SOME BOUNDS ON $\ell_p$ MATRIX AND $\ell_p$ OPERATOR NORMS OF
ALMOST CIRCULANT, CAUCHY-TOEPLITZ AND CAUCHY-
HANKEL MATRICES

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Abstract- Let $C_n$, $T_n$, and $H_n$ denote almost circulant, Cauchy-Toeplitz and Cauchy-
Hankel matrices, respectively. We find some upper bounds for $\ell_p$ matrix norm and $\ell_p$
operator norm of this matrices. Moreover, we give some results for Kronecker products
$C_n \otimes T_n$ and $C_n \otimes H_n$.

Keywords- Circulant matrix, Cauchy-Toeplitz matrix, Cauchy-Hankel matrix, norm, Kronecker product.

1. INTRODUCTION AND PRELIMINARIES

Let $C = [1/(x_i - y_j)]_{i,j=1}^n$ ($x_i \neq y_j$) be a Cauchy matrix and $T_n = [t_{j-i}]_{i,j=0}^n$ be a
Toeplitz matrix. In generally Cauchy-Toeplitz matrices are being defined as

$$T_n = \left[ \frac{1}{g + (i-j)h} \right]_{i,j=1}^n$$

(1.1)

where $h \neq 0$, $g$ and $h$ are some numbers and quotient $g/h$ is not integer. Toeplitz
matrices are precisely those matrices that one constant along all diagonals parallel to the
main diagonal, and thus a Toeplitz matrix is determined by its first row and column.

Closely related to Toeplitz matrices are the so-called circulant matrices. An
$(n \times n)$ matrix $C$ is called a circulant matrix if it is of the form

$$C_n = \begin{bmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
    c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{bmatrix}$$

For each $i,j=1,\ldots,n$ and $k=0,1,2,\ldots,n-1$, all the elements $(i,j)$ such that
$j - i \equiv k (\text{mod } n)$ have the same value $c_k$; these elements form the so-called $k$th stripe of
C. Obviously, a circulant matrix is determined by its first row (or column). It is clear that every circulant matrix is a Toeplitz matrix, but the converse is not necessarily true.

Let $H_n = [h_{i,j}]_{i,j=0}^n$ be Hankel matrix. A matrix

$$H_n = \left[ \frac{1}{g + (i+j)h} \right]_{i,j=0}^n$$

is called Cauchy-Hankel matrix. Hankel matrices are symmetric.

E. E. Tyrkin found a lower bound for the spectral norm of Cauchy-Toeplitz matrix such that $h=1$ and $g=1/2$. D. Bozkurt has given an upper and lower bounds for the $\ell_p$ matrix norm of almost Cauchy-Toeplitz matrix.

In this study, we have defined almost circulant matrix in the following form:

$$C_n = \begin{cases} a, & i = j \\ \frac{1}{k} & (j - i \equiv k(\text{mod } n)), i \neq j \end{cases}$$

where $a \in \mathbb{R} \setminus \{0\}$ ($\mathbb{R}$ is the set of real numbers) and $k=1,2,\ldots,n-1$.

In section 2, firstly we will establish upper bound for the $\ell_p$ matrix and $\ell_p$ operator norms of the matrix $C_n$. Secondly, we will establish upper bounds for the $\ell_p$ matrix norm of Cauchy-Toeplitz and Cauchy-Hankel matrices defined with (1.1) and (1.2). Finally we will give results for $\ell_p$ norms of Kronecker products $C_n \otimes T_n$ and $C_n \otimes H_n$.

For $1 \leq p < \infty$, the $\ell_p$ matrix norm of a matrix $A = [a_{ij}]_{nm}$ is defined as

$$\|A\|_p = \left( \sum_{i,j=1}^n |a_{ij}|^p \right)^{1/p}$$

If $p=\infty$, then

$$\|A\|_\infty = \lim_{p \to \infty} \|A\|_p = \max_{i,j} |a_{ij}|.$$ 

In addition, $\ell_p$ operator norm of an matrix $A = [a_{ij}]_{nm}$ is defined as

$$\|A\|_p = \max \{ \|Ax\|_p : x \in \mathbb{C}^n, \|x\|_p = 1 \}.$$ 

Let $A$ and $B$ be arbitrary $n \times m$ matrices. Kronecker product of this matrices given to be

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$$
A function $\Psi$ is called as psi (or digamma) function if $\Psi(x) = \frac{d}{dx} \log[\Gamma(x)]$ where $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. It is called as Polygamma function the $n$ th derivatives of $\Psi$ function [4] i.e.

$$\Psi(n, x) = \frac{d}{dx^n} \Psi(x) = \frac{d}{dx^n} \left[ \frac{d}{dx} \ln[\Gamma(x)] \right]$$

If $n=0$, then $\Psi(0, x) = \Psi(x) = \frac{d}{dx} \ln[\Gamma(x)]$. In addition, if $a>0$ and $b$ any number and $n \in \mathbb{Z}^+$ is positive integer, then

$$\lim_{n \to \infty} \Psi(a, n + b) = 0.$$  \hspace{1cm} (1.7)

In this study, $\mathbb{Z}$ and $\mathbb{R}$ will represent the sets of integers and real numbers, respectively.

2. $\ell_p$ MATRIX AND $\ell_p$ OPERATOR NORMS OF ALMOST CIRCULANT, CAUCHY-TOEPLITZ AND CAUCHY-HANKEL

**Theorem 2.1.** Let the matrix $C_n$ be as in (1.3). Then

$$n^{-1/p} \|C_n\|_p = \left\{ \|a\|_p + \zeta(p) \right\}^{1/p}$$  \hspace{1cm} (2.1)

is valid for the $\ell_p$ matrix norm of the matrix $C_n$ where $2 \leq p < \infty$, $a \in \mathbb{R} \setminus \{0\}$ and $\zeta$ is Riemann’s zeta function.

**Proof.** From (1.4) we have

$$\|C_n\|_p^p = n \left( \|a\|_p^p + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(n-1)^p} \right) = n\|a\|_p^p + n \sum_{s=1}^{n-1} \frac{1}{s^p}.$$  \hspace{1cm} (2.2)

If we divide both side of the (2.2) by $n$, then

$$\frac{1}{n}\|C_n\|_p^p = \|a\|_p^p + \sum_{s=1}^{n-1} \frac{1}{s^p}$$

can obtain. If we evaluate the right hand side of this equality, we have

$$\lim_{n \to \infty} \sum_{s=1}^{n-1} \frac{1}{s^p} = \zeta(p) .$$

Hence we obtain

$$\frac{1}{n}\|C_n\|_p^p \leq \|a\|_p^p + \zeta(p).$$  \hspace{1cm} (2.3)

If we take $1/p$ th power of inequality (2.3), then the proof is completed.

**Example 2.1.** Let $a=1$ and $p=2$ in Theorem 2.1. Then we have values listed given in the following table.
Table 2.1

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<tr>
<th>( n )</th>
<th>( n^{-1/2} | C_n |_2 )</th>
<th>( \sqrt{1 + \zeta(2)} )</th>
<th>( n )</th>
<th>( n^{-1/2} | C_n |_2 )</th>
<th>( \sqrt{1 + \zeta(2)} )</th>
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</tbody>
</table>

Corollary 2.1. Since \( C_n C_n^{-1} = E_n \) (\( E \) denote \( n \times n \) unit matrix), \( \| C_n C_n^{-1} \|_p = n^{1/p} \)

From this equality, we can write \( n^{1/p} = \| C_n C_n^{-1} \|_p \leq \| C_n \|_p \| C_n^{-1} \|_p \). Hence

\[
1 \leq n^{-1/p} \| C_n \|_p \| C_n^{-1} \|_p \quad \text{and} \quad \frac{1}{\| C_n^{-1} \|_p} \leq n^{-1/p} \| C_n \|_p.
\]

Consequently, from (2.3) inequality we have

\[
\frac{1}{\| C_n^{-1} \|_p} \leq \square\| a \|_p + \zeta(p) \square^{1/p}.
\]

Theorem 2.2. Let the matrix \( C_n \) be as in (1.3). For the \( \ell_p \) operator norm of matrix \( C_n \) is valid in the following:

\[
| C_n x |_p^p \leq [n|a| + n\zeta(p)]^{1/p} n^{1/q} \| x \|_p
\]

where \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( a \in \mathbb{R} - \{0\} \).

Proof. From (1.4) we have

\[
\| C_n x \|_p^p = \left( \sum_{j=1}^n \sum_{i=1}^n c_{ij} x_j \right)^p \leq \left( \sum_{j=1}^n \sum_{i=1}^n |c_{ij}| x_j \right)^p = \sum_{i,j=1}^n |c_{ij}|^p \left( \sum_{j=1}^n |x_j| \right)^p.
\]

Hence we can write

\[
\| C_n x \|_p^p \leq \sum_{i,j=1}^n |c_{ij}|^p \left( \sum_{j=1}^n |x_j| \right)^p.
\]

If we apply Hölder’s inequality to (2.5) inequality, then we have

\[
\left( \sum_{j=1}^n |x_j| \right)^p = \left( \sum_{j=1}^n |x_j| \right)^p \leq \left[ \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \left( \sum_{j=1}^n 1^q \right)^{1/q} \right]^p = \| x \|_p^p n^{p/q}
\]

(2.6)
where $p > 1$ and $q = \frac{P}{P-1}$. For the other expression,

\[
\sum_{i,j=1}^{\infty} |c_{ij}|^p = |n| |d|^p + n \sum_{s=1}^{n} \frac{1}{s^p}.
\]

Since \(\sum_{s=1}^{n} \frac{1}{s^p} \leq \zeta(p)\) we have

\[
\sum_{i,j=1}^{n} |c_{ij}|^p \leq |n| |d|^p + n\zeta(p) \quad (2.7)
\]

where $p > 1$. Consequently from (2.6) and (2.7),

\[
\|C_k \|_p^p \leq [n|d|^p + n\zeta(p)]n^{p/q} \|x\|_p^p. \quad (2.8)
\]

If we take $1/p$ th power of inequality (2.8), then the proof is completed.

**Theorem 2.3.** Let \(T_m = \left[ \frac{1}{1/k + (i-j)} \right]_{i,j=1}^{m} \) (1 < $k$ $\in$ $\mathbb{Z}^+$) be a Cauchy-Toeplitz matrix. Then, for the $\ell_p$ matrix norm of the matrix $T_m$

\[
m^{-1/p} \|T_m\|_p \leq \left[ k^p + \frac{(-1)^p}{(p-1)!} \psi(p-1,1/k) + \psi(p+1,1/k) \right]^{1/p} \quad (2.9)
\]

is valid where $2 \leq p < \infty$.

**Proof.** From (1.4) we have

\[
\|T_m\|_p^p = m^k |d|^p + \sum_{s=1}^{m} (m-s) \left( \frac{1}{1-s^p} + \frac{1}{1+s^p} \right)
\]

\[
= mk^p + \sum_{s=1}^{m} (m-s) \left( \frac{1}{s-1/k}^p + \frac{1}{s+1/k}^p \right)
\]

If we divide both side of this equality by $n$, then

\[
\frac{1}{m} \|T_m\|_p^p = \left[ k^p + \sum_{s=1}^{m} (1-s/m) \left( \frac{1}{s-1/k}^p + \frac{1}{s+1/k}^p \right) \right]. \quad (2.10)
\]

From the properties of polygamma functions the right hand side of this equality is written as the following:

\[
\sum_{s=1}^{m} (1-s/m) \frac{1}{(s+1/k)^p} = \frac{(-1)^{p-1}(km+1)}{(p-1)m} \left[ \psi(p+1,1/k+m) - \psi(p+1,1/k+1) \right]
\]

\[
+ \frac{(-1)^{p-1}}{(p-2)m} \left[ \psi(p+2,1/k+m) - \psi(p+2,1/k+1) \right] \quad (2.11)
\]
and

\[
\sum_{n=1}^{\infty} \frac{1}{(s-1/k)^p} \frac{1}{(s-1/k)^p} = \left\{ \begin{array}{l}
\frac{\Psi(p-1,m-1/k) - \Psi(p-1,1-k)}{1}\frac{\Psi(p-1,m-1/k) - \Psi(p-1,1-k)}{1}
\end{array} \right\}.
\]

(2.12)

If we take limit as \( m \to \infty \) of (2.11) and (2.12) equalities, then we obtain

\[
\lim_{m \to \infty} \sum_{n=1}^{m-1} \frac{1}{(s-1/k)^p} = \left( \frac{-1}{p-1} \right)^p \Psi(p-1,1/k + 1)
\]

(2.13)

and

\[
\lim_{m \to \infty} \sum_{n=1}^{m-1} \frac{1}{(s-1/k)^p} = \left( \frac{-1}{p-1} \right)^p \Psi(p-1,1-1/k).
\]

(2.14)

from (1.7). If we write (2.13) and (2.14) equalities into (2.10) equality, then we have

\[
\frac{1}{m} \left\| T_n \right\|_p \leq \left( \frac{1}{p-1} \right) \left[ \Psi(p-1,1/k + 1) + \Psi(p-1,1-1/k) \right].
\]

(2.15)

If we take \( 1/p \) th power of this inequality, then the proof is completed.

**Example 2.2.** Let \( k=2 \) and \( p=2 \) in Theorem 2.3. Then we have values listed in the following table, where \( \Delta = \sqrt{2^2 + \Psi(1,1+1/2) + \Psi(1,1-1/2)} \).

<table>
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<tr>
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<th>( n^{-1/2} \left| T_n \right|_2 )</th>
<th>( \Delta )</th>
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</tr>
</tbody>
</table>

**Theorem 2.4.** Let \( H_m = \left[ \frac{1}{1/k + (i+j)} \right]_{i,j=1}^m \) \((1 < k \in \mathbb{Z}^+)\) be a Cauchy-Hankel matrix. Then, for the \( \ell_p \) matrix norm of the matrix \( H_m \)
Almost Circulant, Cauchy-Toeplitz and Cauchy-Hankel Matrices

\[ \|H_n\|_p \leq \left\{ k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2,1/k) + (k+1)\Psi(p-1,1/k)]}{(p-1)!k} \right\}^{1/p} \]

is valid where \( 3 \leq p < \infty \).

**Proof.** From (1.4) we have

\[ \|H_n\|_p^p = \sum_{s=1}^{m} \frac{s}{k + s + 1} + \sum_{s=1}^{m-1} \frac{m - s}{k + s + m + 1} \]

\[ = \sum_{s=1}^{m} \frac{s}{(1/k + s + 1)^p} + \sum_{s=1}^{m-1} \frac{m - s}{(1/k + s + m + 1)^p} \]

From the properties of polygamma functions the right hand side of this equality is wrote as the following:

\[ \sum_{s=1}^{m} \frac{s}{(1/k + s + 1)^p} = \frac{(-1)^p}{(p-2)!} \frac{k+1}{(p-1)^k} \left[ \Psi(p-1, m+1 + (k+1)/k) - \Psi(p-1,1 + (k+1)/k) \right] \]

\[ + \Psi(p-2, m+1 + (k+1)/k) - \Psi(p-2,1 + (k+1)/k) \]

and

\[ \sum_{s=1}^{m-1} \frac{(m-s)}{(1/k + m + s + 1)^p} = \frac{(-1)^p}{(p-2)!} \frac{2km+k+1}{(p-1)^k} \left[ \Psi(p-1,1 + (km+k+1)/k) - \Psi(p-1,m + (km+k+1)/k) \right] \]

\[ + \Psi(p-2,1 + (km+k+1)/k) - \Psi(p-2,m + (km+k+1)/k) \].

If we take limit as \( m \to \infty \) of these equalities, then we have

\[ \lim_{m \to \infty} \sum_{s=1}^{m} \frac{s}{(1/k + s + 1)^p} = k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2,1/k) + (k+1)\Psi(p-1,1/k)]}{(p-1)!k} \]

\[ \lim_{m \to \infty} \sum_{s=1}^{m-1} \frac{m - s}{(1/k + s + m + 1)^p} = 0. \]

from (1.7). If we write (2.18) and (2.19) equalities into (2.17) equality, then we have

\[ \|H_n\|_p^p \leq \left\{ k^p + \frac{(-1)^{p+1} [(p-1)\Psi(p-2,1/k) + (k+1)\Psi(p-1,1/k)]}{(p-1)!k} \right\}^{1/p} \]

If we take \( 1/p \) th power of this inequality, then the proof is completed.

**Corollary 2.2.** For Kronecker product of \( C_n \) and \( T_m \) matrices defined by Theorem 2.1 and Theorem 2.3, following inequality is valid:
\[
\frac{1}{(mn)^{1/p}} \left\| C_n \otimes T_m \right\|_p \leq \left\| \left[ |a|^p + \xi(p) \right]^{1/p} \left[ k^p + \frac{(-1)^p}{(p-1)!} \frac{|\Psi(p(+1/2)| + |\Psi(p-1/2)|}{(p-1)!k} \right] \right\|^{1/p}
\]

**Proof.** Since

\[
\left\| C_n \otimes T_m \right\|_p = \left\| C_n \right\|_p \left\| T_m \right\|_p
\]

the proof is trivial from (2.3) and (2.15).

**Corollary 2.3.** For Kronecker product of \( C_n \) and \( H_n \) matrices defined by Theorem 2.1 and Theorem 2.4, following inequality is valid.

\[
\frac{1}{n^{1/p}} \left\| C_n \otimes H_n \right\|_p \leq \left\| \left[ |a|^p + \xi(p) \right]^{1/p} \left[ k^p + \frac{(-1)^p}{(p-1)!} \frac{|\Psi(p(+1/2)| + |\Psi(p-1/2)|}{(p-1)!k} \right] \right\|^{1/p}
\]

**Proof.** Since

\[
\left\| C_n \otimes H_n \right\|_p = \left\| C_n \right\|_p \left\| H_n \right\|_p
\]

the proof is trivial from (2.3) and (2.20).

**REFERENCES**