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ON GENERALIZED MIXED QUASIVARIATIONAL INCLUSIONS

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Abstract : In this paper, we study a class of generalized mixed quasivariational inclusions. By using the properties of the resolvent operator associated with maximal monotone mapping in real Hilbert space, we have established an existence theorem of solutions for generalized mixed quasivariational inclusions, suggesting a new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions which strongly converge to the exact solutions of the generalized mixed quasivariational inclusions. The results proved in this paper are more general than the previously known results in this area.

1. INTRODUCTION

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in mechanics, mathematical programming, optimization and control problems, equilibrium theory of economics, management science, operations research, and other branches of mathematics and engineering sciences, etc., see [1-16] and the references therein.

In a recent paper [7], Hassouni and Moudafi introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusion. This technique is further extended and generalized in many different directions by Adly [1], Ding [4], Siddiqi et al [13,16], and Noor et al [10].

In this paper, we introduce and study a class of generalized mixed quasivariational inclusions. By applying the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert spaces, it is shown that the quasivariational inclusion problems are equivalent to fixed point problems. A new iterative algorithm and a perturbed proximal point algorithm for finding approximate solutions which strongly converge to the exact solution of the generalized mixed quasivariational inclusion are proposed and analysed.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $<.,.>$. Let $C(H)$ be a family of nonempty compact subsets of H. Let $T, A : H \to C(H)$ be set-valued mappings, $g : H \to H$ be a single-valued mapping, and $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(., y) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $g(H) \cap$ dom $\partial\phi(., y) \neq \Phi$ for each $y \in H$. Then for a given nonlinear operator $N(.,.) : H \times H \to H$, the problem of finding $x \in H$, $u \in T(x)$, $v \in A(x)$ such that $g(x) \in$ dom $\partial\phi(., x)$ and

$$< N(u, v), y - g(x) > \ \geq \ \phi(g(x), x) - \phi(y, x), \ \forall y \in H, \tag{2.1}$$

is called the generalized mixed quasivariational inclusion problem (GMQVIP).

Special Cases : (1) If $N(u, v) = u - v$ then GMQVIP reduces to the problem of finding $x \in H, u \in T(x)$ and $v \in A(x)$ such that $g(x) \in$ dom $\partial\phi(., x)$ and

$$< u - v, y - g(x) > \ \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in H$$

which is known as the generalized quasivariational inclusion problem, studied by Ding [4].

(2) If $N(u, v) = u - v$ and $\phi(x, y) = \phi(x)$ for all $y \in H$ and T and A are both single-valued mappings, then problem (2.1) reduces to the variational inclusion problem (1.1) considered by Hassouni and Moudafi [7].

(3) If $N(u, v) = u - v$ and K is a closed convex subset of H and $\phi = I_K$ is the indicator function of K,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.1) reduces to the generalized strongly nonlinear variational inequality problems, i.e., find $x \in H$, $u \in T(x)$, and $v \in A(x)$ such that $g(x) \in K$ and

$$< u - v, y - g(x) > \ \geq \ 0, \ \forall y \in K.$$

considered by Siddiqi et al [16].

In order to prove our main theorems, we need the following concepts and results.

Definition 2.1. Let X be a Banach space with the dual space X^* and let $\phi : X \to R \cup \{+\infty\}$ be a proper functional. Then ϕ is said to be subdifferential at a point $x \in X$ if there exists an $f^* \in X^*$ such that

$$\phi(y) - \phi(x) \ \geq \ < f^*, y - x >, \ \forall y \in X,$$

where f^* is called the subgradient of ϕ at x. The set of all subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi : X \to 2^{X^*}$ defined by

$$\partial\phi(x) = \{f^* \in X^* : \phi(y) - \phi(x) \ \geq \ < f^*, y - x >, \ \forall y \in X\}$$

is said to be the subdifferential of ϕ.

Definition 2.2.. Let H be a Hilbert space and let $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G : H \to H$ defined by

$$J_\rho^G = (I + \rho G)^{-1}(x), \quad \forall x \in H$$

is said to be the resolvent operator of G where I is the identity mapping on H.

Lemma 2.1. Let X be a reflexive Banach space endowed with a strictly convex norm and $\phi : X \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then $\partial\phi : X \to 2^{X^*}$ is a maximal monotone mapping.

Lemma 2.2. Let $G : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^G : H \to H$ of G is nonexpensive, i.e., for all $x, y \in H$,

$$\| J_\rho^G(x) - J_\rho^G(y) \| \ \leq \ \| x - y \| .$$

Definition 2.3. For all $x, y \in H$, the operator $N(.,.)$ is said to be strongly monotone and Lipschitz continuous with respect to the first argument, if there exists constants $\alpha > 0$, $\beta > 0$ such that

$$\begin{aligned} < N(u_1,.) \ &- \ N(u_2,.), x - y > \\ &\geq \ \alpha \| x - y \|^2, \ \ u_1 \in T(x), u_2 \in T(y), \end{aligned}$$

$$\| N(u_1,.) - N(u_2,.) \| \leq \beta \| u_1 - u_2 \| .$$

In a similar way, we can define the strong monotonicity and Lipschitz continuity of the operator $N(.,.)$ with respect to the second argument.

Definition 2.4. A mapping $g : H \to H$ is said to be
(i) $\lambda-$strongly monotone if there exists a constant $\lambda > 0$ such that

$$< g(x) - g(y), x - y > \geq \lambda \| x - y \|^2, \ \forall x, y \in H;$$

(ii) $\sigma-$Lipschitz continuous if there exists a constant $\sigma \geq 0$ such that

$$\| g(x) - g(y), x - y \| \geq \sigma \| x - y \|^2, \ \forall x, y \in H.$$

Definition2.5. A set-valued mapping $T : H \to C(H)$ is said to be $\eta-$Lipschitz continuous if there exists a constant $\eta \geq 0$ such that

$$\delta(T(x), T(y)) \leq \eta \| x - y \|, \ \forall x, y \in H,$$

where $\delta(A, B) = sup\{\| a - b \|: a \in A, b \in B\}, \ \forall A, B \in C(H)$.

3. MAIN RESULTS

In this section, we shall prove an existence theorem of solutions for the GMQVIP (2.1) and suggest a new iterative algorithm and a purturbed proximal point algorithm for finding approximate solutions of the problem (2.1). Then we show that the sequence of approximate solutions strongly converges to the exact solution of the problem (2.1).

We first transfer the problem (2.1) in a fixed point problem.

Theorem 3.1. The function (x^*, u^*, v^*) is a solution of (2.1) if and only if (x^*, u^*, v^*) statisfy the relation

$$g(x) = J_\rho^{\partial\phi(.,x)}(g(x) - \rho N(u, v)), \ \forall x \in H, \tag{3.1}$$

where $\rho > 0$ is a constant, $J_\rho^{\partial\phi(.,x)} = (1 + \rho\partial\phi(., x))^{-1}$ is the resolvent operator of $\partial\phi(., x)$, and I is the identity mapping on H.

Proof. Let (x^*, u^*, v^*) satisfy the relation (3.1), that is,

$$g(x^*) = J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*)).$$

The equality holds if and only if

$$N(v^*, u^*) \in \partial\phi(., x^*)(g(x^*)),$$

by the definition of $J_\rho^{\partial\phi(.,x^*)}$. This relation holds if and only if

$$\phi(y, x^*) - \phi(g(x^*), x^*) \geq < N(v^*, u^*), y - g(x^*) >, \ \forall y \in H,$$

by the definition of the subdifferential $\partial\phi(., x^*)$. Hence (x^*, u^*, v^*) is the solution of

$$< N(u^*, v^*), y - g(x^*) > \geq \phi(g(x^*), x^*) - \phi(y, x^*), \ \forall y \in H.$$

Remark 3.1. From Theorem 3.1, it follows that the GMQVIP (2.1) is equivalent to the fixed point problem

$$x = x - g(x) + J_\rho^{\partial\phi(.,x)}[g(x) - \rho N(u, v)]. \tag{3.2}$$

This fixed point formulation enables us to suggest the following algorithms.

Algorithm 3.1. For any given $x_0 \in H$, $\overline{u_0} \in T(x_0)$, and $\overline{v_0} \in A(x_0)$, let

$$y_0 = (1 - \beta_0)x_0 + \beta_0\left[x_0 - g(x_0) + J_\rho^{\partial\phi(.,x_0)}(g(x_0) - \rho N(\overline{u_0}, \overline{v_0}))\right].$$

Take any fixed $u_0 \in T(y_0)$ and $v_0 \in A(y_0)$, and let

$$x_1 = (1 - \alpha_0)x_0 + \alpha_0\left[y_0 - g(y_0) + J_\rho^{\partial\phi(.,y_0)}(g(y_0) - \rho N(u_0, v_0))\right].$$

Continuing in this way, we can define sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty$ as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\left[y_n - g(y_n) + J_\rho^{\partial\phi(.,y_n)}(g(y_n) - \rho N(u_n, v_n))\right],$$

$$y_n = (1 - \beta_n)x_n + \beta_n\left[x_n - g(x_n) + J_\rho^{\partial\phi(.,x_n)}(g(x_n) - \rho N(\overline{u_n}, \overline{v_n}))\right], \tag{3.3}$$

for $n = 0, 1, 2, \cdots$ where $u_n \in T(y_n)$, $v_n \in A(y_n)$, $\overline{u_n} \in T(x_n)$, and $\overline{v_n} \in A(x_n)$ can be chosen arbitrarily, $0 \leq \alpha_n, \beta_n \leq 1$, $\sum_{n=0}^\infty \alpha_n$ diverges, and $\rho > 0$ is a constant.

Using the fixed point formulation (3.2), we have the following algorithm.

Algorithm 3.2. For any given $x_0 \in H$, compute the sequences $\{x_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, and $\{v_n\}_{n=0}^\infty$, by the iteration schemes

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\phi(.,x_n)}[g(x_0) - \rho N(u_n, v_n)] \tag{3.4}$$

for $n = 0, 1, 2, \cdots$, where $u_n \in T(x_n)$, $v_n \in A(x_n)$ can be chosen arbitrarily and $\rho > 0$ is a constant.

To perturb the Algorithm (3.2), we first add, in the right-hand side of (3.4), an error e_n to take into account a possible inexact computation of the proximal point and we consider another perturbation by replacing ϕ in (3.4) by ϕ_n, where each $\phi_n : H \times H \to R \cup \{+\infty\}$ is such that for each fixed $y \in H$, $\phi_n(.,y)$ is a proper convex lower semicontinuous function on H and the sequence $\{\phi_n\}$ approximates ϕ on $H \times H$. Then we obtain the following perturbed proximal point algorithm.

Algorithm 3.3. For any given $x_0 \in H$, compute the sequences $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, and $\{v_n\}_{n=0}^{\infty}$, by the iteration schemes

$$x_{n+1} = x_n - g(x_n) + J_\rho^{\partial\phi(.,x_n)}(g(x_n) - \rho N(u_n, v_n)) + e_n, \tag{3.5}$$

where $\{e_n\}_{n=0}^{\infty}$ is an error sequence in H, $u_n \in T(x_n)$ and $v_n \in A(x_n)$ can be chosen arbitrarily, and $\rho > 0$ is a constant.

Now we show that the existence of solutions of the GMQVIP (2.1).

Theorem 3.2. Let $N(.,.) : H \times H \to H$ be $\alpha-$strongly monotone and $\beta-$Lipschitz continuous with respect to the first argument. Let $T : H \to C(H)$ be $\eta-$Lipschitz continuous, $A : H \to C(H)$ be $\gamma-$Lipschitz continuous, $g : H \to H$ be $\lambda-$strongly monotone and $\sigma-$Lipschitz continuous. Let $N(.,.)$ be $\xi-$Lipschitz continuous with respect to the second argument, and $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(.,y)$ is a proper convex lower semicontinuous function on H, $g(H) \cap \partial\phi(.,y) \neq \Phi$ and for each $x, y, z \in H$,

$$\| J_\rho^{\partial\phi(.,x)}(z) - J_\rho^{\partial\phi(.,y)}(z) \| \leq \mu \| x - y \| .$$

Suppose that there exists a constant $\rho > 0$ such that

$$k = \mu + 2\sqrt{1 - 2\lambda + \sigma^2} < 1 ,$$

$$\alpha > (1-k)\xi\gamma + \sqrt{k(\beta^2\eta^2 - \xi^2\gamma^2)(2-k)} ,$$

$$\rho\,\xi\,\gamma < 1 - k,$$

$$\left| \rho - \frac{\alpha - (1-k)\xi\gamma}{\beta^2\eta^2 - \xi^2\gamma^2} \right| < \frac{\sqrt{[\alpha - (1-k)\xi\gamma]^2 - k(\beta^2\eta^2 - \xi^2\gamma^2)(2-k)}}{\beta^2\eta^2 - \xi^2\gamma^2}. \tag{3.6}$$

Then the GMQVIP (2.1) has a solution (x^*, u^*, v^*).

Proof. By Theorem 3.1, it is sufficient to prove that there exists $x^* \in H$, $u^* \in T(x^*)$, $v^* \in A(x^*)$ such that (3.1) holds. Define a set-valued mapping $F : H \to C(H)$ by

$$F(x) = \bigcup_{u\in T(x)} \bigcup_{v\in A(x)} \left[x - g(x) + J^{\partial\phi(.,x)}\rho(g(x) - \rho N(u,v))\right]$$

for all $x \in H$. For arbitrary $x, y \in H$, $a \in F(x)$, and $b \in F(y)$, there exists $u_1 \in T(x)$, $v_1 \in A(x)$, $u_2 \in T(y)$, $v_2 \in A(y)$ such that

$$a = x - g(x) + J_\rho^{\partial\phi(.,x)}(g(x) - \rho N(u_1, v_1)),$$

$$b = y - g(y) + J_\rho^{\partial\phi(.,y)}(g(y) - \rho N(u_2, v_2)).$$

By the assumption of ϕ and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\| a - b \| \\
&\quad \le \| x - y - (g(x) - g(y)) \| + \| J_\rho^{\partial\phi(.,x)}(g(x) - \rho N(u_1, v_1)) \\
&\qquad - J_\rho^{\partial\phi(.,x)}(g(y) - \rho N(u_2, v_1)) \| \\
&\qquad + \| J_\rho^{\partial\phi(.,x)}(g(y) - \rho N(u_2, v_1)) - J_\rho^{\partial\phi(.,y)}(g(y) - \rho N(u_2, v_2)) \| \\
&\quad \le 2 \| x - y - (g(x) - g(y)) \| + \| x - y - \rho(N(u_1, v_1) - N(u_2, v_1)) \| \\
&\qquad + \rho \| N(u_2, v_1) - N(u_2, v_2) \| + \mu \| x - y \| . \qquad (3.7)
\end{aligned}$$

By strong monotonicity and the Lipschitz continuity of g, we have

$$\| x - y - (g(x) - g(y)) \| \le \sqrt{1 - 2\lambda + \sigma^2} \| x - y \| . \qquad (3.8)$$

Since $N(.,.)$ is strongly monotone and Lipschitz continuous with respect to the first argument and T is Lipschitz continuous, we have

$$\| x - y - \rho N(u_1, v_1) - N(u_2, v_1) \| \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\eta^2} \| x - y \| . \qquad (3.9)$$

Again since $N(.,.)$ is Lipschitz continuous with respect to second argument and A is Lipschitz continuous, we have

$$\| N(u_2,v_1) - N(u_2,v_2) \| \leq \xi\gamma \| x-y \| . \tag{3.10}$$

By combining (3.7) to (3.10), we obtain

$$\begin{aligned} \delta(F(x),F(y)) &\leq \left[2\sqrt{1-2\lambda+\sigma^2}+\sqrt{1-2\rho\alpha+\rho^2\eta^2\beta^2}+\rho\xi\gamma+\mu\right] \| x-y \| \\ &= [k+t(\rho)+\rho\xi\gamma] \| x-y \| = \theta \| x-y \| \end{aligned} \tag{3.11}$$

where $k = 2\sqrt{1-2\lambda+\sigma^2}+\mu$, $t(\rho) = \sqrt{1-2\rho\alpha+\rho^2\eta^2\beta^2}$ and $\theta = k+t(\rho)+\rho\xi\gamma$. By the condition (3.6) we have $\theta < 1$. It follows from the condition (3.11) and Theorem 3.1 of Siddiqi and Ansari [12] that f has a fixed $x^* \in H$. By the definition of F there exists $u^* \in T(x^*)$ $v^* \in A(x^*)$ such that

$$g(x^*) = J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*,v^*)).$$

Therefore (x^*,u^*,v^*) is a solution of GMQVIP (2.1).

In the following, we shall show the convergence of the Algorithms 3.1 and 3.3.

Theorem 3.3. Let H, N, T, A, g, and ϕ satisfy all conditions in Theorem 3.2. If the condition (3.6) is also satisfied, then the iterative sequences $\{x_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$ defined in the Algorithm 3.1 strongly converge to x^*, u^*, and v^*, respectively, and (x^*,u^*,v^*) is a solution of the GMQVIP (2.1).

Proof. By Theorem 3.2, the GMQVIP (2.1) has a solution (x^*,u^*,v^*). From Theorem 3.1 we have $x^* \in H$, $u^* \in T(x^*)$, $v^* \in A(x^*)$, and for all $n \geq 0$,

$$\begin{aligned} x^* &= x^* - g(x^*) + J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*,v^*)) \\ &= (1-\alpha_n)x^* + \alpha_n\left[x^* - g(x^*) + J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*,v^*))\right] \\ &= (1-\beta_n)x^* + \beta_n\left[x^* - g(x^*) + J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*,v^*))\right]. \end{aligned}$$

By the Algorithm 3.1, using similar arguments as in the proof of Theorem 3.2, we obtain

$$\begin{aligned}
\| x_n - x^* - (g(x_n) - g(x^*)) \| &\leq \sqrt{1-2\lambda+\sigma^2} \| x_n - x^* \|, \\
\| x_n - x^* - \rho\{N(\overline{u_n}, \overline{v_n}) - N(u^*, \overline{v_n}\} \| &\leq \sqrt{1-2\rho\alpha+\rho^2\beta^2\eta^2} \| x_n - x^* \|, \\
\| N(u^*, \overline{v_n}) - n(u^*, v^*) \| &\leq \xi\gamma \| x_n - x^* \|, \\
\| y_n - x^* - (g(y_n) - g(x^*)) \| &\leq \sqrt{1-2\lambda+\sigma^2} \| y_n - x^* \| \\
\| y_n - x^* - \rho\{N(u_n, v_n) - N(u^*, v_n)\} \| &\leq \sqrt{1-2\rho\alpha-\rho^2\beta^2\eta^2} \| y_n - x^* \|, \\
\| N(u^*, v_n) - N(u^*, v^*) \| &\leq \xi\gamma \| y_n - x^* \| .
\end{aligned}$$

Thus, by the Algorithm 3.1, the assumption of ϕ, and Lemmas 2.1 and 2.2, we have $\| y_n - x^* \|$

$$\begin{aligned}
&= \| (1-\beta_n)x_n + \beta_n \left[x_n - g(x_n + J_\rho^{\partial\phi(.,x_n)}(g(x_n) - \rho N(\overline{u_n}, \overline{v_n}))\right] \\
&\quad - (1-\beta_n)x^* - \beta_n \left[x^* - g(x^*) + J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*))\right] \| \\
&\leq (1-\beta_n) \| x_n - x^* \| + \beta_n \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \beta_n \| J_\rho^{\partial\phi(.,x_n)}(g(x_n) - \rho N(\overline{u_n}, \overline{v_n})) - J_\rho^{\partial\phi(.,x^*)}(g(x_n) - \rho N(\overline{u_n}, \overline{v_n})) \| \\
&\quad + \beta_n \| J_\rho^{\partial\phi(.,x^*)}(g(x_n) - \rho N(\overline{u_n}, \overline{v_n})) - J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*)) \| \\
&\leq (1-\beta_n) \| x_n - x^* \| + \beta_n\mu \| x_n - x^* \| + 2\beta_n \| x_n - x^* - (g(x_n - g(x^*)) \| \\
&\quad + \beta_n \| x_n - x^* - \rho\{N(\overline{u_n}, \overline{v_n}) - N(u^*, \overline{v_n}\} \| \\
&\quad + \beta_n\rho \| N(u^*, \overline{v_n}) - N(u^*, v^*) \| \\
&\leq (1-\beta_n) \| x_n - x^* \| + \beta_n k \| x_n - x^* \| + \beta_n t(\rho) \| x_n - x^* \| \\
&\quad + \beta_n\rho\xi\gamma \| x_n - x^* \| \\
&= (1-\beta_n) \| x_n - x^* \| + \beta_n\theta \| x_n - x^* \| \\
&\leq \| x_n - x^* \| . \qquad (3.12)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\| x_{n+1} - x^* \| &= \| (1-\alpha_n)x_n + \alpha_n \left[y_n - g(y_n) + J_\rho^{\partial\phi(.,y_n)}(g(y_n) - \rho N(u_n, v_n)) \right] \\ &\quad - (1-\alpha_n)x^* - \alpha_n \left[x^* - g(x^*) + J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*)) \right] \| \\ &\leq (1-\alpha_n) \| x_n - x^* \| + \alpha_n \theta \| y_n - x^* \| . \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned}\| x_{n+1} - x^* \| &\leq (1-\alpha_n) \| x_n - x^* \| + \alpha_n \theta \| x_n - x^* \| \\ &= [1 - (1-\theta)\alpha_n] \| x_n - x^* \| \\ &\leq \Pi_{i=0}^{n} [1 - (1-\theta)\alpha_i] \| x_0 - x^* \| . \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\Pi_{i=0}^{\infty} [1 - (1-\theta)\alpha_i] = 0$. Hence the sequence $\{x_n\}$ strongly converges to x^*. By (3.12), the sequence $\{y_n\}$ also strongly converges to x^*. Since $u_n \in T(y_n)$, $u^* \in T(x^*)$, and T is $\eta-$Lipschitz continuous, we have

$$\| u_n - u^* \| \leq \eta \| y_n - x^* \| \to 0,$$

and hence the sequence $\{u_n\}$ strongly converges to u^*. Similarly, we can show that the sequence $\{v_n\}$ strongly converges to v^*. This completes the proof.

Theorem 3.4. Let $N(.,.) : H \times H \to H$ be $\alpha-$strongly monotone and $\beta-$Lipschitz continuous with respect to the first argument. Let $T : H \to C(H)$ be $\eta-$Lipschitz continuous, $A : H \to C(H)$ be $\gamma-$Lipschitz continuous, $g : H \to H$ be $\lambda-$ strongly monotone and $\sigma-$Lipschitz continuous. Let $N(.,.)$ be $\xi-$Lipschitz continuous with respect to the second argument, and ϕ, $\phi_n : H \times H \to R \cup \{+\infty\}$, $n = 1, 2, 3, \cdots$, be such that for each fixed $y \in H$, $\phi(.,y)$ and each $\phi_n(.,y)$ are both proper convex lower semicontinuous functions on H, $g(H) \cap dom\partial\phi(.,y) \neq \Phi$, and for each $x, y, z \in H$ and for all $n \geq 1$,

$$\| J_\rho^{\partial\phi_n(.,x)}(z) - J_\rho^{\partial\phi_n(.,y)}(z) \| \leq \mu \| x - y \| .$$

Assume $\lim_{n\to\infty} \| J_\rho^{\partial\phi_n(.,y)}(z) - J_\rho^{\partial\phi(.,y)}(z) \| = 0$ for all $y, z \in H$, $\lim_{n\to\infty} \| e_n \| = 0$, and there exists a constant $\rho > 0$ such that the condition (3.6) in Theorem 3.2 holds. Then the iterative sequences $\{x_n\}$, $\{u_n\}$, and $\{v_n\}$ defined in the Algorithm

3.3 strongly converges to x^*, u^*, and v^*, respectively, and (x^*, u^*, v^*) is a solution of the GMQVIP (2.1).

Proof. By Theorem 3.2, the GMQVIP (2.1) has a solution (x^*, u^*, v^*) such that $u^* \in T(x^*)$, $v^* \in A(x^*)$, and

$$x^* = x^* - g(x^*) + J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*)).$$

By setting $h(x^*) = g(x^*) - \rho N(u^*, v^*)$ and by using the Algorithm 3.3 and assumptions of ϕ and $\phi_n, n = 1, 2, \cdots$, we obtain

$$\begin{aligned}
&\| x_{n+1} - x^* \| \\
&= \| x_n - g(x_n) + J_\rho^{\partial\phi_n(.,x_n)}(g(x_n) - \rho N(u_n, v_n)) + e_n \\
&\quad - x^* + g(x^*) - J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*)) \| \\
&\leq \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \| J_\rho^{\partial\phi_n(.,x_n)}(g(x_n) - \rho N(u_n, v_n)) - J_\rho^{\partial\phi_n(.,x_n)}(g(x^*) - \rho N(u^*, v^*)) \| \\
&\quad + \| J_\rho^{\partial\phi_n(.,x_n)}(g(x^*) - \rho N(u^*, v^*)) - J_\rho^{\partial\phi_n(.,x^*)}(g(x^*) - \rho N(u^*, v^*)) \| \\
&\quad + J_\rho^{\partial\phi_n(.,x^*)}(g(x^*) - \rho N(u^*, v^*)) - J_\rho^{\partial\phi(.,x^*)}(g(x^*) - \rho N(u^*, v^*)) \| \\
&\quad + \| e_n \| \\
&\leq 2 \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \| x_n - x^* - \rho(N(u_n, v_n) - N(u^*, v_n)) \| \\
&\quad + \rho \| N(u^*, v_n) - N(u^*, v^*) \| + \mu \| x_n - x^* \| \\
&\quad + \| J_\rho^{\partial\phi_n(.,x^*)}(h(x^*)) - J_\rho^{\partial\phi(.,x^*)}(h(x^*)) \| + \| e_n \| \\
&\leq (k + t(\rho) + \rho\varepsilon\gamma) \| x_n - x^* \| \\
&\quad + \| J_\rho^{\partial\phi_n(.,x^*)}(h(x^*)) - J_\rho^{\partial\phi(.,x^*)}(h(x^*)) \| + \| e_n \| \\
&= \theta \| x_n - x^* \| + \epsilon_n, \qquad (3.14)
\end{aligned}$$

where $k = \mu + 2\sqrt{1-2\lambda+\sigma^2}$, $t(\rho) = \sqrt{1-2\alpha\rho+\rho^2\beta^2\eta^2}$, $\theta = k + t(\rho) + \rho\xi\gamma$, and $\epsilon_n = \| J_\rho^{\partial\phi_n(.,x^*)}(h(x^*)) - J_\rho^{\partial\phi(.,x^*)}(h(x^*)) \| + \| e_n \|$. By the condition (3.6) in Theorem 3.2, we have $\theta < 1$. It follows from (3.14) that

$$\| x_{n+1} - x^* \| \leq \theta^{n+1} \| x_0 - x^* \| + \sum_{i=1}^{n} \theta^i \epsilon_{n+1-i}.$$

Since $\epsilon_n \to 0$ by the assumption, it follows from Ortega and Rheinboldt [11,p.338] that

$$lim_{n\to\infty} \| x_{n+1} - x^* \| = 0,$$

and hence the sequence $\{x_n\}$ strongly converges to x^*. Since $u_n \in T(x_n)$, $v_n \in A(x_n)$, $u^* \in T(x^*)$, $v^* \in A(x^*)$, we have

$$\| u_n - u^* \| \leq \delta(T(u_n), T(u^*)) \leq \eta \| x_n - x^* \|$$

$$\| v_n - v^* \| \leq \delta(A(v_n), A(v^*)) \leq \gamma \| x_n - x^* \| .$$

It follows that the sequence $\{u_n\}$ and $\{v_n\}$ also strongly converges to u^* and v^*, respectively. This completes the proof.

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