POTENTIAL SYMMETRY GENERATORS OF SOME PERTURBED NONLINEAR EVOLUTION RELATED EQUATIONS

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Abstract- Some recent results on approximate Lie group methods and previously developed concepts on potential symmetries are extended and applied to nonlinear systems perturbed to some order by a small parameter. The potential (or auxiliary) form of the perturbed system necessarily requires knowledge of an "approximate" conservation law of the system. We make an analysis on a perturbed Burgers equation and discuss other nonlinear equations where the results are applicable.

Keywords- potential, approximate symmetries

1. INTRODUCTION

The notions of nonlocal and potential symmetries associated with a partial differential equation (pde) has been widely investigated in the literature-the most popular being that by Bluman and Kumei ([1]). As an example, we consider the nonlinear Burgers equation $R\{x, t, u\}$ (which arises in the modeling of turbulence) given by

$$u_{xx} - uu_x - u_t = 0. \quad (1)$$

Its associated auxilliary system $S\{x, t, u, v\}$ given by

$$v_x = 2u, \quad v_t = 2u_x - u^2 \quad (2)$$

has, amongst others, an infinite parameter Lie point transformation

$$X_0 = e^{\psi/4} [(2\psi_x + \psi u) \frac{\partial}{\partial u} + 4\psi \frac{\partial}{\partial v}], \quad (3)$$

where $\psi(x, t)$ satisfies the heat equation $\psi_t - \psi_{xx} = 0$, so that (3) defines a potential symmetry for (1) (see [2] and [1]). Thus, $Y_0 = e^{\psi/4} \psi \frac{\partial}{\partial v}$ is a Lie point symmetry generator of the integrated Burgers equation $T\{x, t, v\}$

$$v_{xx} - v_t - \frac{1}{4}v_x^2 = 0. \quad (4)$$

Also, Baumann ([3]) obtains a further seven dimensional finite group of generators for (2).

Recently, Kara et al ([4]) extended the notions of potential symmetries to ‘perturbed’ pdes. That is, approximate symmetries of pdes which involve a perturbation of the pde by a small parameter ε , say. It was shown that additional ‘exact’ (invariant) solutions may be obtained for the perturbed pde by considering the approximate symmetries of an associated auxiliary system as they may produce potential symmetries (called approximate potential symmetries) for the perturbed pde. The ideas were applied to a large class of wave equations with variable speeds and diffusion equations.

We now, briefly, present the theory for pdes involving a small parameter of order one.

Suppose an $(n-1)$ -form

$$\omega = T^i \frac{\partial}{\partial x_i} \lrcorner \Omega, \quad (5)$$

where $\Omega = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$, is approximately conserved to order one in ε along the solutions of a r th-order perturbed pde $R\{x, u\}$ given by

$$E(x, u, u_{(1)}, \dots, u_{(r)}; \varepsilon) = 0, \quad (6)$$

where $x = (x^1, x^2, \dots, x^n)$, ε a small parameter, $u_{(1)}, u_{(2)}, \dots, u_{(r)}$ are the collection of various order derivatives, viz., $u_i = D_i u$, $u_{ij} = D_j D_i u$ being the first, second derivatives, respectively, and so on and

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots, \quad i = 1, \dots, n \quad (7)$$

is the total derivative operator with respect to x^i . That is,

$$D\omega|_{(6)} = D_i T^i|_{(6)} = O(\varepsilon^2), \quad (8)$$

where $T^i = T^i(x, u, u_{(1)}, \dots, u_{(r-1)}; \varepsilon) = T_0^i + \varepsilon T_1^i$ and D is the operator of total differentiation. For further details on the geometric analysis, we refer the reader to [8].

In particular, for $n = 2$ (with $x^1 = t$ and $x^2 = x$), we can cast (6) as an auxiliary system $S\{x, t, u, v\}$

$$\begin{aligned} T_0^1 + \varepsilon T_1^1 &= \frac{\partial}{\partial x} v, \\ T_0^2 + \varepsilon T_1^2 &= -\frac{\partial}{\partial t} v. \end{aligned} \quad (9)$$

Definition 1. ([1],[4]) A Lie point symmetry generator (approximate)

$$\begin{aligned} X &= X_0 + \varepsilon X_1 \\ &= \tau_0 \frac{\partial}{\partial t} + \xi_0 \frac{\partial}{\partial x} + \phi_0 \frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial v} \\ &\quad + \varepsilon [\tau_1 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \phi_1 \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial v}] \end{aligned} \quad (10)$$

of (9) is a potential symmetry (approximate) of (6) if and only if coefficients τ_1 , ξ_1 and ϕ_1 depend explicitly on v . In general, all the coefficients depend on x, t, u and v .

We recall (from [4]) an algorithm, due to Baikov et al ([5], [6]) for determining approximate symmetries $X = X_0 + \varepsilon X_1$ for (9). If X_0 is a Lie point symmetry generator of the unperturbed form of (9), i.e., X_0 leaves invariant the system $v_x - T_0^1 = 0$ and $v_t + T_0^2 = 0$, then we determine *auxiliary* functions H_1 and H_2 by

$$H_1 = \frac{1}{\varepsilon} X_0 (v_x - T_0^1 - \varepsilon T_1^1) |_{(9)}, \quad (11)$$

$$H_2 = \frac{1}{\varepsilon} X_0 (v_t + T_0^2 + \varepsilon T_1^2) |_{(9)}.$$

Then, X_1 is determined from

$$X_1 (v_x - T_0^1) + H_1, \quad X_1 (v_t + T_0^2) + H_2 = 0 \quad (12)$$

along the unperturbed system $v_x - T_0^1 = 0$ and $v_t + T_0^2 = 0$.

2. A PERTURBED BURGERS EQUATION

We now consider Burgers equation with a first-order dissipation, viz.,

$$u_{xx} - uu_x - u_t = \varepsilon u_x \quad (13)$$

which has an auxiliary form

$$v_x = 2u, \quad v_t = 2u_x - u^2 - 2\varepsilon u. \quad (14)$$

As mentioned above, X_0 given in (3) is a symmetry generator of (14) for $\varepsilon = 0$ and is a potential symmetry of the unperturbed Burgers equation (1). By (11),

$$H_1 = \frac{1}{\varepsilon} X_0 (v_x - 2u), \quad H_2 = \frac{1}{\varepsilon} X_0 (v_t - 2u_x + u^2 + 2\varepsilon u)$$

along (14), where X_0 is prolonged to first-order, i.e.,

$$X_0 = \tau_0 \frac{\partial}{\partial t} + \xi_0 \frac{\partial}{\partial x} + \phi_0 \frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial v} + \phi_0^x \frac{\partial}{\partial u_x} + \phi_0' \frac{\partial}{\partial u_t} + \zeta_0^x \frac{\partial}{\partial v_x} + \zeta_0' \frac{\partial}{\partial v_t}$$

where

$$\phi_0^x = e^{\sqrt[4]{4}} \left[\frac{1}{2} v_x \psi_x + \frac{1}{4} v_x \psi u + 2\psi_{xx} + \psi_x u + \psi_x u_x \right],$$

$$\zeta_0' = e^{\sqrt[4]{4}} [v_t \psi + 4\psi_t],$$

$$\zeta_0^x = e^{\sqrt[4]{4}} [v_x \psi + 4\psi_x]$$

and ϕ_0' is not required. Thus, $H_1 = 0$ and

$$\begin{aligned} H_2 &= e^{\sqrt[4]{4}} / \varepsilon [2u\psi_x + u^2\psi + 4\psi_{xx} + 2\psi_x u + 2\psi u_x - 4u\psi_x - 2u^2\psi \\ &\quad - 4\varepsilon\psi_x - 2\varepsilon\psi u - 2u_x\psi + u^2\psi + 2\varepsilon u\psi - 4\psi_t] \\ &= -4e^{\sqrt[4]{4}}\psi_x \end{aligned}$$

as $\psi = \psi_{xx}$. We now determine

$$X_1 = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial u} + \zeta \frac{\partial}{\partial v} + \phi^x \frac{\partial}{\partial u_x} + \phi' \frac{\partial}{\partial u_t} + \zeta^x \frac{\partial}{\partial v_x} + \zeta' \frac{\partial}{\partial v_t}$$

given as a first prolongation, in (12) along the solutions of (2). Equation (12b) is

$$\begin{aligned} &\zeta_t + u_t \zeta_u + \zeta_v (2u_x - u^2) - \tau_t (2u_x - u^2) - u_x \tau_u (2u_x - u^2) - \tau_v (4u_x^2 - 4u^2 u_x + u^4) \\ &- \xi_t 2u - \xi_u u_t 2u - \xi_v 2u (2u_x - u^2) \\ &- 2[\phi_x + u_x \phi_u + \phi_v 2u - \tau_x u_t - \tau_u u_x u_t - \tau_v u_t 2u - u_x \xi_x - \xi_u u_x^2 - \xi_v u_x 2u] \\ &+ 2u\phi - 4e^{\sqrt[4]{4}}\psi_x = 0 \end{aligned}$$

which, by separation, yields

$$\begin{aligned}\xi_u &= 2\tau_v, \\ 2\xi_v - 2\tau_t - 2\phi_u + 2\xi_x &= 0, \\ \tau_u u^2 + \xi_u - 2u\xi_u + 2\tau_x + 4u\tau_v &= 0, \\ \xi_t + \tau_t u^2 - 2\xi_t u + 2u^3 \xi_v - 2\phi_x - 4u\phi_v - \xi_v u^2 + 2u\phi - 4e^{y/4} \psi_x &= 0\end{aligned}\tag{15}$$

and (12a) is

$$\begin{aligned}\xi_x + u_x \xi_u + \xi_v 2u - \tau_x (2u_x - u^2) - \tau_u u_x (2u_x - u^2) - \tau_v 2u (2u_x - u^2) \\ - \xi_x 2u - \xi_u u_x 2u - \xi_v 4u^2 - 2\phi = 0\end{aligned}$$

which yields the system

$$\begin{aligned}\tau &= 0, \\ \xi_u - 2\tau_x - 4u\tau_v - 2u\xi_u &= 0, \\ \xi_x + 2u\xi_v - 2u\xi_x - 2\phi &= 0.\end{aligned}\tag{16}$$

The calculations shows that τ is a constant (time translation) and

$$\begin{aligned}\phi &= A^1(x, t)u + A^2(x, t)e^{y/4}u + B(x, t, v), \\ \xi &= 4A^2e^{y/4} + P^1(x, t)v + P^2(x, t)\end{aligned}$$

where

$$\begin{aligned}B &= Q^1(x, t) + Q^2(x, t)e^{y/2} + 2A_x^2e^{y/4}, \\ A_t^2 - A_{xx}^2 &= \psi_x, \\ P_t^i - P_{xx}^i &= 0.\end{aligned}\tag{17}$$

Substituting back yields

$$\begin{aligned}\xi &= \frac{1}{2}Px + F(t), \\ \phi &= \frac{1}{2}Pu + A^2ue^{y/4} + \frac{1}{2}P_x^2 + 2A_x^2e^{y/4} \\ \xi &= 4A^2e^{y/4} + Pv + P^2,\end{aligned}\tag{18}$$

P a constant. For example, we obtain $X_1^1 = \frac{1}{2}x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ and $X_1^2 = \frac{1}{2}P_x^2\frac{\partial}{\partial u} + P^2\frac{\partial}{\partial v}$ which are not 'proper' approximate potential symmetries $X_0 + \varepsilon X_1^i$ (see comments in notes below).

Nevertheless, $e^{\psi/4}\psi\frac{\partial}{\partial v} + \varepsilon\left[\frac{1}{2}x\frac{\partial}{\partial x} + v\frac{\partial}{\partial x}\right]$ and $e^{\psi/4}\psi\frac{\partial}{\partial v} + \varepsilon P^2\frac{\partial}{\partial v}$ are

approximate symmetries of the 'integrated' version of (13), viz., $v_t = v_{xx} - \frac{1}{4}v_x^2 - \varepsilon v_x$.

An approximate potential symmetry of (13) (by (18)) is

$$X = X_0 + e^{\psi/4}\varepsilon\left[(A^2u + 2A_x^2)\frac{\partial}{\partial u} + 4A^2\frac{\partial}{\partial v}\right].$$

Notes

1. We have, here, chosen to define a 'proper' approximate potential symmetry as being one in which τ, ξ and ϕ are explicitly dependent on v . However, these may be (approximate) potential symmetries as X_0 is a potential symmetry of the unperturbed equation.
2. The Lie symmetry generators of (2) which are not potential symmetries of (1) may lead to approximate potential symmetries of (13) using the method described above; one can use any one of the seven generators obtained by Baumann in [3].
3. The approximate invariant solutions are determined in the standard way- see [1], [7].

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