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LINEAR TRANSVERSE VIBRATIONS OF A SIMPLY SUPPORTED BEAM CARRYING CONCENTRATED MASSES

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Abstract- Linear transverse vibrations of an Euler-Bernoulli beam are considered. The beam carries masses and is simply supported at both ends. The equations of motion are obtained and solved. Linear frequency equations are obtained. Natural frequencies are calculated for different number of masses, mass ratios, and mass locations.

Key Words- Beam – mass systems, linear vibration, natural frequency

1. INTRODUCTION

Beam – mass systems are frequently used as design models in engineering. Approximate and exact analysis have been carried out for calculating the natural frequencies of a beam-mass system under various end conditions [1-8]. The relevant work was reviewed up to 1979 by Nayfeh and Mook [9]. For more recent work on the topic, see [10-12].

Finally, for linear vibrations of beam – mass systems, detailed calculations for fundamental frequencies can be seen in references [13-14].

The analysis presented here is closely related to references [10] and [12]. In reference [10], five different end conditions were treated. Linear and non-linear frequencies were investigated for a single mass. In reference [12], linear and non-linear frequencies given in ref. [10] were calculated using artificial neural networks. In this work, exact natural frequencies were calculated for a simply supported beam carrying many concentrated masses. Natural frequencies were calculated for different number of masses, mass ratios and mass locations.

2. EQUATIONS OF MOTION

Consider the beam-mass system shown in Figure 1. The beam carries n number of concentrated masses.

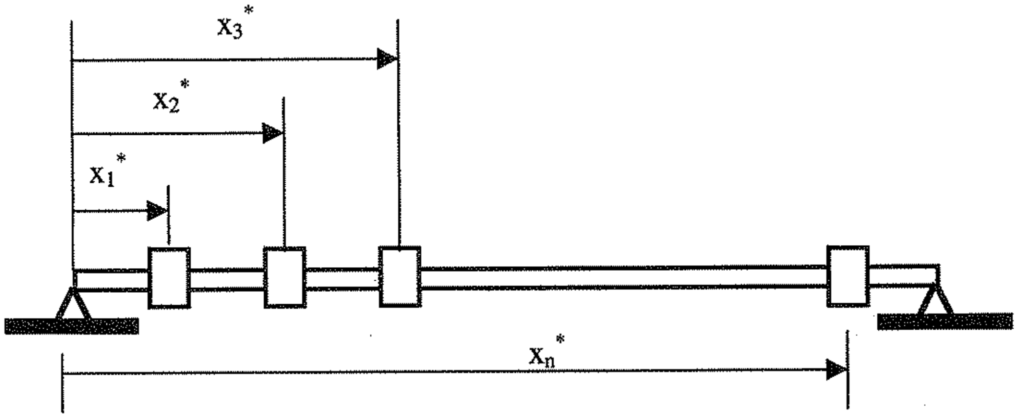

Figure 1: Beam-mass system with both ends simply supported

The Lagrangian of the system can be written as

$$\leq = \frac{1}{2}\rho A \sum_{m=0}^{n} \int_{x_m}^{x_{m+1}} \dot{w}_{m+1}^{*\,2} dx^* + \frac{1}{2}\sum_{m=1}^{n} M_m \dot{w}_m^{*\,2} - \frac{1}{2} EI \sum_{m=0}^{n} \int_{x_m}^{x_{m+1}} w_{m+1}^{*\,\prime\prime 2} dx^*, \quad x_0=0, \quad x_{m+1}=L \qquad (1)$$

Concentrated masses are located at x_m^*. The length of the beam is L. ρ is the constant density of the beam, A is the constant cross-sectional area, EI is the flexural rigidity of the beam and n is the total number of concentrated masses on the beam. w_{m+1}^* is the transverse displacement (corresponding m+1 th segment of beam), ($\dot{}$) and ($'$) denote differentiations with respect to the time variable t^* and the spatial variable x^* respectively. The first two terms in equation (1) are the kinetic energies due to transverse motion of the beam and n masses, and the last term is the elastic potential energy due to bending of the beam segments. Invoking Hamilton's principle,

$$\delta \int_{t_1^*}^{t_2^*} \leq dt^* = 0 \qquad (2)$$

and performing the necessary algebra and eliminating the axial displacements, one finally obtains the following n+1 set of linear differential equations of motion

$$\rho A \ddot{w}_{m+1}^* + EI w_{m+1}^{*\,iv} = 0 \qquad m=0,1,2...n \qquad (3)$$

The boundary conditions are as follows

$$w_1^*(0,t^*) = w_1^{*\,\prime\prime}(0,t^*) = w_{n+1}^*(1,t^*) = w_{n+1}^{*\,\prime\prime}(1,t^*) = 0 \qquad (4)$$

$$w_p^*(x_p^*,t^*) = w_{p+1}^*(x_p^*,t^*)$$

$$w_p^{*\,\prime}(x_p^*,t^*) = w_{p+1}^{*\,\prime}(x_p^*,t^*) \qquad (5)$$

$$w_p^{*\,\prime\prime}(x_p^*,t^*) = w_{p+1}^{*\,\prime\prime}(x_p^*,t^*)$$

$$EI w_p^{*\,\prime\prime\prime}(x_p^*,t^*) - EI w_{p+1}^{*\,\prime\prime\prime}(x_p^*,t^*) - M_p \ddot{w}_p^*(x_p^*,t^*) = 0 \qquad p=1,2,\ldots n \qquad (6)$$

Conditions given in equation (4) are due to the simply supported boundaries, those of equation (5) are due to the equality of displacements, slopes and moments of the beam segments respectively at the left and right side of any concentrated mass. Equation (6) states that the force difference at the left and right side of any concentrated mass is equal to the inertia force of mass. Equations (3-6) are made dimensionless through the definitions

$$x = \frac{x^*}{L}, \quad w_{1,2} = \frac{w_{1,2}^*}{R}, \quad \eta_p = \frac{x_p}{L}, \quad t = \frac{t^*}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad \alpha_p = \frac{M_p}{\rho A L} \qquad (7)$$

where R is the radius of gyration of the cross section with respect to the neutral axis. In terms of the new non-dimensional parameters, the equations of motion and boundary conditions become

$$\ddot{w}_{m+1} + w^{iv}_{m+1} = 0 \qquad m=0,1,2\ldots n \tag{8}$$

$$w_1(0,t) = w_1''(0,t) = w_{n+1}(1,t) = w_{n+1}''(1,t) = 0 \tag{9}$$

$$w_p(\eta_p, t) = w_{p+1}(\eta_p, t), \quad w_p'(\eta_p, t) = w_{p+1}'(\eta_p, t), \quad w_p''(\eta_p, t) = w_{p+1}''(\eta_p, t) \tag{10}$$

$$w_p'''(\eta_p, t) - w_{p+1}'''(\eta_p, t) - \alpha_p \ddot{w}_p(\eta_p, t) = 0 \qquad p=1,2,3,\ldots.n \tag{11}$$

In the next section the equations for linear frequencies will be obtained by solving equations (8-11).

3. ANALYTICAL SOLUTION

In this section the set of equations of motion (8) are solved for boundary conditions given in equations (9)-(11). Assume a solution of the form

$$w_{m+1} = (A \sin \omega t + B \cos \omega t) Y_{m+1}(x) \tag{12}$$

where cc represents the complex conjugate of the preceding terms. Substituting equation (12) into equations (8-11), one has

$$Y^{iv}_{m+1} - \omega^2 Y_{m+1} = 0 \tag{13}$$
$$Y_1(0) = Y_1''(0) = 0, \quad Y_{n+1}(1) = Y_{n+1}''(1) = 0 \tag{14}$$
$$Y_p(\eta_p) = Y_{p+1}(\eta_p), \quad Y_p'(\eta_p) = Y_{p+1}'(\eta_p), \quad Y_p''(\eta_p) = Y_{p+1}''(\eta_p) \tag{15}$$
$$Y_p^{iii}(\eta_p) - Y_{p+1}^{iii}(\eta_p) - \alpha_p \omega^2 Y_p(\eta_p) = 0 \tag{16}$$

A solution can be suggested for equation (13) as follows:

$$Y_{m+1} = C^1{}_{m+1} \sin \beta x + C^2{}_{m+1} \cos \beta x + C^3{}_{m+1} \sinh \beta x + C^4{}_{m+1} \cosh \beta x \tag{17}$$

where

$$\beta = \sqrt{\omega} \tag{18}$$

Substituting equations (17) and (18) into equation (13)-(16) yields natural frequency equation for arbitrary number of masses. For two concentrated masses the transcendental equations giving natural frequencies are

$$
\begin{aligned}
&-2\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta]\mathrm{Cosh}[\beta]+\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta-\eta_1\beta]\mathrm{Cosh}[\beta]\\
&+\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta-2\eta_2\beta]\mathrm{Cosh}[\beta]\\
&+2\alpha_1\alpha_2\beta^2\mathrm{Cos}[(1-\eta_1-\eta_2)\beta]\mathrm{Cosh}[(1-\eta_1-\eta_2)\beta]\\
&-\alpha_1\alpha_2\beta^2\mathrm{Cos}[(1+\eta_1-\eta_2)\beta]\mathrm{Cosh}[(1-\eta_1-\eta_2)\beta]\\
&-2\alpha_1\alpha_2\beta\text{^}2\mathrm{Cos}[(1-\eta_1-\eta_2)\beta]\mathrm{Cosh}[(1+\eta_1-\eta_2)\beta]\\
&+2\alpha_1\alpha_2\beta^2\mathrm{Cos}[(1+\eta_1-\eta_2)\beta]\mathrm{Cosh}[(1+\eta_1-\eta_2)\beta]\\
&+\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta]\mathrm{Cosh}[\beta-2\eta_1\beta]\\
&-\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta-2\eta_2\beta]\mathrm{Cosh}[\beta-2\eta_1\beta]+\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta]\mathrm{Cosh}[\beta-2\eta_2\beta]\\
&-\alpha_1\alpha_2\beta^2\mathrm{Cos}[\beta-2\eta_1\beta]\mathrm{Cosh}[\beta-2\eta_2\beta]-4\alpha_1\beta\mathrm{Cosh}[\beta]\mathrm{Sin}[\beta]\\
&-4\alpha_2\beta\mathrm{Cosh}[\beta]\mathrm{Sin}[\beta]+4\alpha_1\beta\mathrm{Cosh}[\beta-2\eta_1\beta]\mathrm{Sin}[\beta]\\
&+4\alpha_2\beta\mathrm{Cosh}[\beta-2\eta_2\beta]\mathrm{Sin}[\beta]-4\alpha_1\beta\mathrm{Cos}[\beta]\mathrm{Sinh}[\beta]-4\alpha_2\beta\mathrm{Cos}[\beta]\mathrm{Sinh}[\beta]\\
&+4\alpha_1\beta\mathrm{Cos}[\beta-2\eta_1\beta]\mathrm{Sinh}[\beta]+4\alpha_2\beta\mathrm{Cos}[\beta-2\eta_2\beta]\mathrm{Sinh}[\beta]\\
&-16\mathrm{Sin}[\beta]\mathrm{Sinh}[\beta]-\alpha_1\alpha_2\beta^2\mathrm{Sin}[(1+2\eta_1-2\eta_2)\beta]\mathrm{Sinh}[\beta]\\
&-\alpha_1\alpha_2\beta^2\mathrm{Sin}[\beta-2\eta_1\beta]\mathrm{Sinh}[\beta]+\alpha_1\alpha_2\beta^2\mathrm{Sin}[\beta-2\eta_2\beta]\mathrm{Sinh}[\beta]\\
&+\alpha_1\alpha_2\beta^2\mathrm{Sin}[\beta]\mathrm{Sinh}[(1+2\eta_1-2\eta_2)\beta]+\alpha_1\alpha_2\beta^2\mathrm{Sin}[\beta]\mathrm{Sinh}[\beta-2\eta_1\beta]\\
&-\alpha_1\alpha_2\beta^2\mathrm{Sin}[\beta]\mathrm{Sinh}[\beta-2\eta_2\beta]=0
\end{aligned}
\tag{19}
$$

For different number of concentrated masses, mass rations (α_p) and mass location (η_p) values, the first five natural frequencies are calculated. For two and three concentrated masses the natural frequencies are given in Table 1 and Table 2.

Table 1: For the two-mass problem, natural frequencies corresponding to different mass ratios and mass positions

α_1	α_2	η_1	η_2	ω_1	ω_2	ω_3	ω_4	ω_5
1	1	0.1	0.3	6.118	26.506	55.412	99.097	196.790
			0.7	6.183	22.598	60.226	125.021	174.858
		0.5	0.3	4.785	19.802	45.252	95.238	158.080
			0.7	4.730	25.128	60.883	141.289	183.110
	10	0.1	0.3	2.509	26.075	51.069	94.505	194.767
			0.7	2.516	20.060	58.824	124.285	168.185
		0.5	0.3	2.404	13.367	44.785	94.752	158.080
			0.7	2.387	17.925	59.569	136.993	180.905
10	1	0.1	0.3	4.514	18.563	38.578	96.694	195.720
			0.7	4.671	12.429	50.992	121.432	171.647
		0.5	0.3	2.086	15.959	43.170	91.623	158.043
			0.7	2.078	22.036	54.647	140.866	179.431
	10	0.1	0.3	2.357	16.257	29.975	92.863	193.920
			0.7	2.413	8.850	48.934	121.018	164.747
		0.5	0.3	1.771	6.573	42.942	94.643	158.043
			0.7	1.677	9.812	53.516	136.535	177.620

Table 2: For the three – mass problem, natural frequencies corresponding to different mass ratios and mass positions

α_1	α_2	α_3	η_1	η_2	η_3	ω_1	ω_2	ω_3	ω_4	ω_5
1	1	1	0.1	0.4	0.8	5.130	18.915	40.668	101.949	193.298
1	1	10	0.1	0.4	0.8	3.011	11.731	39.445	98.713	193.010
1	10	1	0.1	0.4	0.8	2.182	17.186	37.356	99.323	189.777
10	1	1	0.1	0.4	0.8	4.142	13.021	25.958	99.439	186.121
10	10	10	0.1	0.4	0.8	1.864	6.675	14.161	93.774	181.624
1	1	1	0.2	0.5	0.7	4.411	18.201	39.189	137.980	174.375
1	1	10	0.2	0.5	0.7	2.350	13.469	35.001	134.770	171.032
1	10	1	0.2	0.5	0.7	2.048	18.185	29.378	137.958	169.335
10	1	1	0.2	0.5	0.7	2.858	10.771	35.379	137.274	172.900
10	10	10	0.2	0.5	0.7	1.540	6.383	13.578	134.252	164.439

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