2 \quad (17)$$

$$\theta_0 = -\Gamma \left(\frac{C^2 r^4}{64} + \frac{Cc_1 r^2}{4} + \frac{c_1^2 \ln(r)^2}{2} \right) + d_1 \ln(r) + d_2 \quad (18)$$

where c_1, c_2, d_1 and d_2

$$c_1 = \frac{C(r_i^2 - r_o^2)}{4 \ln(r_o/r_i)}, \quad c_2 = -\frac{C}{4} (r_i^2) - c_1 \ln(r_i) \quad (19)$$

$$d_1 = -\frac{\Gamma}{\ln(r_o/r_i)} \left(\frac{C^2(r_i^4 - r_o^4)}{64} + \frac{Cc_1(r_i^2 - r_o^2)}{4} - \frac{c_1^2}{2} (\ln(r_i)^2 - \ln(r_o)^2) \right) \quad (20)$$

$$d_2 = \Gamma \left(\frac{C^2}{64} r_i^4 + \frac{Cc_1}{4} r_i^2 + \frac{c_1^2}{2} \ln(r_i)^2 \right) - d_1 \ln(r_i) \quad (21)$$

Substituting these solutions to order ϵ equations one finally obtains

$$v_1 = \frac{t_0}{r^2} + t_1 r^2 - t_2 r^4 - t_3 r^6 + t_4 r^2 \ln(r) + t_5 \ln(r)^2 - t_6 r^2 \ln(r)^2 - t_7 \ln(r)^3 + h_1 \ln(r) + h_2 \quad (22)$$

$$\begin{aligned}\theta_1 = & k_1 r^8 + k_2 r^6 + r^4 (k_3 \ln(r)^2 + k_4 \ln(r) + k_5) + r^2 (k_6 \ln(r)^2 + k_7 \ln(r) + k_8) + k_9 \ln(r)^4 \\ & + k_{10} \ln(r)^3 + k_{11} \ln(r)^2 + \frac{k_{12}}{r^2} + m_1 \ln(r) + m_2\end{aligned}\quad (23)$$

where $t_0, t_1, \dots, t_7, h_1, h_2, k_1, k_2, \dots, k_{12}, m_1$ and m_2

$$t_0 = \frac{c_1^3 \lambda}{2} \quad (24)$$

$$t_1 = \frac{C(-6Cc_1\lambda - 2d_1m + 4d_2m - 3c_1^2\Gamma m)}{16} \quad (25)$$

$$t_2 = \frac{C^2(8C\lambda + 9c_1\Gamma m)}{256} \quad (26)$$

$$t_3 = \frac{C^3\Gamma m}{768} \quad (27)$$

$$t_4 = \frac{Cm(2d_1 + c_1^2\Gamma)}{8} \quad (28)$$

$$t_5 = \frac{c_1 d_1 m}{2} \quad (29)$$

$$t_6 = \frac{Cc_1^2\Gamma m}{8} \quad (30)$$

$$t_7 = \frac{c_1^3\Gamma m}{6} \quad (31)$$

$$\begin{aligned}h_1 = & \frac{1}{\ln(r_0/r_1)} \left[t_0 \left(\frac{1}{r_i^2} - \frac{1}{r_o^2} \right) + t_1(r_i^2 - r_o^2) - t_2(r_i^4 - r_o^4) - t_3(r_i^6 - r_o^6) + t_4(r_i^2 \ln(r_i) - r_o^2 \ln(r_o)) \right. \\ & \left. + t_5(\ln(r_i)^2 - \ln(r_o)^2) - t_6(r_i^2 \ln(r_i)^2 - r_o^2 \ln(r_o)^2) - t_7(\ln(r_i)^3 - \ln(r_o)^3) \right] \quad (32)\end{aligned}$$

$$h_2 = \left[-\frac{t_0}{r_i^2} - t_1 r_i^2 + t_2 r_i^4 + t_3 r_i^6 - t_4 r_i^2 \ln(r_i) - t_5 \ln(r_i)^2 + t_6 r_i^2 \ln(r_i)^2 + t_7 \ln(r_i)^3 - h_1 \ln(r_i) \right] \quad (33)$$

$$k_1 = \left(-\frac{m\Gamma^2 C^4}{16384} + \frac{3\Gamma C t_3}{32} \right) \quad (34)$$

$$k_2 = \left(-\frac{\lambda \Gamma C^4}{576} - \frac{5m\Gamma^2 C^3 c_1}{2304} + \frac{\Gamma c_1 t_3}{3} + \frac{\Gamma C t_2}{9} \right) \quad (35)$$

$$k_3 = \left(-\frac{m\Gamma^2 C^2 c_1^2}{128} + \frac{\Gamma C t_6}{8} \right) \quad (36)$$

$$k_4 = \left(\frac{m\Gamma^2 C^2 c_1^2}{128} + \frac{m\Gamma d_1 C^2}{64} - \frac{\Gamma C t_4}{8} \right) \quad (37)$$

$$k_5 = \left(-\frac{\lambda \Gamma C^3 c_1}{32} + \frac{m\Gamma d_2 C^2}{64} - \frac{\Gamma C t_6}{64} - \frac{m\Gamma d_1 C^2}{128} - \frac{\Gamma C t_1}{8} - \frac{5m\Gamma^2 C^2 c_1^2}{256} + \frac{\Gamma c_1 t_2}{2} \right) \quad (38)$$

$$k_6 = \left(-\frac{m\Gamma^2 C c_1^3}{8} + \Gamma c_1 t_6 + \frac{3\Gamma C t_2}{4} \right) \quad (39)$$

$$k_7 = \left(\frac{m\Gamma^2 C c_1^3}{4} - \Gamma c_1 t_4 - \frac{\Gamma C t_5}{2} - \Gamma c_1 t_6 - \frac{3\Gamma C t_7}{2} + \frac{m\Gamma d_1 c_1 C}{4} \right) \quad (40)$$

$$k_8 = \left(\frac{\Gamma C t_5}{2} - \Gamma c_1 t_1 - \frac{3\lambda \Gamma c_1^2 C^2}{8} + \frac{m\Gamma d_2 C c_1}{4} - \frac{\Gamma C k_1}{4} + \frac{\Gamma c_1 t_6}{2} + \frac{\Gamma c_1 t_4}{2} - \frac{m\Gamma^2 C c_1^3}{4} - \frac{m\Gamma c_1 C d_1}{4} + \frac{9\Gamma C t_7}{8} \right) \quad (41)$$

$$k_9 = \left(\frac{\Gamma c_1 t_7}{2} - \frac{m\Gamma^2 c_1^4}{24} \right) \quad (42)$$

$$k_{10} = \left(-\frac{2\Gamma c_1 t_5}{3} + \frac{m c_1^2 \Gamma d_1}{6} \right) \quad (43)$$

$$k_{11} = \left(\frac{m\Gamma d_2 c_1^2}{2} - \Gamma c_1 k_1 + \Gamma C t_0 - \lambda \Gamma c_1^3 C \right) \quad (44)$$

$$k_{12} = \left(\frac{-\lambda \Gamma c_1^4 + 4\Gamma c_1 t_0}{4} \right) \quad (45)$$

$$m_1 = \frac{1}{\ln(r_o/r_i)} \left[k_1 (r_i^8 - r_o^8) + k_2 (r_i^6 - r_o^6) + (r_i^4 - r_o^4) (k_3 (\ln(r_i)^2 - \ln(r_o)^2) + k_4 (\ln(r_i) - \ln(r_o)) + k_5) + (r_i^2 - r_o^2) (k_6 (\ln(r_i)^2 - \ln(r_o)^2) + k_7 (\ln(r_i) - \ln(r_o)) + k_8) + k_9 (\ln(r_i)^4 - \ln(r_o)^4) + k_{10} (\ln(r_i)^3 - \ln(r_o)^3) + k_{11} (\ln(r_i)^2 - \ln(r_o)^2) + k_{12} \left(\frac{1}{r_i^2} - \frac{1}{r_o^2} \right) \right] \quad (46)$$

$$m_2 = - \left[k_1 r_i^8 + k_2 r_i^6 + r_i^4 (k_3 \ln(r_i)^2 + k_4 \ln(r_i) + k_5) + r_i^2 (k_6 \ln(r_i)^2 + k_7 \ln(r_i) + k_8) + k_9 \ln(r_i)^4 + k_{10} \ln(r_i)^3 + k_{11} \ln(r_i)^2 + k_{12}/r_i^2 \right] - m_1 \ln(r_i) \quad (47)$$

For $M=0$, solutions reduced to those of constant viscosity case.

4. NUMERICAL RESULTS

In this section, approximate analytical solutions obtained in Sections 3 are plotted for C , Γ , Λ and M . In Figures 1 and 2, variation of velocity and temperature profiles with the pressure gradient is shown. Results are plotted for $|C| \leq 1$. As expected, when pressure gradient decreases absolutely, the velocity and temperature profiles also decrease. In Figures 3 and 4, variation of Γ parameter (related to Eckert and Prandtl number) is considered. As Γ increases both quantities also increase. When non-Newtonian effect increases (as Λ increases), in Figures 5 and 6, velocity and temperature decreases. Figures 7 and 8 show the variation with respect to the parameter M . As M increases, the velocity and temperature also increase.

Figure 1. Effect of C on velocity profile (C values indicated on the curves) ($R=1/2$, $M=\Lambda=\Gamma=1$)

Figure 4. Effect of Γ on temperature profile (Γ values indicated on the curves) ($R=4/5$, $M=\Lambda=1$, $C=-1$)

Figure 2. Effect of C on temperature profile (C values indicated on the curves) ($R=4/5$, $M=\Lambda=\Gamma=1$)

Figure 5. Effect of Λ on velocity profile (Λ values indicated on the curves) ($R=1/2$, $M=\Gamma=1$, $C=-1$)

Figure 3. Effect of Γ on velocity profile (Γ values indicated on the curves) ($R=1/2$, $M=\Lambda=1$, $C=-1$)

Figure 6. Effect of Λ on temperature profile (Λ values indicated on the curves) ($R=4/5$, $\Gamma=M=1$, $C=-1$)

Figure 7. Effect of M on velocity profile (M values indicated on the curves) ($R=1/2, \Lambda=\Gamma=1, C=-1$)

Figure 8. Effect of M on temperature profile (M values indicated on the curves) ($R=4/5, \Gamma=\Lambda=1, C=-1$)

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