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THE GCD-RECIPROCAL LCM MATRICES ON GCD-CLOSED SETS

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Abstract- We have given a structure theorem for the GCD-Reciprocal LCM matrix and then we have calculated the value of the determinant of the GCD-Reciprocal LCM matrix. We have obtained formula for the determinant and the inverse of the GCD-Reciprocal LCM matrix defined on gcd-closed sets.

Key words- Greatest common divisor, GCD matrices, LCM matrices.

1.INTRODUCTION

Let $S=\{x_1,x_2,...,x_n\}$ be an ordered set of distinct positive integers. The matrix (S) whose ij-entry is the greatest common divisor (x_i,x_j) of x_i and x_j is called the GCD matrix on S [4]. The LCM matrix [S] on S is defined analogously [5].

In 1876, H.J.S. Smith [1] shown that the determinant of the GCD matrix defined on S={1,2,…,n }(Smith's determinant) is equal to $\varphi(1)\varphi(2)...\varphi(n)$, where φ is Euler's totient function. He also noted that this result remains valid if S is replaced by a factor closed (FC) set (i.e., all positive factors of any member of S belong to S). The S is said to be gcd-closed if $(x_i,x_j)\in$ S whenever $x_i,x_j\in$ S [3].

In this paper, we define an $n\times n$ matrix $A=[a_{ij}]$, where $a_{ij}=\dfrac{(x_i,x_j)}{[x_i,x_j]}$ and call it to be the GCD-Reciprocal LCM matrix on $S=\{x_1,x_2,...,x_n\}$. In the second section, we give a structure theorem for the GCD-Reciprocal LCM matrix defined on S and then we have calculated the value of the determinant of the GCD-Reciprocal LCM matrix. If S gcd-closed, we have calculated the determinant and the inverse of the GCD-Reciprocal LCM matrix defined on S.

2. THE STRUCTURE OF GCD-RECIPROCAL LCM MATRİX

Definition 1. Let $S=\{x_1,x_2,...,x_n\}$ be an ordered set of distinct positive integers. The $n\times n$ matrix $[A]=(a_{ij})$ having $a_{ij}=\dfrac{(x_i,x_j)}{[x_i,x_j]}$ as its ij-entry is called GCD-Reciprocal LCM matrix on S where (x_i,x_j) is the greatest common divisor of x_i and x_j, $[x_i,x_j]$ is the least common multiple of x_i and x_j.

Clearly GCD-Reciprocal LCM matrices are symmetric. Furthermore rearrangements of the elements of S yield similar matrices. Hence we may always assume $x_1<x_2<...<x_n$.

We let $B(x_i)$ denote the sum,

$$B(x_i)=\sum_{\substack{d|x_i\\ d\nmid x_t\\ t<i}} J_2(d)$$

for all i = 1,2,...,n; where $J_2(n)$ is Jordan's totient function as

$$J_2(n)=\sum_{d|n}\mu(d)\left(\frac{n}{d}\right)^2 .$$

We note that $B(x_i)=J_2(x_i)$ for all i = 1,2,...,n if and only if S is factor closed .

Now we prove a structure theorem for GCD-Reciprocal LCM matrices defined on $S=\{x_1,x_2,...,x_n\}$.

Theorem 1. Let $\bar{S}=\{d_1,d_2,...,d_m\}$ be the minimal gcd-closed ordered set containing $S=\{x_1,x_2,...,x_n\}$ where $x_1<x_2<...<x_n$ and $d_1<d_2<...d_m$. Define the $n\times m$ matrix $E=(e_{ij})$ by

$$e_{ij}=\begin{cases}\frac{1}{x_i}, & d_j\mid x_i\\ 0, & otherwise\end{cases}$$

and the $m\times m$ diagonal matrix by

$$\Lambda=diag(B(d_1),B(d_2),...,B(d_m)).$$

Then, $[A]=E\Lambda E^T$.

Proof. The ij-entry of $E\Lambda E^T$ is equal to

$(E\Lambda E^T)_{ij}=\sum_{k=1}^{m}B(d_k)e_{ik}e_{jk}=\sum_{\substack{d_k|x_i\\ d_k|x_j}}B(d_k)\frac{1}{x_i}\frac{1}{x_j}=\frac{1}{x_ix_j}\sum_{d_k|(x_i,x_j)}B(d_k)=\frac{1}{x_ix_j}\sum_{d|(x_i,x_j)}J_2(d),$ it is well known $\sum_{d|n}J_2(d)=n^2$. Then,

$$(E\Lambda E^T)_{ij}=\frac{1}{x_ix_j}(x_i,x_j)^2=\frac{(x_i,x_j)}{[x_i,x_j]}=a_{ij}\,.$$

Theorem 2. Let $\bar{S}=\{d_1,d_2,...,d_m\}$ be the minimal gcd-closed ordered set containing $S=\{x_1,x_2,...,x_n\}$ where $d_1<d_2<...d_m$ and $x_1<x_2<...<x_n$. Then

$$\det[A]=\sum_{1\le k_1<k_2<...<k_n\le m}\left(\det E_{(k_1,k_2,...,k_n)}\right)^2 B(x_{k_1})B(x_{k_2})...B(x_{k_n})$$

where $E_{(k_1,k_2,...,k_n)}$ is the submatrix of E consisting of the k_1 th, k_2 th,..., k_n th columns of E.

Proof. Theorem 1 says that $[A]=(E\Lambda^{1/2})(E\Lambda^{1/2})^T$. We can write $C=E\Lambda^{1/2}$. Then $[A]=CC^T$. So by the Cauchy-Binet formula, we obtain

$$\det[A]=\det(CC^T)=\sum_{1\le k_1<k_2<...<k_n\le m}\det C_{(k_1,k_2,...,k_n)}\det C^T{}_{(k_1,k_2,...,k_n)}$$

$$=\sum_{1\le k_1<k_2<...<k_n\le m}\left(\det C_{(k_1,k_2,...,k_n)}\right)^2$$

where $C_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of C consisting of the k_1 th, k_2 th, …, k_n th columns of C.

$$\det C_{(k_1,k_2,\ldots,k_n)} = \sqrt{B(x_{k_1})B(x_{k_2})\ldots B(x_{k_n})}\,\det E_{(k_1,k_2,\ldots,k_n)}$$

and hence,

$$\det[A] = \sum_{1\le k_1<k_2<\ldots<k_n\le m} \left(\det E_{(k_1,k_2,\ldots,k_n)}\right)^2 B(x_{k_1})B(x_{k_2})\ldots B(x_{k_n}).$$

Corollary 1. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. If S is gcd-closed, then the determinant of the GCD-Reciprocal LCM matrix [A] defined on S is

$$\det[A] = \prod_{k=1}^{n} \frac{B(x_k)}{{x_k}^2}.$$

Theorem 3. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. If S is gcd-closed set, then the inverse of the GCD-Reciprocal LCM matrix [A] defined on S is the matrix $N = (n_{ij})$, where

$$n_{ij} = \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{p_{ik}\, p_{jk}}{B(x_k)}.$$

Proof. We will calculate the inverse of the GCD-Reciprocal LCM matrix defined on a gcd-closed set. Let $E = (e_{ij})$ and $P = (p_{ij})$ be defined as follows:

$$e_{ij} = \begin{cases} \dfrac{1}{x_i}, & x_j \mid x_i \\ 0, & otherwise \end{cases}$$

$$p_{ij} = x_i \sum_{\substack{dx_i|x_j \\ dx_i\nmid x_t \\ x_t<x_j}} \mu(d) \quad \text{or} \quad p_{ij} = x_i \sum_{\substack{d|\frac{x_j}{x_i} \\ d\nmid\frac{x_t}{x_i} \\ x_t<x_j}} \mu(d)$$

(If $\dfrac{x_j}{x_i}$ is not an integer then no d divides $\dfrac{x_j}{x_i}$.)

Calculating the ij-entry of the product EP^T gives

$$(EP^T)_{ij} = \sum_{k=1}^{n} e_{ik}\, p_{jk} = \sum_{x_k|x_i} \frac{1}{x_i} x_j \sum_{\substack{dx_j|x_k \\ dx_j\nmid x_t \\ x_t<x_k}} \mu(d) = \frac{x_j}{x_i} \sum_{dx_j|x_i} \mu(d) = \frac{x_j}{x_i} \sum_{d|\frac{x_i}{x_j}} \mu(d) = \begin{cases} 1, & x_i = x_j \\ 0, & otherwise \end{cases}.$$

Hence $E^{-1} = P^T$.

If $\Lambda = \mathrm{diag}\big(B(x_1), B(x_2), \ldots, B(x_n)\big)$ then by Theorem 1, we write $[A] = E\Lambda E^T$. Therefore, $[A]^{-1} = (E^T)^{-1}\,\Lambda^{-1}\,E^{-1} = (E^{-1})^T\,\Lambda^{-1}\,E^{-1} = P\,\Lambda^{-1}\,P^T = (\,n_{ij}\,)$ where

$$n_{ij} = \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{p_{ik}\,p_{jk}}{B(x_k)}.$$

3. NUMERICAL RESULTS

Example 1. Let $S=\{4,6,8\}$. The GCD-Reciprocal LCM matrix [A] defined on S is

$$[A]=\begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & 1 & \frac{1}{12} \\ \frac{1}{2} & \frac{1}{12} & 1 \end{bmatrix}.$$

S is not gcd-closed set and $\overline{S} = \{2,4,6,8\}$ is the minimal gcd-closed set containing S. The 3×4 matrix $E = (e_{ij})$ defined in Theorem 1 is

$$E=\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}.$$

From Theorem 2,

$$\det[A] = \sum_{1\le k_1<k_2<k_3\le 4} \big(\det E_{(k_1,k_2,k_3)}\big)^2 B(x_{k_1})B(x_{k_2})B(x_{k_3}).$$

$$\det[A] = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{8} & 0 \end{vmatrix}^2 B(2)B(4)B(6) + \begin{vmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{6} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{vmatrix}^2 B(2)B(4)B(8) +$$

$$+ \begin{vmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} \end{vmatrix}^2 B(2)B(6)B(8) + \begin{vmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} \end{vmatrix}^2 B(4)B(6)B(8)$$

where

$$B(2) = \sum_{d\mid 2} J_2(d) = J_2(1) + J_2(2) = 4 \qquad B(4) = \sum_{\substack{d\mid 4 \\ d\nmid 2}} J_2(d) = J_2(4) = 12$$

$$\mathrm{B}(6)=\sum_{\substack{d\mid 6\\ d\nmid 2,4}} J_2(d)=J_2(3)+J_2(6)=32 \qquad \mathrm{B}(8)=\sum_{\substack{d\mid 8\\ d\nmid 2,4,6}} J_2(8)=48.$$

Thus

$$\det[A]=\frac{35}{48}.$$

Example 2. Let S={5,15,30}. The GCD-Reciprocal LCM matrix [A] defined on S is

$$[\mathrm{A}]=\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & 1 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & 1 \end{bmatrix}.$$

S is gcd-closed set where

$$\mathrm{B}(5)=\sum_{d\mid 5} J_2(d)=J_2(1)+J_2(5)=25 \ , \ \mathrm{B}(15)=\sum_{\substack{d\mid 15\\ d\nmid 5}} J_2(d)=J_2(3)+J_2(15)=200$$

$$\mathrm{B}(30)=\sum_{\substack{d\mid 30\\ d\nmid 5,15}} J_2(d)=J_2(2)+J_2(6)+J_2(10)+J_2(30)=675.$$

Then

$$\det[A]=\prod_{k=1}^{3}\frac{B(x_k)}{x_k^{\ 2}}=\frac{B(5).B(15).B(30)}{5^2\,15^2\,30^2}=\frac{2}{3}.$$

Example 3. Let S={5,15,30}. The GCD-Reciprocal LCM matrix [A] defined on S is

$$[\mathrm{A}]=\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & 1 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & 1 \end{bmatrix}.$$

S is gcd-closed set but not factor-closed. From Theorem 3, we obtain $[A]^{-1}=(n_{ij})$ where

$$p_{11}=5,\ p_{12}=-5,\ p_{13}=0,\ p_{21}=0,\ p_{22}=15,\ p_{23}=-15,\ p_{31}=0,\ p_{32}=0,\ p_{33}=30$$

and

$$n_{11}=\frac{p_{11}^2}{B(5)}+\frac{p_{12}^2}{B(15)}+\frac{p_{13}^2}{B(30)}=\frac{9}{8} \ , \ n_{12}=\frac{p_{12}p_{22}}{B(15)}+\frac{p_{13}p_{23}}{B(30)}=\frac{-3}{8}=n_{21} \ ,$$

$$n_{13}=\frac{p_{13}p_{33}}{B(30)}=0=n_{31} \qquad n_{22}=\frac{p_{22}^2}{B(15)}+\frac{p_{23}^2}{B(30)}=\frac{35}{24} \ , \ n_{23}=\frac{p_{23}p_{33}}{B(30)}=\frac{-2}{3}=n_{32} \ ,$$

$$n_{33}=\frac{p_{33}^2}{B(30)}=\frac{4}{3}.$$

Thus

$$N = \begin{bmatrix} \frac{9}{8} & \frac{-3}{8} & 0 \\ \frac{-3}{8} & \frac{35}{24} & \frac{-2}{3} \\ 0 & \frac{-2}{3} & \frac{4}{3} \end{bmatrix}.$$

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