Mathematical & Computational Applications, Vol. 7, No. 3, pp. 229-234, 2002

THE HARNACK INEQUALITIES FOR NONNEGATIVE SOLUTIONS OF AN ELLIPTIC AND ULTRA-HYPERBOLIC EQUATION

Nuri Özalp and Fatma Taşdelen
Ankara University, Faculty of Science, Department of Mathematics,
06100 Ankara, Turkey

Abstract- In this study, in means of the Harnack inequalities of the harmonic functions, some Harnack type inequalities are given for the solutions of GASPT equation with singular coefficients and for two expanded equations of it.

Keywords- Harnack inequality, harmonic functions, GASPT equation.

1. INTRODUCTION

Let , in xoy-plane, $u^*(x,y)$ be a nonnegative harmonic function in a disk D of radius a with center M. Then for any $P \in D$, the following Harnack inequality

$$\frac{a-\rho}{a+\rho} u^*(M) \le u^*(P) \le \frac{a+\rho}{a-\rho} u^*(M) \tag{1}$$

is hold between the values of $u^*(x,y)$ at the point P and at the center M. (Figure 1) [1,3,4,5].

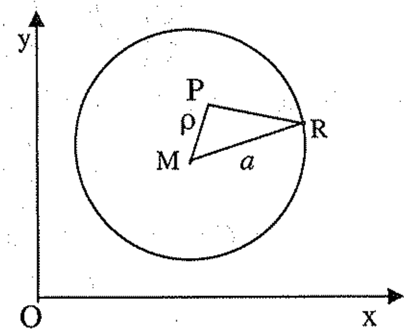

Figure 1.

It should be noted that the Harnack inequality is hold also for n – dimensional case with the inequality

$$\frac{a-\rho}{(a+\rho)^{n-1}} a^{n-2} u^*(M) \le u^*(P) \le \frac{a+\rho}{(a-\rho)^{n-1}} a^{n-2} u^*(M) \tag{2}$$

where M is the center of the n-dimensional ball $B^n : x_1^2 + \dots + x_n^2 < a^2$, $P(x_1,\dots,x_n) \in B^n$ is a point at distance $\rho < a$ from the center, and u^* is a nonnegative harmonic function in B^n.

In this study, we obtain Harnack type inequalities for some solutions of the GASPT (Generalized Auxiliary Symmetric Potential Theory) equation

$$\sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial u}{\partial x_i} \right) + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

where α_i $(i=1,2,\ldots,n)$ are constants, and for some solutions of the equation

$$\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2}+\sum_{j=1}^{m}\left(b_j^2\frac{\partial^2 u}{\partial y_j^2}+\frac{\alpha_j}{y_j-y_j^0}\frac{\partial u}{\partial y_j}\right)=0 \tag{4}$$

where α_j $(j=1,2,\ldots,m)$ are constants, b_j $(j=1,2,\ldots,m)$ are nonzero constants and $(y_1^0,\ldots,y_m^0)$ is a constant point in the domain of u.

Finally, we give Harnack inequality for some solutions of the ultra-hyperbolic equation

$$\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\alpha_i}{x_i}\frac{\partial u}{\partial x_i}\right)-\sum_{j=1}^{m}\left(\frac{\partial^2 u}{\partial y_j^2}+\frac{\beta_j}{y_j}\frac{\partial u}{\partial y_j}\right)+\frac{\partial^2 u}{\partial z^2}=0 \tag{5}$$

where α_i $(i=1,2,\ldots,n)$ and β_j $(j=1,2,\ldots,m)$ are constants [2].

2. HARNACK TYPE INEQUALITIES

Let us make the transformation $x^2=x_1^2+\ldots+x_n^2$ in (3). Then we have

$$\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\alpha_i}{x_i}\frac{\partial u}{\partial x_i}\right)=\frac{\partial^2 u}{\partial x^2}+\frac{n-1+\sum_{i=1}^{n}\alpha_i}{x}\frac{\partial u}{\partial x}.$$

Hence, if the constants α_i $(i=1,2,\ldots,n)$ are chosen as to satisfy the relation

$$n-1+\sum_{i=1}^{n}\alpha_i=0, \tag{6}$$

then, the equation (3) is reduced to the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{7}$$

Thus, we can give the following theorem.

Theorem 1. Let, in (3), α_i $(i=1,\ldots,n)$ be constants satisfying the relation (6), and let $u(x_1,x_2,\ldots,x_n,y)$ be a nonnegative solution of the equation (3) in the ball $B_R: x_1^2+\ldots+x_n^2+y^2<R^2$. If the point $P(x_1,\ldots,x_n,y)$ is at distance $r<R$ from the center O of the ball then,

$$\frac{R-r}{R+r}u(O)\le u(P)\le\frac{R+r}{R-r}u(O). \tag{8}$$

Proof. In the light of above discussion, the transformation $x^2=x_1^2+\ldots+x_n^2$ reduces the equation (3) into the two-dimensional Laplace equation. Since $u(x_1,x_2,\ldots,x_n,y)$ is a nonnegative solution of (3) then,

$$u(x_1,x_2,\ldots,x_n,y)=u^*(\sqrt{x_1^2+\ldots+x_n^2},y)=u^*(x,y)$$

is a nonnegative solution of (7). Thus u^* satisfies the inequality (1) and hence u satisfies the inequality (8).

Theorem 1 gives a bound for u at any point P in the ball B_R in terms of the center O of the ball. Now let u be a nonnegative solution of the equation (3) in a domain D, S be a closed bounded, connected subset of D and R be smaller than the distance of any point in S to the boundary of D. Then, we can give a bound for u at any point P in terms of another point Q in D which states the Harnack inequality in general sense. For if, let Q_2 be any point in a ball $B_R \subset S$ which is at distance less than αR, $(\alpha < 1)$ from the center Q_1. Then, by (8), we have

$$\frac{R-\alpha R}{R+\alpha R}u(Q_1) \le u(Q_2) \le \frac{R+\alpha R}{R-\alpha R}u(Q_1)$$

or

$$\frac{1-\alpha}{1+\alpha}u(Q_1) \le u(Q_2) \le \frac{1+\alpha}{1-\alpha}u(Q_1)$$

Now, if Q_3 is another point at distance less than αR from Q_2, we may apply the same idea with Q_2 and hence we derive

$$\frac{1-\alpha}{1+\alpha}u(Q_2) \le u(Q_3) \le \frac{1+\alpha}{1-\alpha}u(Q_2)$$

which, in turns, gives

$$\left(\frac{1-\alpha}{1+\alpha}\right)^2 u(Q_1) \le u(Q_3) \le \left(\frac{1+\alpha}{1-\alpha}\right)^2 u(Q_1).$$

For any finite number of point, we may continue this process, and for the points Q_1 and Q_{k-1} in S, we obtain

$$\left(\frac{1-\alpha}{1+\alpha}\right)^{k-1} u(Q_1) \le u(Q_{k-1}) \le \left(\frac{1+\alpha}{1-\alpha}\right)^{k-1} u(Q_1).$$

(Since S is closed bounded and connected subset, we can cover S by a finite number of balls). Thus, in general, any two pair of points P and Q in S can be connected by a chain of a finite , say k-1, number of balls in D, such that the first ball has its center at P and the last ball has its center at Q, and the centers of two successive balls are at distance less than αR from each other (Figure 2) [3]. Thus, we can give the following result.

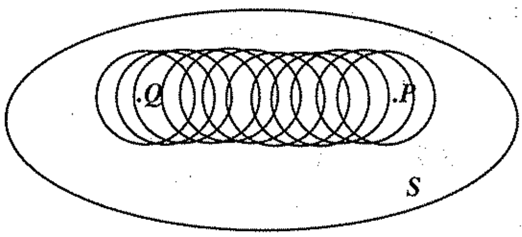

Figure 2.

Result 1. If u is a nonnegative solution of (3) defined in a domain D under the condition (6), then, there is a positive constant A depending on S and D but not on u such that for every pair of points P and Q in S, we have,

$$Au(Q) \le u(P) \le A^{-1}u(Q) \tag{9}$$

An immediate consequence of Result 1 is the following result.

Result 2. Let $\{u_k\}$ be a nonincreasing sequence of nonnegative solutions of (3) in a domain D. If the sequence converges at a single point Q of D, it converges uniformly on every closed bounded subset of D.

Proof. Letting $v_{ij} = u_i - u_j$, for $i < j$, v_{ij} is a nonnegative solution of (3). Hence if S is a closed bounded subset of D and P is a point of S then by (9), we have

$$0 \le v_{ij}(P) \le A^{-1} v_{ij}(Q) = A^{-1}\left[u_i(Q) - u_j(Q)\right]$$

Since u_k converges at Q, for every P, we have $v_{ij} \to 0,\ (i, j \to \infty)$ and the result follows.

Remark 1. The Harnack inequality (9) can be applied to the solutions which are bounded from below or above. For if u is bounded from below by a constant m, then the function $v = u - m$ satisfies the equation (3) and is nonnegative and thus the Harnack inequality (9) is valid for it. Similarly, if u is bounded from above by a constant M, then the function $w = M - u$ satisfies (9) and is nonnegative.

Remark 2. If u, under the condition (6), is any solution of (3), bounded from below or above in all of n+1-dimensional space, then u is constant.

Example. Let $u = \sqrt{x_1^2 + x_2^2}\, y$. Then, u satisfies the equation (3) with n=2 and $\alpha_1 + \alpha_2 = -1$. On the other hand, in the disk $x_1^2 + x_2^2 + y^2 < 1$, $m = -1/2$ is a lower bound for u. Hence, by applying Remark 1 and Theorem 1 to v=u+$1/2$, we obtain the Harnack inequality

$$\frac{1-r}{1+r} v(0,0,0) \le v(P) \le \frac{1+r}{1-r} v(0,0,0)$$

or

$$\frac{1}{2}\frac{1-r}{1+r} \le v(P) \le \frac{1}{2}\frac{1+r}{1-r}$$

where r<1 and P is any point on the sphere $x_1^2 + x_2^2 + y^2 = r^2$.

Simplifying the last inequality, we get

$$\frac{1}{2}\frac{1-r}{1+r} \le \sqrt{x_1^2 + x_2^2}\, y + \frac{1}{2} \le \frac{1}{2}\frac{1+r}{1-r}$$

or

$$\frac{1-r}{1+r} \le 2\sqrt{x_1^2 + x_2^2}\, y + 1 \le \frac{1+r}{1-r}$$

Hence, for any point on the sphere $x_1^2 + x_2^2 + y^2 = r^2$, we have the inequality

$$\frac{-r}{1+r} \le \sqrt{x_1^2 + x_2^2}\, y \le \frac{r}{1-r}$$

Now, we state the Harnack inequality for the solutions of (4).

Theorem 2. Let the constants of the equation (4) satisfy the relation

$$m-1+\sum_{j=1}^{m}\frac{\alpha_j}{b_j^2}=0 \tag{10}$$

and let $u(x_1,x_2,...,x_{n-1},y_1,y_2,...,y_m)$ be a nonnegative solutions of the equation (4) in the ellipsoidal domain $x_1^2+x_2^2+...+x_{n-1}^2+\left(\frac{y_1-y_1^0}{b_1}\right)^2+...+\left(\frac{y_m-y_m^0}{b_m}\right)^2<R^2$. If the point $P(x_1,x_2,...,x_{n-1},y_1,y_2,...,y_m)$ is at distance $r<R$ from the center O of the ellipsoid, then

$$\frac{R-r}{(R+r)^{n-1}}R^{n-2}u(O)\le u(P)\le\frac{R+r}{(R-r)^{n-1}}R^{n-2}u(O) \tag{11}$$

Proof. Let us make the transformation

$$x_n^2=\sum_{j=1}^{m}\left(\frac{y_j-y_j^0}{b_j}\right)^2 \tag{12}$$

in the equation (4). A straightforward computation shows that

$$\sum_{j=1}^{m}\left(b_j^2\frac{\partial^2 u}{\partial y_j^2}+\frac{\alpha_j}{y_j-y_j^0}\frac{\partial u}{\partial y_j}\right)=\frac{\partial^2 u}{\partial x_n^2}+\frac{m-1+\sum_{j=1}^{m}\frac{\alpha_j}{b_j^2}}{x_n}\frac{\partial u}{\partial x_n}=\frac{\partial^2 u}{\partial x_n^2}$$

Hence, under the transformation (12) and the relation (10), the equation (4) is reduced to the n-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2}+...+\frac{\partial^2 u}{\partial x_n^2}=0 \tag{13}$$

Since $u(x_1,x_2,...,x_{n-1},y_1,y_2,...,y_m)$ is a nonnegative solution of (4), then

$$u(x_1,x_2,...,x_{n-1},y_1,y_2,...,y_m)=u^*\left(x_1,x_2,...,x_{n-1},\sqrt{\left(\frac{y_1-y_1^0}{b_1}\right)^2+...+\left(\frac{y_m-y_m^0}{b_m}\right)^2}\right)$$
$$=u^*(x_1,x_2,...,x_{n-1},x_n)$$

is a nonnegative solution of (13). Thus, u^* satisfies the inequality (2) and hence u satisfies the inequality (11).

Now, let us consider the equation (5). Under the transformation

$$x^2=\sum_{i=1}^{n}x_i^2-\sum_{j=1}^{m}y_j^2>0 \tag{14}$$

we have

$$\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\alpha_i}{x_i}\frac{\partial u}{\partial x_i}\right)-\sum_{j=1}^{m}\left(\frac{\partial^2 u}{\partial y_j^2}+\frac{\beta_j}{y_j}\frac{\partial u}{\partial y_j}\right)=\frac{\partial^2 u}{\partial x^2}+\frac{n+m-1+\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j}{x}\frac{\partial u}{\partial x}$$

Thus, if we assume that the relation

$$n+m-1+\sum_{i=1}^{n}\alpha_i+\sum_{j=1}^{m}\beta_j=0 \tag{15}$$

holds between the coefficients of the equation (5), then the equation is transformed to the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}=0 \tag{16}$$

Hence we can give the following result.

Theorem 3. Let the relation (15) holds for the constants in (5) and let $u(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m,z)$ be a nonnegative solution of the equation (5) in the hyperboloidal domain $x_1^2+x_2^2+\ldots+x_n^2-y_1^2-\ldots-y_m^2+z^2<R^2$. If the point $P(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_m,z)$ is at distance $r<R$ from the origin O, then we have the Harnack inequality

$$\frac{R-r}{R+r}u(O)\le u(P)\le\frac{R+r}{R-r}u(O) \tag{17}$$

Remark 3. The results mentioned in the Remarks 1, 2 and in the Result 1,2 are also hold for the solutions of the equation (4) and the equation (5).

Acknowledgement-We are thankful to Prof. Dr. Abdullah ALTIN for his valuable suggestions.

REFERENCES

1. A. Mohammed, Weak Harnack's Inequality for non-negative solutions of elliptic equations with potential, *Proc. of Amer. Math. Soc.***129**, 2617-2621, 2001.

2. A. Altın, Particular solutions for iterated GASPT equations in terms of Bessel Functions, *Bull. Inst. Math. Acad. Sinica* **12**, 379-387, 1984.

3. H.M. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Springer Verlag, New York Inc., 1984.

4. J. B. Serrin, On the Harnack inequality for linear elliptic equations, *J. d'Analyse Math.* **4**, 292-308, 1954-1956.

5. J. B. Serrin, The Harnack inequality for elliptic partial differential equations in more than two independent variables, *Notices of the Amer. Math. Soc.*, 52-53, 1958.