VIBRATIONS OF A SIMPLY SUPPORTED BEAM
WITH A NON–IDEAL SUPPORT AT AN INTERMEDIATE POINT

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Abstract— A simply supported Euler-Bernoulli beam with an intermediate support is considered. The concept of non-ideal boundary conditions is applied to the beam problem. In accordance, the intermediate support is assumed to allow small deflections. Approximate analytical solution of the problem is found using the Method of Multiple Scales, a perturbation technique. Ideal and non-ideal frequency response curves are contrasted.

Keywords— nonlinear beam vibrations, non-ideal boundary conditions, perturbation methods

1. INTRODUCTION

In vibration of continuous systems, types of support conditions are important and have direct effect on the solutions and natural frequencies. Boundary conditions of real systems are idealized by different types of supports such as simply supported, clamped, sliding, free, etc. The real system is modeled by choosing one of the nearest ideal boundary conditions. It is always assumed that those ideal conditions are satisfied exactly. However, small deviations from ideal conditions in real systems indeed occur. For example, a beam connected at its ends to rigid supports by pins is modeled using simply supported boundary conditions which require deflections and moments to be zero. However, the hole and pin assembly may have small gaps and/or friction which may introduce small deflections and/or moments at the ends. Similarly, a real built-in beam may have very small variations in deflection and/or slope. These types of boundary conditions with small deviations from the ideal conditions are defined here as non-ideal boundary conditions.

Non-ideal boundary conditions are modeled using perturbations. The idea is applied to a simple-simple beam having a non-ideal simple support at an intermediate point. Effect of non-ideal boundary condition at the intermediate point on the natural frequencies and frequency-response relations are examined using the Method of Multiple Scales. The pioneering work on non-ideal boundary conditions was due to Pakdemirli and Boyaci [1] for linear beam problems and Pakdemirli and Boyaci [2] for a non-linear beam problem.

2. PROBLEM FORMULATION AND SOLUTION

The simple-simple Euler-Bernoulli beam considered here has a simple support at an intermediate point at \( x = \eta \) \((0<\eta<1)\), where \( x \) is the spatial co-ordinate (See Fig.1). The equations of motion and the boundary conditions are,
$$\ddot{w}_1 + w_1'' = -2\bar{\mu}\dot{w}_1 + F_1 \cos \Omega t, \quad \ddot{w}_2 + w_2'' = -2\bar{\mu}\dot{w}_2 + F_2 \cos \Omega t$$  \hspace{1cm} (1, 2)
$$w_1(0,t) = w_1'(0,t) = w_2(1,t) = w_2'(1,t) = 0 \quad w_1(\eta,t) = w_2(\eta,t) = \epsilon \alpha(t)$$  \hspace{1cm} (3)
$$w_1'(\eta,t) = w_2'(\eta,t) \quad w_1''(\eta,t) = w_2''(\eta,t)$$

where $w_1$ is the left hand side deflection and $w_2$ is the right hand side deflection, $t$ is the time variable, $\bar{\mu}$ is the damping coefficient and, $F_i$'s and $\Omega$ are the magnitudes and the frequency of the external excitations. (') denotes derivation with respect to time variable $t$ and (')' denotes derivation with respect to spatial variable $x$. $\epsilon$ is a small perturbation parameter. All variables are dimensionless. Here, small deflections at the intermediate point $\eta$ are permitted to indicate deviations from the ideal boundary condition.

![Fig. 1 - A simply supported beam with a non-ideal boundary condition at an intermediate point](image)

Now, approximate solutions of equations (1) and (2) with associated boundary conditions (3) are sought. The Method of Multiple Scales (perturbation technique) [3] is applied directly to the partial differential systems and boundary conditions. Expansions are assumed of the forms

$$w_i(x,t;\epsilon) = w_{i1}(x,T_0,T_1) + \epsilon w_{i2}(x,T_0,T_1),$$  \hspace{1cm} (4, 5)
$$w_{i2}(x,t;\epsilon) = w_{i21}(x,T_0,T_1) + \epsilon w_{i22}(x,T_0,T_1)$$

where $T_0 = t$ is the fast time scale and $T_1 = \epsilon t$ is the slow time scale. Only primary resonance case is considered and hence, the forcing and damping terms are ordered as

$$\bar{\mu} = \epsilon \mu \quad \text{and} \quad \bar{F}_i = \epsilon F_i$$  \hspace{1cm} (6)

The time derivatives are written as

$$(') = D_0 + \epsilon D_1, \quad ('rometer) = D_0^2 + 2\epsilon D_0 D_1, \quad D_n = \partial / \partial T_n$$  \hspace{1cm} (7)

Inserting equations (4)-(7) into equations (1)-(3) and equating coefficients of like powers of $\epsilon$, one obtains at order 1,

$$D_0^2 w_{i1} + w_{i1}'' = 0, \quad D_0^2 w_{i2} + w_{i2}'' = 0$$  \hspace{1cm} (8, 9)
$$w_{i1}(0,T_0,T_1) = w_{i2}(0,T_0,T_1) = w_{i21}(1,T_0,T_1) = w_{i22}(1,T_0,T_1) = 0$$
$$w_{i1}(\eta,T_0,T_1) = w_{i2}(\eta,T_0,T_1) = w_{i21}(\eta,T_0,T_1) = w_{i22}(\eta,T_0,T_1) = 0$$  \hspace{1cm} (10)
$$w_{i1}'(\eta,T_0,T_1) = w_{i2}'(\eta,T_0,T_1)$$

and, at order $\epsilon$,

$$D_0^2 w_{i2} + w_{i2}'' = -2D_0 D_1 w_{i1} - 2\mu D_0 w_{i1} + F_1 \cos \Omega T_0,$$  \hspace{1cm} (11, 12)
$$D_0^2 w_{i2} + w_{i2}'' = -2D_0 D_1 w_{i2} - 2\mu D_0 w_{i2} + F_2 \cos \Omega T_0$$
$$w_{i2}(0,T_0,T_1) = w_{i2}(0,T_0,T_1) = w_{i21}(1,T_0,T_1) = w_{i22}(1,T_0,T_1) = 0$$
$$w_{i2}(\eta,T_0,T_1) = w_{i2}(\eta,T_0,T_1) = \alpha(T_0,T_1) \quad w_{i2}'(\eta,T_0,T_1) = w_{i2}'(\eta,T_0,T_1)$$  \hspace{1cm} (13)
$$w_{i2}''(\eta,T_0,T_1) = w_{i2}''(\eta,T_0,T_1)$$
At order 1, a solution of the form

\[ w_{11} = (A(T_1) e^{i\omega_0 t} + cc) Y_1(x) \]

\[ w_{21} = (A(T_1) e^{i\omega_0 t} + cc) Y_2(x) \]  \hspace{1cm} (14, 15)

is assumed, where \( cc \) stands for the complex conjugate of the preceding terms. Substituting equations (14) and (15) into equations (8)–(10), one has

\[ Y_1^{(n)} - \omega^2 Y_1 = 0, \quad Y_2^{(n)} - \omega^2 Y_2 = 0 \]  \hspace{1cm} (16, 17)

\[ Y_1'(0) = Y_1'(\eta) = 0, \quad Y_2'(1) = Y_2'(\eta) = 0 \]  \hspace{1cm} (18)

Solving equations (16)–(18) exactly yields the mode shapes,

\[ Y_1(x) = C \sin \beta (1 - \eta) \left[ \sin \beta x - \frac{\sin \beta \eta}{\sinh \beta \eta} \sinh \beta x \right] \]  \hspace{1cm} (19, 20)

\[ Y_2(x) = C \sin \beta \eta \left[ \sin \beta (1 - x) - \frac{\sin \beta (1 - \eta)}{\sinh \beta (1 - \eta)} \sinh \beta (1 - x) \right] \]

and natural frequencies \( \omega \) which satisfy the transcendental equation

\[ \sin \beta \eta \sinh \beta \eta [\cos \beta (1 - \eta) \sinh \beta (1 - \eta) - \sin \beta (1 - \eta) \cosh \beta (1 - \eta)] \]

\[ + \sin \beta (1 - \eta) \sinh \beta (1 - \eta) [\cos \beta \eta \sinh \beta \eta - \sin \beta \eta \cosh \beta \eta] = 0 \]  \hspace{1cm} (21)

where

\[ \beta = \sqrt{\omega} \]  \hspace{1cm} (22)

Equation (21) is solved numerically for the first seven modes and results are given in Table-1 for different \( \eta \) values. Due to the symmetry of the problem, results are given up to \( \eta = 0.5 \). Inspecting equation (21), one finds that for some specific \( \beta \) values \( \sin \beta \eta \) and \( \sin \beta (1 - \eta) \) vanishes and for those degenerate cases the mode shapes take the simpler form

\[ Y_1(x) = C \sin \beta x, \quad Y_2(x) = -C \sin \beta (1 - x) \]  \hspace{1cm} (23, 24)

Because the homogeneous equations (8)–(10) have a non-trivial solution, the non-homogeneous problem (11)–(13) will have a solution only if a solvability condition [3] is satisfied. To determine this condition, the secular and non-secular terms are separated by assuming a solution of the form

\[ w_{12} = \phi_1(x, T_1) e^{i \omega_0 t} + W_1(x, T_0, T_1) + cc, \quad w_{22} = \phi_2(x, T_1) e^{i \omega_0 t} + W_2(x, T_0, T_1) + cc \]  \hspace{1cm} (25, 26)

Substituting this solution into equations (11)–(13), secular and non-secular terms separate (up to this order, only the secular ones are of interest)

\[ \phi_1^{(n)} - \omega^2 \phi_1 = -2i \omega D_1 AY_1 - 2 \mu \omega AY_1 + \frac{F_1}{2} e^{i \omega t}, \]  \hspace{1cm} (27, 28)

\[ \phi_2^{(n)} - \omega^2 \phi_2 = -2i \omega D_1 AY_2 - 2 \mu \omega AY_2 + \frac{F_2}{2} e^{i \omega t} \]

In obtaining these equations, the order 1 solutions (14) and (15) are substituted into equations (11)–(13). It is also assumed that the external excitation frequency is close to one of the natural frequencies of the system such that;

\[ \Omega = \omega + \epsilon \sigma \]  \hspace{1cm} (29)

Here \( \sigma \) is a detuning parameter of order 1. After algebraic manipulations, the solvability conditions for equations (27) and (28) are obtained.
\[ 2i\omega(D, A + \mu A) + kA[Y_2(\eta) - Y_1(\eta)] - \frac{1}{2} f e^{i\sigma t} = 0 \]  
(30)

where

\[ f = \int_0^\eta F_1Y_1dx + \int_\eta^\eta F_2Y_2dx \]  
(31)

In obtaining equation (30) the normalization condition \( \int_0^\eta Y_1^2dx + \int_\eta^\eta Y_2^2dx = 1 \) is employed.

The amplitude of the deflection allowed at the intermediate point is assumed to be of the same form as that of order 1 solution, namely

\[ a(T_0, T_1) = kA(T_1) e^{i\alpha T_1} + cc \]  
(32)

where \( k \) is an arbitrary constant of order 1. Equation (30) determines the modulations in the complex amplitudes. The polar form

\[ A = \frac{1}{2} a(T_1) e^{i(\alpha T_1)} \]  
(33)

is to be used to calculate real amplitudes and phases. After separating real and imaginary parts, one obtains

\[ \alpha x' = -\mu \alpha x + \frac{1}{2} f \sin \gamma, \quad \alpha x' = \alpha x - Ka + \frac{1}{2} f \cos \gamma \]  
(34, 35)

where \( \gamma \) and \( K \) are defined as

\[ \gamma = \sigma T_1 - \theta, \quad K = \frac{k}{2} [Y_2(\eta) - Y_1(\eta)] \]  
(36)

In the steady state case, \( \alpha' = \gamma' = 0 \) and solving for the detuning parameter yields

\[ \sigma = \frac{K}{\omega} \pm \sqrt{\frac{f^2}{4\omega^2 a^2} - \mu^2} \]  
(37)

For free undamped vibrations, non-ideal natural frequencies are obtained from

\[ \omega_{ni} = \omega + \epsilon \frac{K}{\omega} \]  
(38)

In Table 1, first five ideal and non-ideal natural frequencies are calculated and given for different \( \eta \) up to 0.5. The non-ideal frequencies may increase, decrease or remain unchanged depending on the position parameter \( \eta \) and number of modes.

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( \omega_{1i} )</th>
<th>( \omega_{1ni} )</th>
<th>( \omega_{2i} )</th>
<th>( \omega_{2ni} )</th>
<th>( \omega_{3i} )</th>
<th>( \omega_{3ni} )</th>
<th>( \omega_{4i} )</th>
<th>( \omega_{4ni} )</th>
<th>( \omega_{5i} )</th>
<th>( \omega_{5ni} )</th>
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<td>61.4112</td>
<td>122.1138</td>
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<td>72.6689</td>
<td>147.9077</td>
<td>145.6414</td>
<td>246.7401</td>
<td>242.2972</td>
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<td>355.3058</td>
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</tr>
</tbody>
</table>
Vibrations Of A Simply Supported Beam With A Non-Ideal Support At An Intermediate Point

One can also obtain amplitude-excitation frequency relation from (37) and (29)

$$\Omega = \omega + \varepsilon \frac{K}{\omega} \pm \frac{\varepsilon^2 \gamma^2}{4\omega'^2 a^2} \left( \varepsilon \mu \right)^2$$  \hspace{1cm} (39)

In Figure 2, four different plots corresponding to the first four natural frequencies are given for the parameter values, $\eta = 0.3$, $\varepsilon \mu = 0.01$, $\varepsilon k/2 = 0.1$ and $\varepsilon f = 1$.

![Graphs showing frequency response for different modes](image)

Figure 2- Frequency-response graphs; a) 1st mode, b) 2nd mode, c) 3rd mode, d) 4th mode (--- ideal, — non-ideal)

Beam deflections to the first order approximation are given as

$$w_1 = a \cos(\Omega - \gamma) Y_1(x) + O(\varepsilon), \quad w_2 = a \cos(\Omega - \gamma) Y_2(x) + O(\varepsilon)$$  \hspace{1cm} (40, 41)

3. CONCLUDING REMARKS

Non-ideal boundary conditions are defined and formulated using perturbation theory. A simply supported beam with a non-ideal simple support at an intermediate point is treated. Approximate analytical solution of the problem is presented using the Method of Multiple Scales. Ideal and non-ideal natural frequencies are given for different intermediate support locations. Depending on the mode numbers and locations, the frequencies may increase, decrease or remain unchanged. Mode shapes are also affected by the non-ideal boundary condition. To determine those variations, calculations should be carried up to order $\varepsilon^2$. 
REFERENCES

