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ON THE INTEGRAL INVARIANTS OF A TIME-LIKE RULED SURFACE

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Abstract- In this study, we discuss the dual Lorentzian spherical motions and calculate the real integral invariants of a closed time-like ruled surfaces in R_1^3. Then, we define the dual angle of pitch of a closed time-like ruled surface, and give a relation between the dual Steiner vector of the dual spherical motion and dual angle of pitch of the time-like ruled surface. Finally, we obtain a relation between the dual angle of pitch and the real integral invariants of time-like ruled surface.

Keywords- Time-Like Ruled Surface, Real Pitch, Real Angle of Pitch, Dual Angle of Pitch

1. INTRODUCTION

The Geometry of ruled surface is very important in the study of kinematics or spatial mechanisms in R^3, [4],[8],[12]. Moreover, it is well known, from H.R.Müller [6] that a v_1- closed ruled surface generated by v_1-oriented line of rigid body has two real integral invariants; the real pitch, l_{v_1} and the real angle of pitch , λ_{v_1}.

In recent years, by using the real integral invariants of closed ruled surfaces , many investigate are done,[4],[5],[7].

Then, by using Lorentzian inner product, many theorems and definitions in R^3 are defined again in Minkowski space R_1^3,[1],[2],[3]. For instance, the distribution parameter of timelike ruled surface are investigeted by A.Turgut and H.H.Hacısalihoğlu.

In this paper, by using Lorentzian metric, we defined real integral invariants of a timelike ruled surface and then by using dual magnitudes, dual integral invariant , Λ_{v_1} , of a v_1-closed timelike ruled surface were introduced.

Let us consider Minkowski 3-space R_1^3 = [R^3 ,(+,+,-)] and let the Lorentzian inner product of $a=(a_1, a_2, a_3)$ and $b=(b_1, b_2, b_3) \in R^3$ be $\langle a,b\rangle = a_1b_1 + a_2b_2 - a_3b_3$. Under this condition, following definitions can be given : For a vector $x= a=(a_1, a_2, a_3) \in R_1^3$, if $\langle x,x\rangle\rangle 0$, $\langle x,x\rangle\langle 0$ and $\langle x,x\rangle = 0$ for non zero x , then x is called, the space-like ,the time-like vector and the light –like (null) vector, respectively,[9].

By considering the Lorentzian inner product, we may write inner product of A and B as follows:

$$\langle A,B\rangle = \langle a,b\rangle + \varepsilon\,(\langle a_o,b\rangle + \langle a,b_o\rangle)\ ,\ A= a+\varepsilon a_o\quad ,\quad B= b+\varepsilon b_o \tag{1}$$

We call it dual Lorentzian space which is defined and denote by ID_1^3,[10].

Definition 1 : Let $A= (a,a_o) = a + \varepsilon a_o \in ID_1^3$. The dual vector A is said to be space -like if the vector a is space -like , time-like if the vector a is time-like , and light-like (or null) if the vector a is light-like, [10].

Definition 2 : Let A , B $\in ID_1^3$. The Lorentzian cross product of dual vectors B and A is

$$A \wedge B = -\begin{vmatrix} E_1 & E_2 & -E_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad , \; E_i = e_i + \varepsilon \, e_{i0} \quad i = 1,2,3 \tag{2}$$

where $A=(A_1, A_2, A_3)$, $B=(B_1, B_2, B_3)$ $E_1 \wedge E_2 = E_3$, $E_2 \wedge E_3 = -E_1$, $E_3 \wedge E_1 = -E_2$, [10].

Theorem 1: There is one to one correspondence between directed space-like (resp.,time-like) lines of R_1^3 and ordered pair of vectors (a, a_0) such that [10],

$$a^2 = 1 \quad , \quad a\, a_0 = 0 \tag{3}$$

The dual Lorentzian and dual hyperbolic sphere of radius 1 in R_1^3 are defined by

$$S_1^2 = \left\{ A = a + \varepsilon\, a_0 \,\middle|\, \|A\| = (1,0); a, a_0 \in R_1^3, \textit{and the } a \textit{ is a space-like} \right\} \tag{4}$$

$$H_0^2 = \left\{ A = a + \varepsilon\, a_0 \,\middle|\, \|A\| = (1,0); a, a_0 \in R_1^3, \textit{ and the } a \textit{ is a time-like} \right\} \tag{5}$$

respectively,[10].

Theorem 1 gives us, the points on dual Lorentzian unit sphere is one-to-one correspondence with the oriented space-like lines in the Minkowski 3-space R_1^3.

Definition 3: A time-like closed ruled surface X(s) is one-to-one correspondence with a dual closed curve on the dual Lorentzian sphere,[10].

Definition 4: Let R be a matrix with dual coefficient. R is said to be dual Lorentzian orthogonal matrix if

$$R^{-1} = S\, R^T S \qquad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{6}$$

where S is signature matrix in ID_1^3, [11].

2. THE CLOSED DUAL SPHERICAL MOTIONS IN ID_1^3

The closed dual spherical motions and real integral invariants of the closed ruled surfaces are investigated in [5]. In this part, we want to investigate closed dual spherical motions in D_1^3.

A motion of a rigid body about a fixed point O uniquely defines a dual motion K/K′ of the moving dual Lorentzian sphere K with the fixed center O over the fixed Lorentzian sphere K′ of the same center.

Let $\{V_1, V_2, V_3\}$ *and* $\{E_1, E_2, E_3\}$ be two right-handed sets of orthonormal unit vectors that are rigidly linked to the spheres K and K′ and denoted by

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad , \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{7}$$

respectively. We may write following relation between these dual orthonormal systems by

$$V = R\, E \tag{8}$$

where

$$R=\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{9}$$

is a dual orthogonal matrix and the elements R_{ij} of the matrix R will be regarded as functions of a real single parameter t. In this study, we assume that, V_1, V_2, E_1 and E_2 are space-like dual vectors, V_3 and E_3 are time like dual vectors.

Since the matrix R is a dual Lorentzian orthogonal matrix, by definition 4, we may write

$$S R^T S\, R = I \tag{10}$$

where I is the unix matrix and S is signature matrix.
The equation (10), by differentiation with respect to t, yields

$$(S\ dR^T S R) = -(S\ R^T S\, dR) \tag{11}$$

The relation (10) shows that the matrix

$$\Omega = (S\ dR^T S R) \tag{12}$$

is not skew-symetric. Then, we may write

$$\Omega = \begin{bmatrix} 0 & \Omega_1^2 & \Omega_1^3 \\ -\Omega_1^2 & 0 & \Omega_2^3 \\ \Omega_1^3 & \Omega_2^3 & 0 \end{bmatrix} \tag{13}$$

Differential of the equation (8) with respect to t, yields

$$\begin{aligned} dV_1 &= \ \Omega_1^2 V_2 + V_3 \Omega_1^3 \\ dV_2 &= -\Omega_1^2 V_1 + V_3 \Omega_2^3 \\ dV_3 &= \ \Omega_1^3 V_1 + V_2 \Omega_2^3 \end{aligned} \tag{14}$$

Now, we define a dual vector Ψ with following form:

$$\Psi = \Omega_2^3 V_1 - \Omega_1^3 V_2 - \Omega_1^2 V_3 \tag{15}$$

Ω_1^2, Ω_1^3, Ω_2^3 are nonzero elements of the matrix Ω and Ψ is called the dual instantaneous Pfaffian vector of the dual motion K/K$'$. The Pfaffian vector Ψ at a given instant t of a one –parameter motion on a Lorentzian sphere is an analogue to the Darboux vector in the Differential Geometry of space curves. Hence, (14) is written as follows:

$$dV_i = \Psi \wedge V_i \qquad i=1,2,3 \tag{16}$$

Definition 5: Let Ψ be a dual pfaff vector during the dual motion K/K$'$. The dual vector **D** which is defined as

$$D = \oint \Psi = \oint (\Omega_2^3 V_1 - \Omega_1^3 V_2 - \Omega_1^2 V_3) = d + \varepsilon\ d_0 \tag{17}$$

is called dual Steiner vector of dual motion.

By seperating real and dual parts of (17), we have

$$d = \oint \psi \tag{18}$$

$$d_0 = \oint \psi_0 \tag{19}$$

3. THE REAL INTEGRAL INVARIANTS OF THE CLOSED TIME-LIKE RULED SURFACE

Let a moving a space-like lines space H be represented by the moving frame $\{O;v_1,v_2,v_3\}$ and H' be represented by the fixed frame $\{O';e_1,e_2,e_3\}$. We know that, any space-like line in H is drawing a closed time-like ruled surface in H' along the motion. Thus, the equation of close time-like ruled surface is

$$x(t,v) = r(t) + v\, v_1(t)\ , \quad x(t+2\pi,v) = x(t,v) \quad \|v_1\| = 1 \tag{20}$$

During the motion, we assume that v_1 and v_2 are space-like and v_3 is a time-like vector. This time-like closed ruled surface is generated by the axes- v_1. Where, v_2 is a normal vector of time-like ruled surface. By taking differential from (20), we may write differential equation of the ortogonal trajectory of v_1-closed time-like ruled surface by

$$\langle dx, v_1\rangle = 0 \quad , \quad \|v_1\| = 1 \tag{21}$$

and also from (21) , we have

$$dv = -\langle dr, v_1\rangle . \tag{22}$$

Definition 6: The pitch (öffnungsstrecke) of v_1(t) – closed time-like ruled surface is defined by

$$l_{v_1} = \oint dv = -\oint\langle dr, v_1\rangle . \tag{23}$$

This definition means that, an orthogonal trajectory of v_1(t) – closed time-like ruled surface intersects after one periodic of the axis v_1 at the P_1 different from P_0. Thus , $l_{v_1} = \overline{P_0P_1}$

Now, in order to rewrite (23) in terms of the elements of the dual Steiner vector, we use the following expression for the diffential velocity of the fixed point $r(t_0)$ of the moving space H with respect to the fixed space H':

$$dr = \psi_0 + \psi \wedge r \tag{24}$$

ψ_0 is the moment vector with respect to a fixed point. In (24), since ψ and ψ_0 are respectively the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of the motion H/H', the instantaneous dual Pfaffian vector $\Psi = \psi + \varepsilon\psi_0$ of the corresponding dual spherical motion K/K'. Then, replacing (24) in (23), we obtain

$$l_{v_1} = -(\oint v_1\psi_0 + \oint v_1(\psi \wedge r)) \tag{25}$$

Denote the moment vector of r , with respect to origin 0 , by r_0 then

$$v_{10} = r \wedge v_1 \tag{26}$$

and the last equation reduces to

$$l_{v_1} = -(\oint v_1\psi_0 + \oint \psi\, v_{10}) \tag{27}$$

On the other hand the Plückerian normalized line coordinates v_i , v_{i0} (i=1,2,3) of the fixed line V in H are independent of the motion H/H'. They depend on the choice of V in H.

Then the last expression becomes

$$l_{v_1} = -(v_1\oint\psi_0 + v_{10}\oint\psi) \tag{28}$$

According to the equations (18) and (19) , we may write that

$$\oint \psi = d \qquad \oint \psi_0 = d_0 \tag{29}$$

and then (28) reduces to

$$l_{v_1} = -(\langle v_1, d_0\rangle + \langle v_{10}, d\rangle) \tag{30}$$

Let us consider a unit time-like vector n_2 and space like vector n_3 on (v_2, v_3) which is defined as follows:

$$\begin{aligned} n_2 &= sh\varphi \, v_2 + ch\varphi \, v_3 \\ n_3 &= ch\varphi \, v_2 + sh\varphi \, v_3 \end{aligned} \tag{31}$$

The time like unit vector n_2 generates a space-like ruled surface along the ortogonal trajectory of v_1-closed time-like ruled surface during the closed motion. Where φ is the hyperbolic angle between unit time-like vectors n_2 and v_3. Thus, the equation of the space-like ruled surface is

$$T = x + w\, n_2 \qquad w \in IR \tag{32}$$

Let H' be a fixed space and denoted by $\{n_1, n_2, n_3\}$. Thus, from (31), we may write

$$\begin{aligned} v_2 &= n_3\, ch\varphi - n_2\, sh\varphi \\ v_3 &= -n_3\, sh\varphi + n_2\, ch\varphi\,, \qquad \varphi = \varphi(t) \end{aligned} \tag{33}$$

Then , from (33), by taking differential according to the parameter t , we obtain

$$\begin{aligned} dv_2 &= dn_3\, ch\varphi - dn_2\, sh\varphi + (n_3\, sh\varphi - n_2\, ch\varphi)\, d\varphi \\ dv_3 &= -dn_3\, sh\varphi + dn_2\, ch\varphi + (-n_3\, ch\varphi + n_2\, sh\varphi)\, d\varphi \end{aligned} \tag{34}$$

where n_2 and n_3 are the adges of fixed system $\{n_1, n_2, n_3\}$. Therefore

$$dn_2 = 0 \qquad dn_3 = 0 \tag{35}$$

is obtained. Thus, from (34) and (35), we have

$$\begin{aligned} dv_2 &= (n_3\, sh\varphi - n_2\, ch\varphi)\, d\varphi \\ dv_3 &= (-n_3\, ch\varphi + n_2\, sh\varphi)\, d\varphi \end{aligned} \tag{36}$$

is obtained. Then, by using the equation (33),

$$\begin{aligned} dv_2 &= -v_3\, d\varphi \\ dv_3 &= -v_2\, d\varphi \end{aligned} \tag{37}$$

are written. So, from (37), $d\varphi$ is calculated as

$$d\varphi = \langle dv_2, v_3\rangle = -\langle dv_3, v_2\rangle \tag{38}$$

and the total change of φ during the motion H/H' is

$$\lambda_{v_1} = \oint_{(r)} d\varphi \tag{39}$$

or

$$\lambda_{v_1} = \oint_{(r)} d\varphi = \oint \langle dv_2, v_3\rangle = -\oint \langle dv_3, v_2\rangle \tag{40}$$

On the other hand, if we consider real parts of (14) and (15), we obtain that

$$\begin{bmatrix} dv_1 \\ dv_2 \\ dv_3 \end{bmatrix} = \begin{bmatrix} 0 & w_3 & w_2 \\ -w_3 & 0 & w_1 \\ w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{41}$$

and

$$\psi = w_1 v_1 - w_2 v_2 - w_3 v_3 \tag{42}$$

Then, from (41),

$$\langle dv_2, v_3 \rangle = -\langle dv_3, v_2 \rangle = -w_1 \tag{43}$$

is obtained. Thus, the angle of pitch is written as follows:

$$\lambda_{v_1} = -\oint w_1 \tag{44}$$

Moreover, from (42) , we know that , the steiner vector of the motion is

$$d = \oint (w_1 v_1 - w_2 v_2 - w_3 v_3) \tag{45}$$

Hence,

$$\langle d, v_1 \rangle = \oint w_1 \tag{46}$$

and

$$\lambda_{v_1} = -\langle d, v_1 \rangle \tag{47}$$

are obtained.

Thus , the angle of pitch of the closed time-like ruled surface is defined total change of the hyperbolic angle φ is given following relation:

$$\lambda_{v_1} : \oint d\varphi = \oint \langle d\mathrm{v}_2, \mathrm{v}_3 \rangle \tag{48}$$

The pitch and the hyperbolic angle of the pitch are integral invariants of a closed time-like ruled surface .

4.THE DUAL ANGLE OF PITCH OF THE CLOSED TIME-LIKE RULED SURFACES

Let K be a moving dual Lorentzian unit sphere and K' be a fixed dual Lorentzian dual sphere. The two dual Lorentzian spheres K and K' are represented by the orthonormal systems $\{V_1, V_2, V_3\}$ and $\{E_1, E_2, E_3\}$. By definition 3 , a differentiable dual closed curve on the dual unit Lorentzian sphere depending on a real paremeter t represent a differentiable family of space-like lines in IR_1^3. Where $\{V_1, V_2, V_3\}$ and $\{E_1, E_2, E_3\}$ are orthonormal systems.

Let $V_1 = V_1(t)$ be a closed time-like ruled surface which is the first axes of moving trihedron during the one parameter closed dual spherical motion. Let

$$N_2 = sh\Phi\, V_2 + ch\Phi\, V_3 \tag{49}$$

be a dual unit time –like vector in dual plane (V_2, V_3) where Φ is dual hyperbolic angle between time-like dual unit vectors N_2 and V_3. While the first axes V_1 is drawing the closed time-like ruled surface during the dual closed spherical motion K/K', the dual unit vector N draws a ruled surface along the ortogonal trajectors.

Definition 7: The total change of hyperbolic dual angle Φ is called the hyperbolic dual angle of pitch of the closed time ruled surface $V_1(t)$ and is given by

$$\Lambda_{V_1} = \oint d\Phi \tag{50}$$

where integral is dual line integral.

Now, let us calculate $d\Phi$ according to one forms of dual motion.

Let N: $\{N_1, N_2, N_3\}$ and V: $\{V_1, V_2, V_3\}$ be two dual orthonormal systems. Let a dual orthogonal matrix which is satisfy definition 4 be as follows:

$$R = \begin{bmatrix} 0 & 1 & 0 \\ ch\Phi & 0 & -sh\Phi \\ -sh\Phi & 0 & ch\Phi \end{bmatrix} \tag{51}$$

This matrix is a translate matrix from the orthonormal system V to N. Thus, we have

$$V = R\,N \tag{52}$$

From (52), by taking differential

$$dV = dR\,R_s V \tag{53}$$

is obtained. Then, the dual matrix dRR_t is calculated following forms:

$$dRR_s = \begin{bmatrix} 0 & 0 & 0 \\ d\Phi sh\Phi & 0 & -d\Phi\, ch\Phi \\ -d\Phi\, ch\Phi & 0 & d\Phi sh\Phi \end{bmatrix} \begin{bmatrix} 0 & ch\Phi & sh\Phi \\ 1 & 0 & 0 \\ 0 & sh\Phi & ch\Phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -d\Phi \\ 0 & -d\Phi & 0 \end{bmatrix} \tag{54}$$

Thus, the differential of the adge V_2 is obtained as

$$dV_2 = -\,d\Phi\, V_3 \tag{55}$$

Therefore, $d\Phi$ is given by

$$d\Phi = \langle dV_2, V_3 \rangle = -\langle V_2, dV_3 \rangle \tag{56}$$

and from (14) and (56)

$$d\Phi = \langle -\Omega_1^2 V_1 + \Omega_2^3 V_3, V_3 \rangle \tag{57}$$

$$= -\,\Omega_2^3 \tag{58}$$

are written

Theorem 2 : During the closed motion K/K', the dual space-like vector X_1, which is fixed in the orthonormal system $\{V_1, V_2, V_3\}$, draws a closed time-like ruled surface. The dual angle of pitch of this time-like ruled surface is

$$\Lambda_{X_1} = -\langle D, X_1 \rangle \tag{59}$$

where D is dual Steiner vector of the motion K/K'.

Proof: Let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{60}$$

be a dual right system. Where, X_1, X_2 is space-like and X_3 is time-like dual unit vectors. From (56) and definition 7, the dual angle of pitch of the time-like ruled surface $X_1 = X_1(t)$ is obtained by

$$\Lambda_{X_1} = \langle dX_2, X_3 \rangle \tag{61}$$

Let $M = [M_{ij}]$, $1 \le i, j \le 3$, be a dual orthogonal matrix. Thus, we may write following transformation between the dual trihedron X and V

$$X = M\ V \tag{62}$$

On the other hand, following diagram is also commutative in ID_1^3. Namely ;

$$\begin{array}{ccc} V & \xrightarrow{M} & X \\ \Omega \downarrow & & \downarrow \Omega_t \\ dV & \xleftarrow{SM^TS} & dX \end{array} \tag{63}$$

Where, the matrixs Ω and Ω_s shall plays an important part in the theory of dual motion as in the angular velocity matrix. Then, from (63) , we may write

$$\Omega_s = M\ \Omega S\ M^T S \tag{64}$$

If we consider

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \quad , \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{65}$$

we have following form for Ω_s :

$$\Omega_s = \begin{pmatrix} M_{11}W_1 + M_{12}W_2 - M_{13}W_3 & M_{21}W_1 + M_{22}W_2 - M_{23}W_3 & -M_{31}W_1 - M_{32}W_2 + M_{33}W_3 \\ M_{11}W_4 + M_{12}W_5 - M_{13}W_6 & M_{21}W_4 + M_{22}W_5 - M_{23}W_6 & -M_{31}W_4 - M_{32}W_5 + M_{33}W_6 \\ M_{11}W_7 + M_{12}W_8 - M_{13}W_8 & M_{21}W_7 + M_{22}W_8 - M_{23}W_9 & -M_{31}W_7 - M_{32}W_8 + M_{33}W_9 \end{pmatrix} \tag{66}$$

where

$$\begin{array}{lll} W_1 = -\Omega_1^2 M_{12} + \Omega_1^3 M_{13} & W_4 = -\Omega_1^2 M_{22} + \Omega_1^3 M_{23} & W_7 = -\Omega_1^2 M_{32} + \Omega_1^3 M_{33} \\ W_2 = \Omega_1^2 M_{11} + \Omega_2^3 M_{13} & W_5 = \Omega_1^2 M_{21} + \Omega_2^3 M_{23} & W_8 = \Omega_1^2 M_{31} + \Omega_2^3 M_{33} \\ W_3 = \Omega_1^3 M_{11} + \Omega_2^3 M_{12} & W_6 = \Omega_1^3 M_{21} + \Omega_2^3 M_{22} & W_9 = \Omega_1^3 M_{31} + \Omega_2^3 M_{32} \end{array} \tag{67}$$

Thus , from the following matrix form

$$dX = \begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix} = \begin{bmatrix} M_{11}W_1 + M_{12}W_2 - M_{13}W_3 & M_{21}W_1 + M_{22}W_2 - M_{23}W_3 & -M_{31}W_1 - M_{32}W_2 + M_{33}W_3 \\ M_{11}W_4 + M_{12}W_5 - M_{13}W_6 & M_{21}W_4 + M_{22}W_5 - M_{23}W_6 & -M_{31}W_4 - M_{32}W_5 + M_{33}W_6 \\ M_{11}W_7 + M_{12}W_8 - M_{13}W_9 & M_{21}W_7 + M_{22}W_8 - M_{23}W_9 & -M_{31}W_7 - M_{32}W_8 + M_{33}W_9 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{68}$$

$d\mathbf{X_2}$ is written as

$$\begin{aligned} dX_2 = {} & (M_{11}W_4 + M_{12}W_5 - M_{13}W_6)X_1 + (M_{21}W_4 + M_{22}W_5 - M_{23}W_6)X_2 \\ & + (-M_{31}W_4 - M_{32}W_5 + M_{33}W_6)X_3 \end{aligned} \tag{69}$$

Then,

$$\langle dX_2, X_3 \rangle = -\left\{ [M_{31}M_{22} - M_{32}M_{21}]\Omega_1^2 + [-M_{32}M_{23} + M_{33}M_{22}]\Omega_2^3 + [-M_{31}M_{23} + M_{33}M_{21}]\Omega_1^3 \right\} \tag{70}$$

or

$$\langle dX_2, X_3 \rangle = -\left\{ M_{13}\,\Omega_1^2 + M_{11}\,\Omega_2^3 - M_{12}\,\Omega_1^3 \right\} \tag{71}$$

are obtained. By considering the relation (61) , the dual angle of pitch of the space-like line X_1 is calculated as

$$\Lambda_{X_1} = \oint \langle dX_2, X_3 \rangle = \oint (-M_{13}\,\Omega_1^2 - M_{11}\,\Omega_2^3 + M_{12}\,\Omega_1^3) \tag{72}$$

On the other hand , the dual Steiner vector of the motion is

$$D = \oint (\Omega_2^3 V_1 - \Omega_1^3 V_2 - \Omega_1^2 V_3) \tag{73}$$

From (62), we may write

$$X_1 = M_{11}V_1 + M_{12}V_2 + M_{13}V_3 \tag{74}$$

Then, by taking inner product of (73) and (74), we have

$$\langle D, X_1 \rangle = \oint (M_{11}\,\Omega_2^3 - M_{12}\,\Omega_2^3 + M_{13}\,\Omega_1^2) \tag{75}$$

Thus ,

$$\Lambda_{X_1} = \oint \langle dX_2, X_3 \rangle = -M_{13}\oint \Omega_1^2 - M_{11}\oint \Omega_2^3 + M_{12}\oint \Omega_1^3 \tag{76}$$

$$= -\langle D, X_1 \rangle \tag{77}$$

can written. Finally, from (76) and (77) , we get the theorem.

Separating real and dual parts of (59), we have the following relation:

$$\Lambda_{X_1} = -\langle d, x \rangle - \varepsilon\ (\langle d_0, x \rangle + \langle d, x_0 \rangle) \tag{78}$$

Theorem 3: The dual angle of pitch, Λ_{X_1} , a dual integral invariant of the time-like ruled surface $X_1 = X_1(t)$, has the relationship

$$\Lambda_{X_1} = \lambda_{X_1} + \varepsilon\, l_{X_1} \tag{79}$$

with the real integral invariants.

Proof: If we consider (29), (47) and (78) , we have theorem 3.

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