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RESPONSE OF A PARAMETRICALLY EXCITED SYSTEM WITH QUADRATIC AND CUBIC NON-LINEARITIES

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Absrtact- An investigation is presented of the response of a three-degree-of-freedom system with quadratic and cubic non-linearities under parametric excitations. The problem of suppressing the vibration of a structure that is subjected to combination parametric excitation is considered , where the vibration amplitudes resulting from such resonance can not be controlled. The fixed points of the three equations are obtained and their stability are determined. Numerical solutions are conducted to obtain the response of the three modes and their stability. Effects of the different parameters on both response and stability of the system are also investigated .

Keywords- Resonance, Power series, Stability, System response, Multi excitation force

1-INTRODUCTION

A non-linear system of three-degree-of- freedom with quadratic and cubic damping and stiffness representing the vibration of a cantilever beam is studied and solved using multiple scale perturbation technique. The oscillations of this beam are modeled by three differential equations with non-linear quadratic damping and cubic stiffness under the interaction of parametric excitation [1]. The governing equations of motion are:

$$\ddot{X}_n + \varepsilon\, c_n \dot{X}_n + \varepsilon\, \mu_n \left| \dot{X}_n \right| \dot{X}_n + \omega_n^{\,2} X_n + \varepsilon \alpha_n X_n^{\,3} + \varepsilon \sum_{j=1}^{3} (k_{nj}\, x_j) \times \sum_{s=1}^{N} \cos(\Omega_s t) = 0 \qquad (1)$$

where μ_n and c_n are damping coefficients $(n=1,2,3)$, α_n are cubic non-linear parameters, ε is a small perturbation parameter, ω_n and Ω_s are the natural and excitation frequencies, N any natural number and $(s=1,2,3)$,.

Dugundij and Mukhopodhyay [2] studied experimentally and theoretically the influence of some types of combination resonances involving bending and torsion modes of vibration on the instability region near forcing frequencies. Cartmell and Roberts [3] derived an expression to describe the stability boundary for another type of combination resonance, in which only bending is apparent.

The study of three-degree-of-freedom, non-linear systems has not received much attention. The response of non-linear system to harmonic excitations often exhibits complicated behaviors when their natural frequencies are commensurable [4,5] at internal (auto-parametric) resonance. Nayfeh et. al [6] considered a theoretical and experimental investigation of two-degree-of-freedom structure exhibiting an auto-parametric combination resonance of the additive type.

El-Bassiouny and Eissa [7] studied the response of three-degree-of-freedom system with cubic non-linearities and auto-parametric resonances to a harmonic excitation of the third mode. Effects of the different parameters on the system response, stability and dynamic chaos are studied applying well known numerical techniques [8,9,10].

In this work, the behavior of a three-degree-of freedom-system with non-linear quadratic damping and cubic stiffness is studied. The method of multiple scales [11,12] is applied to study the stability boundary for the combination resonance $\Omega_1 \cong \omega_1 + \omega_2, \Omega_1 \cong \omega_2 + \omega_3$ and $\Omega_1 \cong \omega_1 + \omega_3$ of the three modes of vibrations. The periodic solution and its stability are obtained and studied. Effects of the different parameters on both response and stability of the oscillation are investigated and reported.

2-MATHEMATICAL ANALYSIS

A general uniform expression of the solution of Eq. (1) is sought in the form :

$$X_n(t;\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k x_k(T_0, T_1) \tag{2}$$

where $T_0 = t$ is a fast time scale associated with changes occurring at the frequency ω_n and Ω_s, and $T_1 = \varepsilon t$ is a slow time scale associated with modulations in both the amplitude and phase caused by the non-linearities of both damping, spring stiffness and parametric resonance. Substituting Eq.(2) and its derivatives into Eq.(1),and equating the coefficients of the same power of ε for both sides, we get the following:

$$(D_0^2 + \omega_n{}^2)\, x_{n0} = 0 \tag{3}$$

$$(D_0^2 + \omega_n{}^2)\, x_{n1} = -2D_0 D_1 x_{n0} - C_n\, D_0 x_{n0} \mp \mu_n (D_0 x_{n0})^2 - \alpha_n x_{n0}^3 - \sum_{j=1}^{3}\sum_{s=1}^{N} k_{nj} x_{j0} \cos(\Omega_s T_0) \tag{4}$$

$$(D_0^2 + \omega_n{}^2)\, x_{n2} = -D_1^2 x_{n0} - 2D_0 D_1 x_{n1} - C_n (D_1 x_{n0} + D_0 x_{n1}) \mp 2\mu_n (D_0 x_{n0})(D_1 x_{n0} + D_0 x_{n1}) - 3\alpha_n x_{n0}^2 x_{n1} - \sum_{j=1}^{3}\sum_{s=1}^{N} k_{nj} x_{j1} \cos \Omega_s T_0 \tag{5}$$

The general solution of Eq. (3) can be expressed in the form :

$$x_{n0}(T_0, T_1) = Q_{n0} \exp(i\,\omega_n T_0) + \overline{Q}_{n0} \exp(-i\,\omega_n T_0). \tag{6}$$

where $Q_{n0}, \overline{Q}_{n0}$ are conjugate complex functions in T_1.

Substituting from Eq. (6) into Eq. (4),we get the following:

$$(D_0^2 + \omega_n{}^2)\, x_{n1} = -2i\omega_n (D_1 Q_{n0}) \exp(i\omega_n T_0) - i\, C_n\, \omega_n Q_{n0} \exp(i\omega_n T_0) \pm \mu_n \omega_n^2 (Q_{n0}^2 \exp(2i\omega_n T_0) - Q_{n0}\overline{Q}_{n0}) + \alpha_n (Q_{n0}^3 \exp(3i\omega_n T_0) + 3Q_{n0}^2 \overline{Q}_{n0} \exp(i\omega_n T_0)) - \sum_{j=1}^{3}\sum_{s=1}^{N} [\frac{k_{nj}}{2}(Q_{n0} \exp(i(\Omega_s + \omega_j)T_0) + \overline{Q}_{n0} \exp(i(\Omega_s - \omega_j)T_0))] + cc. \tag{7}$$

Hence, eliminating the secular terms, the general solution of Eq.(7) is obtained as:

$$x_{n1}(T_0,T_1) = Q_{n1}\exp(i\omega_n T_0) \pm \frac{\mu_n}{3} Q_{n0}^2 \exp(2i\omega_n T_0) + \alpha_n Q_{n0}^3 \exp(3i\omega_n T_0)/8\omega_n^2 + B_{n0}$$
$$-\sum_{j=1}^{3}\sum_{s=1}^{N}[E_{nj1}\exp(i(\Omega_s+\omega_j)T_0) + E_{nj2}\exp(i(\Omega_s-\omega_j)T_0)] + cc. \qquad (8)$$

where $E_{n\ell1}, E_{n\ell2}, Q_{n0}$ and B_{n0} are complex conjugate functions in T_1, and cc denotes the complex conjugate of the preceding terms.

Substituting Eqs (6) and (8) into Eq. (5) and following the same procedure, the third order approximate solution is obtained as:

$$x_{n2}(T_0,T_1) = Q_{n2}\exp(i\omega_n T_0) + F_{n2}\exp(2i\omega_n T_0) + F_{n3}\exp(3i\omega_n T_0) + F_{n4}\exp(4i\omega_n T_0) + F_{n5}\exp(5i\omega_n T_0)$$
$$+\sum_{s=1}^{N}\sum_{j=1}^{3}\{E_{nj3}\exp(i(\Omega_s+\omega_j)T_0) + E_{nj4}\exp(i(\Omega_s-\omega_j)T_0) + E_{nj5}\exp(i(\Omega_s+3\omega_j)T_0)$$
$$+ E_{nj6}\exp(i(\Omega_s-3\omega_j)T_0) + E_{nj7}\exp(i(\Omega_s+2\omega_j)T_0) + E_{nj8}\exp(i(\Omega_s-2\omega_j)T_0)$$
$$+ F_{nj1}\exp(i(2\Omega_s+\omega_j)T_0) + F_{nj2}\exp(i(2\Omega_s-\omega_j)T_0) + F_{n6}\exp(i\Omega_s T_0) + B_{n1}$$
$$+ H_{nj1}\exp(i(\Omega_s+\omega_n+\omega_j)T_0) + H_{nj2}\exp(i(\Omega_s-\omega_n+\omega_j)T_0) + H_{nj3}\exp(i(\Omega_s+\omega_n-\omega_j)T_0)$$
$$+ H_{nj3}\exp(i(\Omega_s-\omega_n-\omega_j)T_0) + H_{nj5}\exp(i(\Omega_s+2\omega_n+\omega_j)T_0) + H_{nj6}\exp(i(\Omega_s+2\omega_n-\omega_j)T_0)$$
$$+ H_{nj7}\exp(i(\Omega_s-2\omega_n+\omega_j)T_0) + H_{nj8}\exp(i(\Omega_s-2\omega_n-\omega_j)T_0) + Q_{nj3}\exp(i(\Omega_s+\Omega_{N-s}+\omega_j)T_0)$$
$$+ Q_{nj2}\exp(i(\Omega_s+\Omega_{N-s}-\omega_j)T_0) + Q_{nj3}\exp(i(\Omega_s-\Omega_{N-s}+\omega_j)T_0) + Q_{nj4}\exp(i(\Omega_s-\Omega_{N-s}-\omega_j)T_0)\}$$
$$+cc \qquad (9)$$

where $F_{n2},\dots,F_{n5}, E_{nj3},\dots,E_{nj6}, F_{nj1}, F_{nj2}, H_{nj1},\dots,H_{nj8}, Q_{nj1},\dots,Q_{nj4}$, B_{n1} and Q_{n1} are complex functions in T_1 and cc denotes complex conjugate of the preceding terms.

The general solution of Eq. (1) is obtained as:

$$X_n(t;\varepsilon) = x_{n0}(T_0,T_1) + \varepsilon x_{n1}(T_0,T_1) + \varepsilon^2 x_{n2}(T_0,T_1) + \varepsilon^3(0) \qquad (10)$$

To describe quantitatively the nearness of the combination resonance cases $\Omega_1 \cong \omega_1+\omega_2$ $\Omega_1 \cong \omega_2+\omega_3$, $\Omega_1 \cong \omega_1+\omega_3$, we introduce detuning parameters σ_1, σ_2 and σ_3, such that: $\Omega_1 = \omega_1+\omega_2+\varepsilon\sigma_1$, $\Omega_1 = \omega_2+\omega_3+\varepsilon\sigma_2$, $\Omega_1 = \omega_1+\omega_3+\varepsilon\sigma_3$. (11)

Substituting Eqs. (11) into Eq.(7) and setting the coefficients of secular terms to zero, yields the solvability conditions as:

$$i\omega_1(2D_1Q_{10} + c_1Q_{10}) + 3\alpha_1 Q_{10}^2\overline{Q}_{10} + \frac{1}{2}k_{12}\overline{B}_{20}e^{i\sigma_2 T_1} + \frac{1}{2}k_{13}\overline{B}_{20}e^{i\sigma_3 T_1} = 0 \qquad (12)$$

$$i\omega_2(2D_1Q_{20} + c_2Q_{20}) + 3\alpha_2 Q_{20}^2\overline{Q}_{20} + \frac{1}{2}k_{21}\overline{B}_{10}e^{i\sigma_2 T_1} + \frac{1}{2}k_{23}\overline{B}_{20}e^{i\sigma_3 T_1} = 0 \qquad (13)$$

$$i\omega_3(2D_1Q_{30} + c_3Q_{30}) + 3\alpha_3 Q_{30}^2\overline{Q}_{30} + \frac{1}{2}k_{32}\overline{B}_{10}e^{i\sigma_3 T_1} + \frac{1}{2}k_{32}\overline{B}_{20}e^{i\sigma_2 T_1} = 0 \qquad (14)$$

Substituting the polar form $Q_{n0} = \frac{1}{2}a_n(T_1)\,e^{i\beta_n(T_1)}$, into Eqs. (12) and (13) and separating the coefficients of real and imaginary parts yields the modulation equations:

$$\omega_1 a_1\left(\frac{\sigma_1-\sigma_2+\sigma_3-\gamma_1'+\gamma_2'-\gamma_3'}{2}\right)-\frac{3}{8}\alpha_1 a_1^3-\frac{a_2}{4}k_{12}\cos\gamma_1-\frac{a_3}{4}k_{13}\cos\gamma_3=0 \qquad (15)$$

$$\omega_1 a_1'+\omega_1 c_1 a_1/2+\frac{a_2}{4}k_{12}\sin\gamma_1+\frac{a_3}{4}k_{13}\sin\gamma_3=0 \qquad (16)$$

$$\omega_2 a_2\left(\frac{\sigma_2+\sigma_1-\sigma_3-\gamma_1'-\gamma_2'+\gamma_3'}{2}\right)-\frac{3}{8}\alpha_2 a_2^3-\frac{a_1}{4}k_{21}\cos\gamma_1-\frac{a_3}{4}k_{23}\cos\gamma_2=0 \qquad (17)$$

$$\omega_2 a_2'+\omega_2 c_2 a_2/2+\frac{a_1}{4}k_{21}\sin\gamma_1+\frac{a_3}{4}k_{23}\sin\gamma_2=0 \qquad (18)$$

$$\omega_3 a_3\left(\frac{\sigma_3+\sigma_2-\sigma_1-\gamma_2'+\gamma_1'-\gamma_3'}{2}\right)-\frac{3}{8}\alpha_3 a_3^3-\frac{a_1}{4}k_{31}\cos\gamma_3-\frac{a_2}{4}k_{32}\cos\gamma_2=0 \qquad (19)$$

$$\omega_3 a_3'+\omega_3 c_3 a_3/2+\frac{a_1}{4}k_{31}\sin\gamma_3+\frac{a_2}{4}k_{32}\sin\gamma_2=0 \qquad (20)$$

where, $\gamma_1=\sigma_1 T_1-\beta_2-\beta_1$, $\gamma_2=\sigma_2 T_1-\beta_2-\beta_3$ and $\gamma_3=\sigma_3 T_1-\beta_3-\beta_1$.

For steady-state solutions, $a_n'=\gamma_n'=0$, and the periodic solution at the fixed point corresponding to Eqs. (15)-(20) are obtained. We have the following three possibilities:
(i) $a_1=a_2=a_3=0$
(ii) $a_1\neq 0$, $a_2=0$ and $a_3=0$. Hence from Eq. (15) we get which has solution gieven by:

$$a_1^2=4\omega_1(\sigma_1-\sigma_2+\sigma_3)/3\alpha_1 \qquad (21)$$

(iii) $a_1\neq 0$, $a_2\neq 0$ and $a_3=0$, and hence from Eqs. (15) and (16) , we get

$$\frac{9}{16}\alpha_1^2 a_1^6-\frac{3}{4}\alpha_1\omega_1(\sigma_1-\sigma_2+\sigma_3)a_1^4+[\omega_1^2(\sigma_1-\sigma_2+\sigma_3)^2+\omega_1^2c_1^2)]a_1^2-\frac{k_{12}^2}{4}a_2^2=0. \qquad (22)$$

and from Eqs. (17) and (18) ,we get

$$\frac{9}{16}\alpha_2^2 a_2^6-\frac{3}{4}\alpha_2\omega_2(\sigma_2-\sigma_1+\sigma_3)a_2^4+[\omega_2^2(\sigma_2-\sigma_1+\sigma_3)^2+\omega_2^2c_2^2)]a_2^2-\frac{k_{21}^2}{4}a_1^2=0 \qquad (23)$$

3-STABILITY OF THE PERIODIC SOLUTION

To determine the local stability of the various fixed points and hence the various steady state solutions of the linear equation ,one can let $Q_{n0}=\frac{1}{2}(p_n-iq_n)e^{iv_nT_1}$ where p_n *and* q_n are real and $v_1=(\sigma_1-\sigma_2+\sigma_3)/2, v_2=(\sigma_2-\sigma_3+\sigma_1)/2$ and $v_3=(\sigma_3-\sigma_1+\sigma_2)/2$. Substituting into equations (12) and (13) and equating the real and imaginary parts we get

$$p_1'+c_1p_1/2+\frac{k_{12}}{4\omega_1}q_2+v_1q_1/2+\frac{k_{13}}{4\omega_1}q_3=0, \quad q_1'-v_1p_1/2+c_1q_1/2+\frac{k_{12}}{4\omega_1}p_2+\frac{k_{13}}{4\omega_1}q_3=0,$$

$$p_2'+c_2p_2/2+\frac{k_{21}}{4\omega_2}q_1+v_2q_2/2+\frac{k_{23}}{4\omega_2}q_3=0, \quad q_2'-v_2p_2/2+c_2q_2/2+\frac{k_{21}}{4\omega_2}p_1+\frac{k_{23}}{4\omega_2}q_3=0$$

$$p_3' + c_3 p_3/2 + \frac{k_{31}}{4\omega_3} q_1 + v_3 q_3/2 + \frac{k_{32}}{4\omega_3} q_2 = 0,\quad q_3' - v_3 p_3/2 + c_3 q_3/2 + \frac{k_{31}}{4\omega_3} p_1 + \frac{k_{32}}{4\omega_3} p_2 = 0$$

The stability of a particular fixed point with respect to a perturbation proportional to $exp(\lambda T_1)$ is determined by zeros of the characteristic equation:

$$\begin{vmatrix} \lambda + c_1/2 & v_1 & 0 & \alpha_{12}/4\omega_1 & 0 & \alpha_{13}/4\omega_1 \\ -v_1 & \lambda + c_1/2 & \alpha_{12}/4\omega_1 & 0 & \alpha_{13}/4\omega_1 & 0 \\ 0 & \alpha_{21}/4\omega_2 & \lambda + c_1/2 & v_2 & 0 & \alpha_{23}/4\omega_2 \\ \alpha_{21}/4\omega_2 & 0 & -v_2 & \lambda + c_2/2 & \alpha_{23}/4\omega_2 & 0 \\ 0 & \alpha_{31}/4\omega_3 & 0 & \alpha_{32}/4\omega_3 & \lambda + c_3/2 & v_2 \\ \alpha_{31}/4\omega_3 & 0 & \alpha_{32}/4\omega_3 & 0 & -v_2 & \lambda + c_3/2 \end{vmatrix} = 0 \quad (24)$$

4- RESULTS AND DISCUSSION

The frequency response equation (21), is a non-linear algebraic equation in the amplitude a_1. This equation is solved numerically and the results are shown in Fig. 1, which represent the variation of the amplitude a_1 against the detuing parameter σ_1 and at the given values of the other parameters. From Figs.1a, we find that the amplitude a_1 is a monotonic increasing function in the natural frequency ω_1, while the amplitude a_1 against the detuing parameter σ_2, Fig 1b, we find that the amplitude a_1 is a monotonic decreasing function in the natural frequency ω_1. But Figs.1c and 1d, we find that the amplitude a_1 is a monotonic decreasing function in the non-linear coefficient α_1.

The frequency response equation (22), is a non-linear algebraic equation in the amplitude a_1. This equation is solved numerically and the results are shown in Fig.1 which represent the variation of the amplitude a_1 against the detuing parameters for the given values of the other parameters . From Fig.2a, we find that the amplitude a_1 is a monotonic decreasing function in the damping coefficient c_1. Fig.2b, shows that the amplitude a_1 is a monotonic decreasing function in the natural frequency ω_1.

The time history of the given system has been studied applying Runge- Kutta fourth order method. The numerical solution and its stability for both modes of vibration are obtained as shown in Figs 3a, 3b and 3c . From these figures , it can be noticed that for three modes, we have decreasing amplitudes with some chaos . The steady state amplitude for x_1 is about ± 0.08 , for x_2 is about ± 0.06 and for x_3 is about ± 0.1.

From Eqn.(9) the theoretical resonance conditions are determined and classified into the following categories:

(i) *Primary resonance cases:* $\Omega_s \cong \omega_n$.(*n=1,2,3 and s* =1,2).

(ii) *Sub - harmonic resonance:* $\Omega_s \cong 2\omega_n$, $\Omega_s \cong 3\omega_n$, $\Omega_s \cong 4\omega_n$.

(iii) *Super - harmonic resonance:* $\Omega_s \cong \omega_n/2$.

(iv) *Combined resonance* ; some of them are :

$$\Omega_s \cong \pm(\omega_1 \mp \omega_2),\quad \Omega_s \cong \pm(\omega_1 \mp \omega_3),\quad \Omega_s \cong \pm(\omega_2 \mp \omega_3),$$

$$\Omega_s \cong \pm(\omega_1 \pm \omega_2)/2,\quad \Omega_s \cong \pm(\omega_1 \pm \omega_3)/2,\quad \Omega_s \cong \pm(\omega_2 \pm \omega_3)/2,$$

$$\Omega_s \cong \pm(2\omega_1 \pm \omega_2),\ \Omega_s \cong \pm(2\omega_1 \pm \omega_3),\ \Omega_s \cong \pm(2\omega_2 \pm \omega_1),\ \Omega_s = \pm(2\omega_2 \pm \omega_3)$$

$$\Omega_1 \cong \Omega_2, \Omega_1 - \Omega_2 \cong 2\omega_1, \Omega_1 - \Omega_2 \cong 2\omega_2, \Omega_1 - \Omega_2 \cong 2\omega_1.$$

(v) *Simultaneous or incident resonance*

Any combination of the two resonance cases are classified as simultaneous resonance.

In the following, some selected resonance cases are discussed, while Table1, summarizes other possible resonance cases :

i) combined resonance

a) If $\Omega_1 = \omega_1 + \omega_2$ Fig.4a, illustrates the results of this case. It can be seen that the steady state amplitudes of both the first and second modes have increased to about 625% and 1150% compared with the basic case in Figs. 3a and 3b respectively , with increasing dynamic chaos. The steady state amplitude of the third mode has no significant changes.

b) If $\Omega_1 = 3\omega_3 + \omega_1$, . It can be seen from Fig.4b, that the steady state amplitudes of the first and third modes have become time dependent the maximum steady state amplitudes are increased to about 625% and 360% compared to the basic case shown in Figs.3a and 3c respectively, with increasing dynamic chaos. The steady state amplitude of the second mode has no significant changes.

ii) simultaneous resonance

a) If $\Omega_1 = \omega_1$ *and* $\Omega_2 = \omega_1 + \omega_2$. Fig.5a, illustrates the results of this case. It can be seen that the steady state amplitude of the first and second modes have increased to about 400% and 330% compared to the basic case shown in Figs.3a and 3b respectively, with increasing the dynamic chaos, while the steady state amplitude of the third mode has no significant effects.

b) If $\Omega_1 = \Omega_2 = \omega_1 + \omega_2$. Fig 5b illustrates the results of this case. It can be seen that the steady state amplitude of the first mode has increased to about 350% with increasing the dynamic chaos, while the steady state amplitude of the second mode is increased to about 330% with increasing the dynamic chaos, and the steady state amplitude of the third mode is increased to about 270% compared to the basic case shown in Figs 3a, 3b and 3c respectively .

c) If $\Omega_1 = \omega_1 + \omega_2$ *and* $\Omega_2 = \omega_2 + \omega_3$. Fig.5c illustrates the results of this case. It can be seen that the steady state amplitude of the first mode has increased to about 625% with increasing the dynamic chaos, while the steady state amplitude of the second mode is increased to about 300% with increasing the dynamic chaos, and the steady state amplitude of the third mode is increased to about 200% compared to the basic case shown in Figs. 3a, 3b and 3c respectively.

Other possible resonance cases are summerized in Table 1.

5-CONCLUSIONS

A system of non-linear parametrically coupled differential equations representing the vibration of a cantilever beam have been solved applying the multiple time scale method . Different resonance cases are obtained and the numerical solutions of the given system and its stability are determined. The effects of the different parameters are investigated . From this study the following conclusions may be written:

1.The amplitude a_1 is a monotonic increasing function in the natural frequency ω_1.

2.The amplitude a_1 is a monotonic decreasing function in the non-linear coefficient α_1

3. For the damping coefficients c_n we find that the amplitude of the state is monotonic decreasing function in the damping coefficients .

4.We find that the amplitude of the state is monotonic decreasing function in the damping coefficients μ_n .

5.From the reported resonance cases, it is clear that the worst cases are the combined and simultanous cases, we can see this from Table1, which must be avoided in the design of such system.

Table1

	Resonance cases	%Increases of x_1	Phase plane of x_1	%Increase of x_2	Phase plane of x_2	%Increases of x_3	Phase plane of x_3
1	$\Omega_1 \cong \omega_1$	125%	Multi limit cycle chaotic	-	Chaotic	-	Chaotic
2	$\Omega_1 \cong \omega_2$	-	Chaotic	200%	Multi limit cycle chaotic	-	Chaotic
3	$\Omega_1 \cong \omega_3$	-	Chaotic	-	Chaotic	220%	Multi limit cycle chaotic
4	$\Omega_1 \cong 2\omega_1$	400%	Tuned multi limit cycle	-	Increasing chaotic	-	Chaotic
5	$\Omega_1 \cong 2\omega_2$	1000%	More chaotic and multi limit cycle	330%	More chaotic and multi limit cycle	-	Chaotic
6	$\Omega_1 \cong 2\omega_3$	125%	More chaotic and multi limit cycle	-	Chaotic	580%	Tuned multi limit cycles
7	$\Omega_2 \cong 2\omega_1$	400%	More chaotic and multi limit cycle	-	Increasing chaotic	-	Chaotic

8	$\Omega_1 = 3\omega_1$	250%	Multi limit cycles	-	Chaotic	-	Chaotic
9	$\Omega_1 = 4\omega_1$	250%	Multi limit cycles	-	Chaotic	-	Chaotic
10	$\Omega_1 = \omega_1 + \omega_3$	400%	Tuned multi limit cycles	-	Chaotic	500%	Tuned multi limit cycle
11	$\Omega_1 \cong \omega_1 + \omega_2$	625%	Tuned multi limit cycles	1150%	Tuned multi limit cycle	-	Chaotic
12	$\Omega_1 \cong \omega_2 + \omega_3$	-	Chaotic	330%	Tuned multi limit cycle	-	Tuned multi limit cycle
13	$\Omega_2 \cong \omega_1 + \omega_3$	400%	Tuned multi limit cycle	-	Chaotic	500%	Tuned multi limit cycle
14	$\Omega_2 \cong \omega_2 + \omega_3$	125%	Chaotic	250%	Multi limit cycle	500%	Tuned multi limit cycles
15	$\Omega_2 \cong 3\omega_2 + \omega_1$	330%	Multi limit cycle	450%	More chaotic	-	Chaotic
16	$\Omega_2 \cong 3\omega_2 + \omega_3$	160%	Multi limit cycle	330%	Multi limit cycle	1000%	Multi limit cycle
17	$\Omega_2 \cong 3\omega_3 + \omega_1$	850%	Multi limit cycle	-	More chaotic	280%	Multi limit cycle
18	$\Omega_1 \cong \omega_3 - \omega_2$	-	Non chaotic	1000%	Non chaotic	200%	Non chaotic
19	$\Omega_1 \cong 3\omega_2 + \omega_1$	250%	Chaotic	330%	Chaotic	-	Chaotic
20	$\Omega_1 \cong 3\omega_3 + \omega_1$	625%	Increase chaotic	-	Chaotic	360%	Increase chaotic
21	$\Omega_1 \cong 3\omega_3 + \omega_2$	125%	Increase chaotic	280%	more chaotic and multi limit cycles	700%	more chaotic and multi limit cycles
22	$\Omega_1 \cong 3\omega_1 + \omega_3$	250%	More chaotic and multi limit cycle	-	Multi limit cycle	270%	More chaotic and multi limit cycle
23	$\Omega_1 + \Omega_2 \cong 2\omega_1$	125%	Tuned and multi limit cycle	-	Tuned and multi limit cycle	-	Chaotic
24	$\Omega_1 - \Omega_2 \cong \omega_1$	300%	Chaotic	-	-	-	-
25	$\Omega_1 - \Omega_2 \cong 2\omega_2$	125%	Tuned and multi limit cycle	560%	chaotic and multi limit cycle	-	Chaotic
26	$\Omega_1 - \Omega_2 \cong 2\omega_3$	125%	Tuned and multi limit cycle	-	Chaotic	500%	Chaotic and multi limit cycle
27	$\Omega_1 - \Omega_2 \cong \omega_1 + \omega_3$	400%	More chaotic and multi limit cycle	-	Chaotic	300%	More chaotic and multi limit cycle
28	$\Omega_1 \cong \omega_1, \Omega_2 \cong 2\omega_1$	350%	Tuned and multi limit cycle	-	-	-	-
29	$\Omega_1 \cong \omega_1$, $\Omega_2 \cong \omega_1 + \omega_2$	400%	Tuned and multi limit cycle	330%	Tuned and multi limit cycle	-	Chaotic
30	$\Omega_1 \cong \omega_1$, $\Omega_2 \cong (\omega_1 + \omega_2)/2$	600%	Tuned and multi limit cycles	415%	Tuned and multi limit cycles	-	Chaotic
31	$\Omega_1 \cong \omega_1 + \omega_2$, $\Omega_2 \cong \omega_2 + \omega_3$	450%	Tuned and multi limit cycle	300%	Tuned and multi limit cycle	200%	Tuned and multi limit cycle
32	$\Omega_1 = \Omega_2 \cong \omega_1 + \omega_2$	350%	Tuned and multi limit cycle	330%	Tuned and multi limit cycle	270%	Tuned and multi limit cycle

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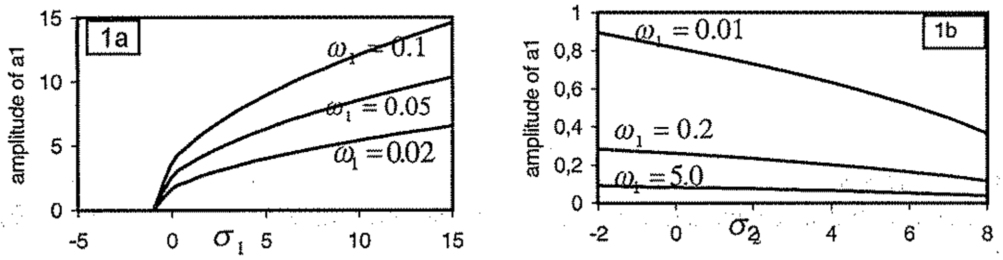
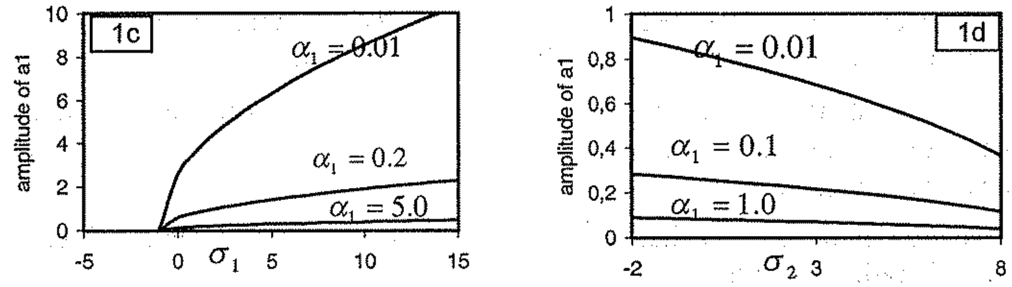
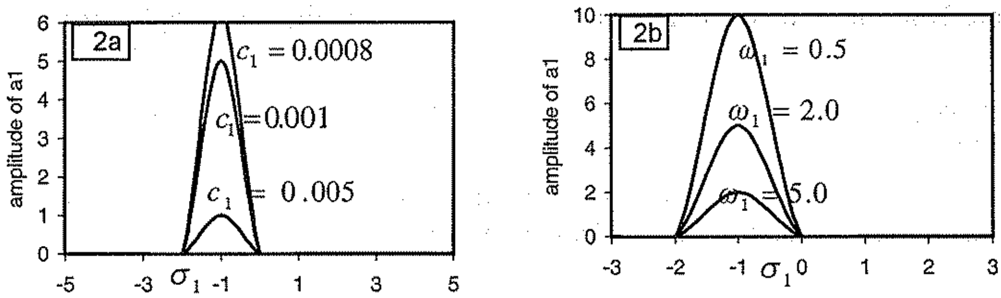


1a) $\sigma_2 = 1.0, \sigma_3 = 2.0, \alpha_1 = 0.01$ **1b)** $\sigma_2 = 1.0, \sigma_3 = 2.0, \alpha_1 = 0.01$



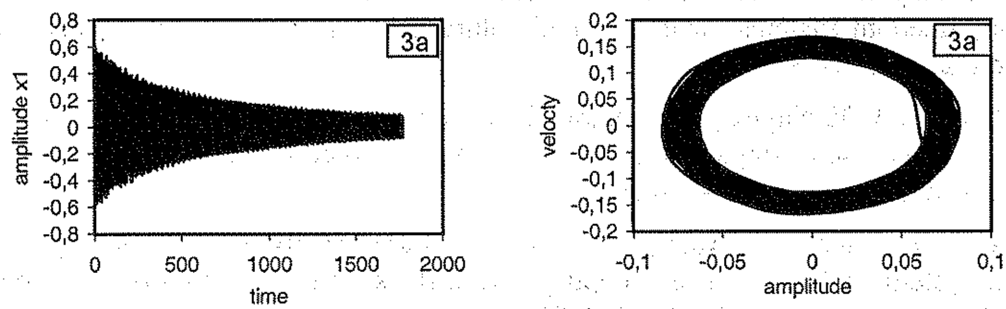
1c) $\sigma_2 = 1.0, \sigma_3 = 2.0, \omega_1 = 0.01$ **1d)** $\sigma_1 = 1.0, \sigma_3 = 2.0, \omega_1 = 0.01$

Fig.1 Response curves of the first case: $a_2 = a_3 = 0$ and $a_1 \neq 0$



2a) $\sigma_2 = 2.0, \sigma_3 = 3.0, \omega_1 = 2.0, a_2 = 2.0, k_{12} = 0.01$ **2b)** $\sigma_2 = 2.0, \sigma_3 = 3.0, c_1 = 0.01, a_2 = 2.0, k_{12} = 0.01$

Fig.2 Response curves of the second case $a_3 = 0$ and $a_1 \neq 0$, $a_2 \neq 0$



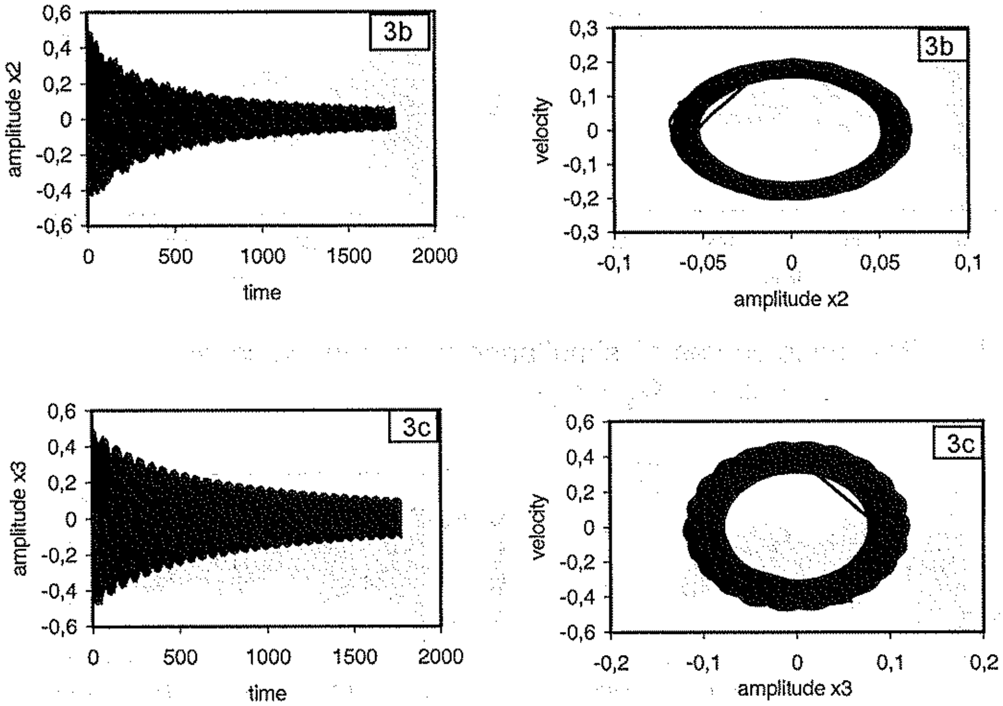


Fig.3 Response curves at non-resonant case which as basic case.

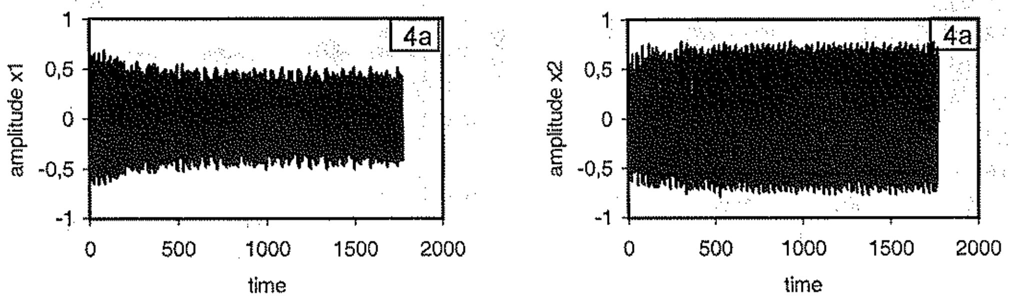


Fig.4a Response curves of combined resonance cases: $\Omega_1 = \omega_1 + \omega_2$

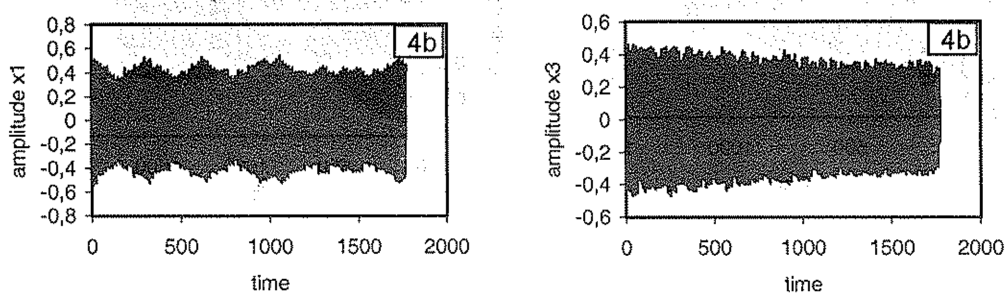


Fig.4b Response curves of combined resonance cases: $\Omega_1 = 3\omega_3 + \omega_1$

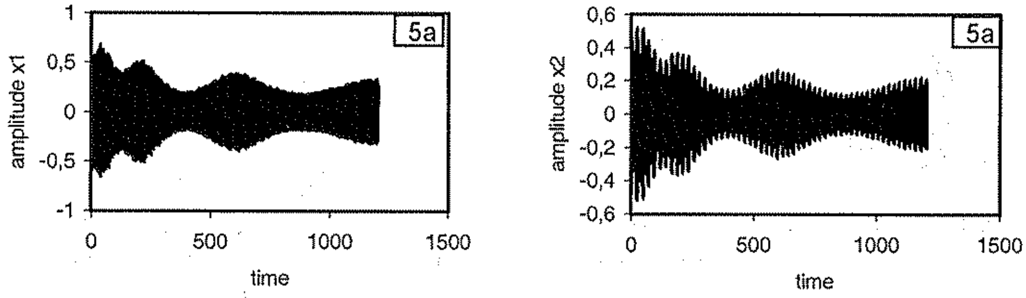
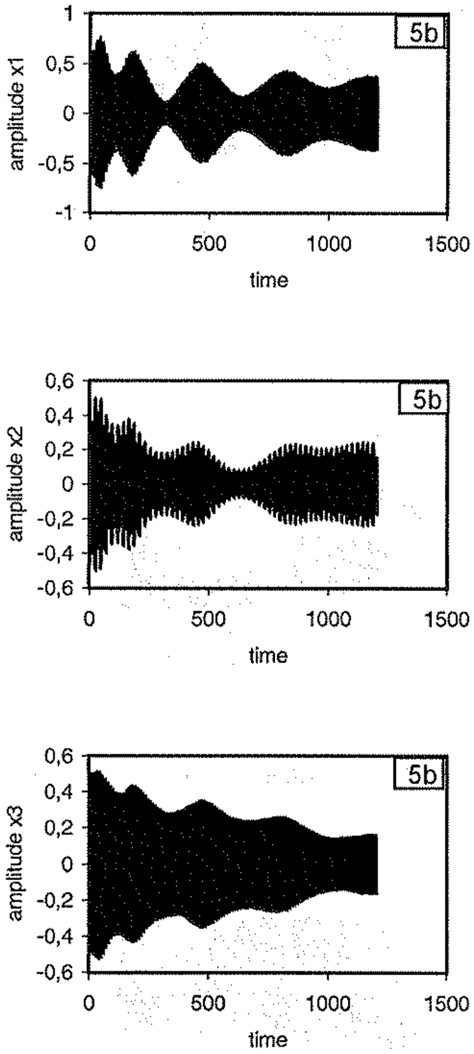


Fig.5a Response curves of simultaneous resonance cases:
$\Omega_1 = \omega_1$ *and* $\Omega_2 = \omega_1 + \omega_2$



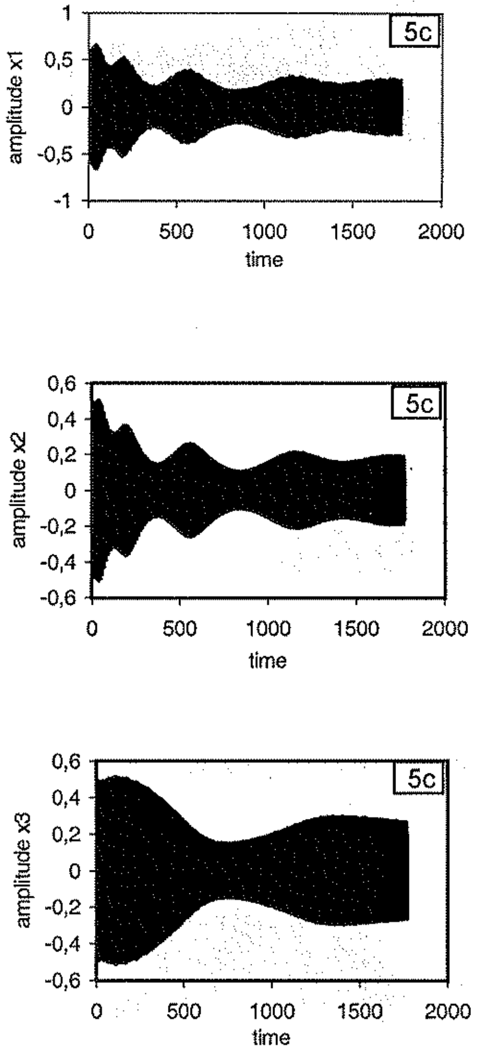


Fig.5b Response curves of simultaneous resonance cases:
$\Omega_1 = \omega_1 + \omega_2$ *and* $\Omega_2 = \omega_2 + \omega_3$

Fig.5c Response curves of simultaneous resonance cases:
$\Omega_1 = \Omega_2 = \omega_1 + \omega_2$