SOME PROPERTIES OF RULED SURFACES UNDER HOMOTHETY IN E^3

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Abstract- It is clearly that, If $f : E^3 \rightarrow E^3$ is a homothety and φ is a surface in three dimensional Euclidean space E^3 , $f(\varphi) = \bar{\varphi}$ is also a surface in E^3 . In this paper, the surface φ has been taken a ruled surface, specially. It was shown that image surface $f(\varphi) = \bar{\varphi}$ has been a ruled surface, too. Furthermore, It was investigated whether some properties related with ruled surfaces on φ have been invariant under the homothety f or not.

Keywords- Ruled Surface, Homothety, Connection Preserving Map.

1. INTRODUCTION

Let M and $\bar{M}$ be two surfaces in E^3 and $f : M \rightarrow \bar{M}$ be a C^∞ map. If there is a C^∞ real valued positive function F on M such that for $\forall P \in M$ and $\forall X_P, Y_P \in T_M(P)$

1.1 $\langle f_*(X_P), f_*(Y_P) \rangle = F(P) \langle X_P, Y_P \rangle$
 then f is called a *conformal map*. If F is constant, f is called a *homothety* of coefficient $F(P)$ where f_* is jacobian map of f .

Let D and $\bar{D}$ be connections on M and $\bar{M}$, respectively. A C^∞ map $f : M \rightarrow \bar{M}$ is *connection preserving* if

1.2 $f_*(D_X Y) = \bar{D}_{f_*(X)} f_*(Y)$

for $\forall X, Y \in \chi(M)$, [2].

Theorem 1.1 Let $f : M \rightarrow \bar{M}$ be a conformal map. Then f is connection preserving iff f is a homothety, [3].

The family with one-parameter of lines in E^3 is called *the ruled surface* and each of line in this family is called *the rulings* of the ruled surface. Thus the ruled surface as parametric is given by

1.3 $\varphi(s, v) = \alpha(s) + ve(s),$

where α and e are *the base curve* and *the unit vector* in the direction of the ruling of the ruled surface, respectively.

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The curve which has been drawn by e on the unit sphere S^2 is called *the spherical indicatrix curve* and the vector e is also called *the spherical indicatrix vector* of the ruled surface, [4].

For a whichever ruling $l = \varphi(s_o, v)$ of φ , the direction of the unit normal vector U of the ruled surface φ as v infinitely decreases is called *the asymptotic normal direction*. For the asymptotic normal direction, we can write

$$1.4 \quad g(s)|_{s=s_o} = \frac{e \times e_s}{\|e_s\|} \Big|_{s=s_o}, \quad e_s = \frac{de}{ds}, [4].$$

The point at which U is perpendicular to g is called *the striction point*. The direction of U at this point is called *the central normal* of the ruled surface φ . For the central normal of φ at striction point $\varphi(s, v_o)$, we can write

$$1.5 \quad t = \frac{e_s}{\|e_s\|}, [4].$$

Thus we have the orthonormal system $\{e, t, g\}$. This system is called *the geodesic Frenet trihedron* of the ruled surface φ . For the geodesic Frenet vectors e, t and g , we can write

$$1.6 \quad \begin{cases} e_q = t \\ t_q = \gamma g - e \\ g_q = -\gamma t, \end{cases}$$

where q and γ are the arc-parameter of the spherical indicatrix curve (e) and the geodesic curvature of (e) with respect to the unit sphere S^2 , respectively.

The equations (1.6) is called *the geodesic Frenet equations* of the ruled surface φ , [4].

2. RULED SURFACES UNDER THE HOMOTHETY

Let φ be a ruled surface which is given by equation (1.3) and $f: E^3 \rightarrow E^3$ be a homothety of coefficient λ .

For point Y which is taken on each ruling d of φ , we can write

$$2.1 \quad Y = \alpha(s) + v e.$$

Thus we obtain

$$f_*(Y) = f_*(\alpha(s)) + v f_*(e).$$

Finally, $f(d)$ is a line passing through the point $f(\alpha(s))$ of image curve $f\alpha$. Then, $f(\varphi) = \bar{\varphi}$ is a ruled surface with base curve $f\alpha$ in E^3 .

Corollary 2.1 The homothety transforms ruled surfaces to ruled surfaces.

If we denote the spherical indicatrix vector of $\bar{\varphi}$ as $\bar{e}$, we can write

$$\bar{\varphi}(s, v) = \bar{\alpha}(s) + v \bar{e}(s)$$

or

$$2.2 \quad \bar{\varphi}(s, v) = f_*(\alpha(s)) + v \frac{1}{\lambda^{1/2}} f_*(e(s)).$$

In this paper, $f : E^3 \rightarrow E^3$ will be taken homothety with coefficient λ such that $f(\varphi) = \bar{\varphi}$, where φ and $\bar{\varphi}$ are ruled surfaces in E^3 .

Theorem 2.1 For all X in $\chi(\varphi)$, we have

$$2.3 \quad [f_*(X)]_s = f_*(X_s).$$

Proof : Let T and $\bar{T}$ be the tangent vectors of the base curves of φ and $\bar{\varphi}$, respectively. Suppose that D and $\bar{D}$ are connections on this surfaces, respectively. Then for $\forall X \in \chi(\varphi)$, we can write

$$2.4 \quad \bar{D}_{\bar{T}} \bar{X} = \bar{D}_{f_*(T)} f_*(X).$$

From Theorem 1.1, we obtain

$$2.5 \quad \bar{X}_s = \bar{D}_{\bar{T}} \bar{X} = f_*(D_T X) = f_*(X_s).$$

Because of equality $\bar{X}_s = [f_*(X)]_s$, we get equation (2.3)

Let $\bar{t}$ and $\bar{g}$ be the central normal and the asymptotic normal of $\bar{\varphi}$, respectively. Because of the equations (1.4) and (1.5), we can write

$$\bar{t} = \frac{\bar{e}_s}{\|\bar{e}_s\|} \quad \text{and} \quad \bar{g} = \frac{\bar{e} \times \bar{e}_s}{\|\bar{e}_s\|}.$$

Due to the equality $\bar{e} = f_*(e)/\lambda^{1/2}$ and Theorem 2.1, we obtain

$$2.6 \quad \bar{t} = \frac{f_*(e_s)}{\|f_*(e_s)\|}$$

and

$$2.7 \quad \bar{g} = \frac{1}{\lambda^{1/2}} \frac{f_*(e) \times f_*(e_s)}{\|f_*(e_s)\|}$$

Thus, from the equation

$$f_*(e) \times f_*(e_s) = \det f_*(e \times e_s), [1].$$

we get

$$2.8 \quad \bar{g} = \frac{\det f_*}{\lambda} g.$$

For the striction curve (c) of the ruled surface φ , we can write

$$2.9 \quad c(s) = \alpha(s) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} e(s), [4].$$

Figure 1

Let $(\bar{c})$ be the striction curve of $\bar{\varphi}$. From Figure 1, we have

$$\bar{c}(s) = \bar{\alpha}(s) + u\bar{e}(s)$$

or

$$2.10 \quad \bar{c}(s) = f_*(\alpha(s)) + u \frac{1}{\lambda^{1/2}} f_*(e).$$

Because of the definition of striction point, we can write

$$2.11 \quad \langle \bar{\varphi}_s|_{\bar{c}(s)} \times \bar{\varphi}_u|_{\bar{c}(s)}, f_*(e) \times f_*(e_s) \rangle = 0.$$

If we calculate the values of $\bar{\varphi}_s|_{\bar{c}(s)}$ and $\bar{\varphi}_u|_{\bar{c}(s)}$ from the equality (2.2), we obtain

$$u = -\lambda^{1/2} \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle}.$$

Then, for the striction curve of $\bar{\varphi}$, we have

$$2.12 \quad \bar{c}(s) = f_*(\alpha(s)) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} f_*(e).$$

From the equations (2.9) and (2.12), we get

$$f_*(c(s)) = \bar{c}(s).$$

Corollary 2.2 The homothety preserves the property of being striction curve.

Let the curve $\beta: I \subset \mathbb{R} \rightarrow E^3$ be a orthogonal trajectory for φ and T_1 be the tangent of β . For $\forall s \in I$, we can write

$$\langle T_1, e \rangle = 0.$$

Because f is homothety, we have

$$\langle f_*(T_1), f_*(e) \rangle = 0.$$

Finally, $f(\beta) = \bar{\beta}$ is also a orthogonal trajectory for $\bar{\varphi}$.

Corollary 2.3 The homothety preserves the property of being the orthogonal trajectory.

For drall P_e of the ruled surface φ , we can write

$$P_e = \frac{\det(\alpha_s, e, e_s)}{\|e_s\|^2}, [1].$$

Using the equations $e_s = e_q q_s$ and (1.6), we obtain

$$2.13 \quad P_e = \frac{1}{q_s} \langle \alpha_s, g \rangle.$$

Similarly, for the drall of $\bar{\varphi}$, we can write

$$P_{\bar{e}} = \frac{\det(\bar{\alpha}_s, \bar{e}, \bar{e}_s)}{\|\bar{e}_s\|^2}.$$

Thus we get

$$P_{\bar{e}} = \frac{\det(f_*(\alpha_s), f_*(e), f_*(t))}{\lambda^{1/2} q_s \|f_*(t)\|^2}.$$

Because of the equality $f_*(e) \times f_*(t) = \lambda^{1/2} f_*(g)$, we obtain

$$P_e = P_{\bar{e}}.$$

Corollary 2.4 The homothety preserves the property of being developable of the ruled surfaces.

If the ruled surface φ is closed, there is a positive integer p such that

$$\varphi(s + p, v) = \varphi(s, v), [1].$$

Then, we can write

$$\alpha(s + p) + v e(s + p) = \alpha(s) + v e(s).$$

Since f_* is linear, we obtain

$$f_*(\alpha(s + p)) + v f_*(e(s + p)) = f_*(\alpha(s)) + v f_*(e(s)).$$

If we take the parametrization of $\bar{\varphi}$ as

$$\bar{\varphi}(s, v) = f_*(\alpha(s)) + v f_*(e(s)),$$

we have

$$\bar{\varphi}(s + p, v) = \bar{\varphi}(s, v).$$

Corollary 2.5 The homothety preserves the property of being closed of the ruled surfaces.

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