FRICITIONLESS CONTACT PROBLEM BETWEEN AN ELASTIC LAYER BONDED TO A RIGID SUPPORT AND A RIGID STAMP

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Abstract-The plane contact problem for an elastic layer bonded to a rigid support on its top surface is considered according to the theory of elasticity. The layer is subjected to a concentrated load at its bottom surface by means of a rigid stamp. Profile of the rigid stamp is taken in the form as circular, parabolic and rectangular. It is assumed that the contact between the layer and the stamp is frictionless and that only compressive normal tractions can be transmitted through the interface. The problem is formulated in terms of a singular integral equation. Numerical results for various dimensionless quantities are presented and discussed.

1. INTRODUCTION

A great deal of research on the contact problem for an elastic layer has been carried out because of its possible application to a variety of structures of practical interest. Foundations, pavements of highway and airfield, railways, rolling mills, ball and roller bearings are some of the application areas of the contact mechanics. In most of the previous studies, the layer usually rests on a continuous foundation which may be either elastic [1-7] or rigid [8-11], and it is pressed between this foundation and a locally applied load. However, there are also some studies in which the layer rests on supports such as simple supports, elastic quarter planes and rigid supports [12-14]. In [15], the layer is supported by a Winkler foundation. Although the effect of gravity is usually neglected, some studies taking it into account may be found in [5-11].

In this study, frictionless contact problem between an infinite layer of constant thickness, which is perfectly bonded to a rigid support on its top surface, and a rigid stamp is considered according to the theory of elasticity. A concentrated load is symmetrically applied to bottom surface of the layer by means of a rigid stamp. The effect of gravity is neglected.

2. FORMULATION OF THE PROBLEM

Consider an isotropic, linearly elastic infinite layer with constant thickness, which is perfectly bonded to a rigid support on its top surface, shown in Figure 1. The layer is symmetrically subjected to a concentrated load at its bottom surface by means of a rigid stamp. Thickness in z -direction is taken to be unit. It is assumed that the contact between the layer and the stamp is frictionless and that only compressive tractions can be transmitted through the interface. Since $x=0$ is the symmetry plane, it is sufficient to consider the problem in the region $0 \leq x < \infty$ only. In solution of the problem effect of

body forces is neglected. For numerical calculations, three type of stamp profile are used such as circular, parabolic and rectangular.

Figure 1. Geometry of the problem (a) Rounded stamp (b) Rectangular stamp

In the absence of body forces, two dimensional Navier equations may be written as the following form [16,17].

$$(\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \nabla^2 u = 0 \quad (1.a)$$

$$(\lambda + \mu) \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \nabla^2 v = 0 \quad (1.b)$$

Where u and v , components of displacement vector. λ and μ represent Lame constant and shear modulus respectively.

It is assumed that components of displacements are written as Fourier sine and cosine transforms of unknown functions $\phi(\alpha, y)$ and $\psi(\alpha, y)$ in the following form.

$$u(x, y) = \frac{2}{\pi_0} \int_0^\infty \phi(\alpha, y) \sin(\alpha x) d\alpha \quad (2.a)$$

$$v(x, y) = \frac{2}{\pi_0} \int_0^\infty \psi(\alpha, y) \cos(\alpha x) d\alpha \quad (2.b)$$

Solving a set of ordinary differential equation obtained by substituting necessary derivatives of equations (2) into Navier equations, and substituting the resulting expressions into equations (2), one may obtain the solutions by the aid of Fourier transform technique [17].

$$u(x, y) = \frac{2}{\pi_0} \int_0^\infty [(A + By)e^{-\alpha y} + (C + Dy)e^{\alpha y}] \sin(\alpha x) d\alpha \quad (3)$$

$$v(x, y) = \frac{2}{\pi_0} \int_0^\infty \left\{ \left[A + \left(\frac{\kappa}{\alpha} + y \right) B \right] e^{-\alpha y} + \left[-C + \left(\frac{\kappa}{\alpha} - y \right) D \right] e^{\alpha y} \right\} \cos(\alpha x) d\alpha \quad (4)$$

Where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, ν is Poisson ratio. The use of κ provides a single formulation for the plane stress and the plane strain cases. Coefficients A , B , C and D are arbitrary constants which will be determined from boundary conditions of the problem. The stress components can be expressed in terms of constant coefficients as follows [17].

$$\sigma_x(x,y) = \frac{4\mu}{\pi} \int_0^\infty \left\{ \left[\alpha(A+By) - \frac{3-\kappa}{2} B \right] e^{-\alpha y} + \left[\alpha(C+Dy) + \frac{3-\kappa}{2} D \right] e^{\alpha y} \right\} \cos(\alpha x) d\alpha \quad (5)$$

$$\sigma_y(x,y) = \frac{4\mu}{\pi} \int_0^\infty \left\{ -\left[\alpha(A+By) + \frac{\kappa+1}{2} B \right] e^{-\alpha y} + \left[-\alpha(C+Dy) + \frac{\kappa+1}{2} D \right] e^{\alpha y} \right\} \cos(\alpha x) d\alpha \quad (6)$$

$$\tau_{xy}(x,y) = \frac{4\mu}{\pi} \int_0^\infty \left\{ -\left[\alpha(A+By) + \frac{\kappa-1}{2} B \right] e^{-\alpha y} + \left[\alpha(C+Dy) - \frac{\kappa-1}{2} D \right] e^{\alpha y} \right\} \sin(\alpha x) d\alpha \quad (7)$$

Boundary conditions of the problem may be written as;

$$u(x,h) = 0 \quad 0 \leq x < \infty \quad (8.a)$$

$$v(x,h) = 0 \quad 0 \leq x < \infty \quad (8.b)$$

$$\tau_{xy}(x,0) = 0 \quad 0 \leq x < \infty \quad (8.c)$$

$$\sigma_y(x,0) = \begin{cases} -p(x) & 0 \leq x < c \\ 0 & c < x < \infty \end{cases} \quad (8.d)$$

$$\frac{\partial}{\partial x} [v(x,0)] = \frac{d}{dx} F(x) = f(x) \quad 0 \leq x < c \quad (8.e)$$

where $p(x)$, c and $F(x)$ represent unknown contact pressure, half-width of the contact area and a function which defines profile of the rigid stamp respectively.

Substituting equations (3) to (7) into conditions (8.a-d), constant coefficients A , B , C and D may be expressed in terms of unknown function $p(x)$ as follows.

$$A = \frac{P(\alpha)}{4\alpha\Delta} \{ [(1-\kappa)(\kappa+2\alpha h) - 4\alpha^2 h^2] e^{-2\alpha h} - (1-\kappa)\kappa \} \quad (9.a)$$

$$B = -\frac{P(\alpha)}{2\Delta} \{ (1-2\alpha h) e^{-2\alpha h} + \kappa \} \quad (9.b)$$

$$C = \frac{P(\alpha)}{4\alpha\Delta} \{ [(1-\kappa)(\kappa-2\alpha h) - 4\alpha^2 h^2] e^{-2\alpha h} - (1-\kappa)\kappa e^{-4\alpha h} \} \quad (9.c)$$

$$D = \frac{P(\alpha)}{2\Delta} \{ (1+2\alpha h) e^{-2\alpha h} + \kappa e^{-4\alpha h} \} \quad (9.d)$$

Where;

$$P(\alpha) = -\frac{1}{\mu_0} \int_0^c p(t) \cos(\alpha t) dt \quad (10.a)$$

$$\Delta = (1 + \kappa^2 + 4\alpha^2 h^2 + \kappa e^{-2\alpha h}) e^{-2\alpha h} + \kappa \quad (10.b)$$

Hence, displacements and stresses are expressed in terms of $p(x)$ unknown contact pressure.

3. THE INTEGRAL EQUATION

In order to obtain $p(x)$ contact pressure, it will be utilised from boundary condition (8.e). If expressions given in (9) are substituted into equation (4) and if the resulting expression is substituted into condition (8.e), after some routine manipulations the following singular integral equation may be obtained.

$$\int_0^1 \left[\frac{1}{s-w} - \frac{1}{s+w} + \frac{c}{h} K(w,s) \right] g(s) ds = m(w) \quad (11.a)$$

Where;

$$K(w,s) = \int_0^\infty \frac{1}{T} (1 + \kappa^2 + 4z(1+z) + 2\kappa e^{-2z}) e^{-2z} \left[\sin \frac{c}{h} (s+w)z - \sin \frac{c}{h} (s-w)z \right] dz \quad (11.b)$$

$$T = (1 + \kappa^2 + 4z^2 + \kappa e^{-2z}) e^{-2z} + \kappa \quad (11.c)$$

$$m(w) = \frac{4\mu\pi}{1+\kappa} f(cw) \quad (11.d)$$

The kernel $K(w, s)$ is bounded in the closed interval $0 \leq s \leq 1$. The equilibrium condition of the stamp may be written as follows.

$$2 \frac{c}{h_0} \int_0^1 g(s) ds = 1 \quad (12)$$

In above expressions, we introduced the following dimensionless quantities to simplify the numerical solution of the integral equation.

$$x = cw, \quad t = cs, \quad M(w) = \frac{m(cw)h}{P}, \quad g(s) = \frac{p(cs)h}{P} \quad (13)$$

Since formulation of the problem is rather general, profile function of the stamp must be known to solve. In this paper, three stamp profiles is treated such as circular, parabolic and rectangular. Derivatives of profile functions of these are given below.

$$\text{For a circular stamp: } F'(x) = f(x) = -x(R^2 - x^2)^{1/2} \quad (14.a)$$

$$\text{For a parabolic stamp: } F'(x) = f(x) = -2Cx \quad (14.b)$$

$$\text{For a rectangular stamp: } F'(x) = f(x) = 0 \quad (14.c)$$

In equations (14), R is radius and C is a constant. In solution of the integral equation, there are some differences between the rounded stamp cases (circular and parabolic) and rectangular stamp case.

3. 1. Rigid rounded stamp case

The contact between the layer and the stamp at the end points of the contact region is smooth and in addition to the contact pressure $p(x)$, the half-width of contact area c is also unknown. For a certain stamp profile, c depends on the magnitude and the distribution of the resultant compressive load. Due to requirement of the smooth contact $g(\pm 1) = 0$ and the index of the integral equation is -1. Therefore, the consistency condition of the integral equation must also be satisfied. Solution of the singular integral equation may be written in the form [17,18,19].

$$g(s) = G(s) \sqrt{1-s^2} \quad (15)$$

Where $G(s)$ is bounded in $0 \leq s \leq 1$. If appropriate Gauss-Chebyshev integration formula is used, equations (11-12) may be reduced to the following forms.

$$\sum_{i=1}^n \frac{1-s_i^2}{2n+1} \left[\frac{1}{s_i - w_j} - \frac{1}{s_i + w_j} + \frac{c}{h} K(w_j, s_i) \right] G(s_i) = M(w_j) \quad (j=1, n+1)$$

$$2\pi \frac{c}{h} \sum_{i=1}^n \frac{1-s_i^2}{2n+1} G(s_i) = 1$$
(16.a)

Where,

$$s_i = \cos\left(\frac{i\pi}{2n+1}\right), \quad w_j = \cos\left(\pi \frac{2j-1}{4n+2}\right) \quad (i=1, n, \quad j=1, n+1)$$
(16.b)

It can be shown that the extra equation in (16.a) corresponds to the consistency condition of the original integral equation in (11.a). In this case, the $(n+1)^{\text{th}}$ equation in (16.a) automatically satisfied [17,18,19]. Thus, the equations given by (16.a) constitute a system of $n+1$ equations for $n+1$ unknowns. Solving this system of equations and using equation (15), $g(s)$ normalised contact pressure distribution and c half-width of contact area are obtained. Note that the system is highly non-linear in c and an interpolation scheme is required to determine this unknown. For different rounded stamp profiles, solution method of integral equation is same as explained above. Only difference is that right hand-side of first expression in (11.a) changes by depending on profile function.

3. 2. Rigid rectangular stamp case

In this case, the contact pressure $p(x)$ has integral singularities at the corners, so the index of the integral equation is $+1$. Writing the solution as follows [17,18,19];

$$g(s) = G(s) / \sqrt{1-s^2}$$
(17)

where $G(s)$ is bounded in $(0 \leq s \leq 1)$, then using the appropriate Gauss-Chebyshev integration formula, equations (11-12) may be reduced to,

$$\sum_{i=1}^n Q_i \left[\frac{1}{s_i - w_j} - \frac{1}{s_i + w_j} + \frac{c}{h} K(w_j, s_i) \right] G(s_i) = 0 \quad (j=1, n-1)$$

$$2\pi \frac{c}{h} \sum_{i=1}^n Q_i G(s_i) = 1$$
(18.a)

Where,

$$Q_i = \frac{\pi}{4n-2} \quad Q_i = \frac{\pi}{2n-1} \quad s_i = \cos\left(\pi \frac{i-1}{2n-1}\right) \quad w_j = \cos\left(\pi \frac{2j-1}{4n-2}\right) \quad (i=1, n \quad j=1, n-1)$$
(18.b)

The system of equations (18.a) contain n linear algebraic equations for n unknowns. Using (17), $g(s)$ normalised contact pressure distribution is obtained.

4. NUMERICAL RESULTS AND DISCUSSION

As a result of numerical solution of the integral equation, it is obtained only contact pressure distribution for rigid rectangular stamp case and obtained both contact pressure distribution and contact area for rigid rounded stamp cases. Then, normal and shear stress distributions in the layer are investigated for the circular stamp case only. Note that all quantities are dimensionless.

Figure 2 shows variation of size of the contact area with load factor and radius of stamp. As expected, c/h increases with increasing of R/h . Since increasing of load factor

corresponds to decreasing of applied concentrated load P , c/h also decreases with decreasing of applied load P .

In Figure 3, it is shown relation between c/h and R/h by depending on material constant κ for a constant value of load factor. c/h increases with increasing of κ . Decreasing of κ corresponds to increasing of Poisson ratio ν , that is; material closes to incompressible case.

In Figure 4, some results obtained for a parabolic stamp case are given. For a parabolic stamp, C is a constant and decreasing of C means that radius of curvature of stamp profile increases and so, much larger contact area is obtained. Comparing size of the contact area obtained from circular stamp case with ones obtained from parabolic stamp case, results obtained from parabolic profile for $Ch=0.05$ and 0.005 are very similar to results obtained from circular profile $R/h=10$ and 100 respectively as in [13].

Figures 5 and 6 show $p(x)h/P$ the dimensionless contact pressure distributions. The contact pressure is maximum at $x = 0$. In the rounded stamp cases, contact pressure distribution depends on applied load as well as the contact area. In Figure 5, for $R/h = 10$ and $\kappa = 2$, relation among $p(x)h/P$, c/h and $\mu h/P$ is shown. As load factor decreases, that is P increases, dimensionless contact pressure decreases because of the fact that this ratio depends on P . Actually, contact stresses also increases as well as contact area when applied load increases. In Figure 6, $\mu h/P=100$ and $\kappa=2$ are fixed and dimensionless contact pressure distribution for $R/h = 10, 100$ and 1000 is given. As R/h increases, size of the contact area increases, so the applied load distributes a much larger area. Therefore, contact pressure decreases.

Figure 2. Variation of size of the contact area for circular stamp by depending on radius and load factor ($\kappa=2$).

Figure 3. Variation of size of the contact area for circular stamp by depending on κ ($\mu h/P=100$)

In Figure 7, contact pressure distribution for the rectangular stamp case is given. In this case contact pressure only depends on the contact area in opposition to the rounded stamp cases. As seen in figure, when c/h increases, dimensionless contact pressure $p(x)h/P$ decreases. At corners of the stamp contact pressure have singularities. It is

observed that contact pressure distribution under the stamp distributes uniformly with increasing c/h except corners of the stamp.

Figure 4. Variation of size of the contact area for parabolic stamp by depending on curvature and load factor ($\kappa=2$)

Figure 5. Contact pressure distribution for circular stamp case ($R/h = 10$, $\kappa=2$)

Figure 6. Contact pressure distribution depending on radius R for circular stamp ($\kappa=2$)

Figure 7. Contact pressure distribution for rectangular stamp case ($\kappa=2$)

Figure 8 shows the dimensionless normal stress $\sigma_x h/P$ along symmetry plane $x=0$ for various values of c/h and $\kappa=2$. The dimensionless normal stress is always compressive in the outer portions of the layer close to surfaces $y=h$ and $y=0$, and as c/h increases, it decreases. In the interior portion of the layer σ_x stresses have different

character. Namely, for small contact areas normal stress is tensile in the interior portion. These tensile stresses decreases with increasing of c/h . As c/h continues increasing, σ_x normal stress becomes compressive along whole cross-section, and these compressive stresses increase with increasing of c/h . Due to shear stresses at rigid supported surface, as shown in Figure 9, the layer behaves like a beam subjected to an effect of axial force at opposite direction to x -axis and bending about z -axis. Also, the layer is compressed in the direction to y -axis. Note that equilibrium condition of the layer under these effects must be satisfied. Shear stresses at supported surface are always in the opposite direction to x -axis and decrease with increasing of c/h . Shear stresses have maximum value at a point beyond symmetry plane $x=0$ and this point goes away from symmetry plane with increasing of c/h . Note that shear stress distribution diminishes to zero as it goes to infinity along x -axis. Similar behaviour is also observed for normal stresses along x -axis as expected from theory of elasticity.

In Figure 10, it is shown the dimensionless stress $\sigma_y h/P$ obtained for various values of c/h along symmetry plane $x=0$. At contact surface, σ_y stress is equal to contact stress so boundary condition is satisfied. σ_y is always compressive along symmetry cross-section. For small values of c/h , σ_y stress becomes larger at contact surface because of closing to the concentrated load case. As c/h increases, σ_y stress decreases along whole cross-section and an almost uniform stress distribution is obtained.

Figure 8. $\sigma_x h/P$ dimensionless stress distribution for various values of c/h along $x=0$ symmetry axis ($\kappa=2$)

Figure 9. $\tau_{xy}h/P$ dimensionless shear stress distribution for various values of c/h along x-axis at support ($\kappa=2$)

Figure 10. σ_yh/P dimensionless stress distribution for various values of c/h along $x=0$ symmetry axis ($\kappa=2$)

5. REFERENCES

- [1] Ratwani, M. and Erdoğan, F., On the Plane Contact Problem for a Frictionless Elastic Layer, *Int. J. of Solids and Structures*, **9**, 921-936, 1973
- [2] Civelek, M.B. and Erdoğan, F., The Axisymmetric Double Contact Problem for a Frictionless Elastic Layer, *Int. J. of Solids and Structures*, **10**, 639-659, 1974
- [3] Adams G.G., A Rigid Punch Bonded to a Half Plane, *J. of Applied Mechanics*, Trans. of ASME, **46**, 844- 848, 1979

- [4] Weitsman, Y., A Tensionless Contact Between a Beam and an Elastic Half Space, *Int. J. of Engineering Science*, **10**, 73-81, 1972
- [5] Geçit, M.R., A Tensionless Contact without Friction between an Elastic Layer and an Elastic Foundation, *Int. J. of Solids and Structures*, **16**, 387-396, 1980
- [6] Geçit, M.R., Axisymmetric Contact Problem for an Elastic Layer and an Elastic Foundation, *Int. J. of Engineering Science*, **19**, 747-755, 1981
- [7] Çakıroğlu, A.O. and Çakıroğlu, F.L., Continuous and Discontinuous Contact Problems for Strips on an Elastic Semi-Infinite Plane, *Int. J. of Engineering Science*, **29**(1), 99-111, 1991
- [8] Civelek, M.B. and Erdoğan, F., The Frictionless Contact Problem for an Elastic Layer under Gravity, *J. of Applied Mechanics*, Trans. of ASME, **42**(97), 136-140, 1975
- [9] Civelek, M.B. and Erdoğan, F., Interface Separation in a Frictionless Contact Problem for an Elastic Layer, *J. of Applied Mechanics*, Trans. of ASME, **43**(98), 175-177, 1976
- [10] Geçit, M.R. and Erdoğan, F., Frictionless Contact Problem for an Elastic Layer under Axisymmetric Loading, *Int. J. of Solids and Structures*, **14**, 771-785, 1978
- [11] Civelek, M.B., Erdoğan, F. and Çakıroğlu, A.O., Interface Separation for an Elastic Layer Loaded by a Rigid Stamp, *Int. J. of Engineering Science*, **16**, 669-679, 1978
- [12] Erdoğan, F. and Ratwani, M., The Contact Problem for an Elastic Layer Supported by Two Quarter Planes, *J. of Applied Mechanics*, Trans. of ASME, **41**(96), 673-678, 1974
- [13] Geçit, M.R. and Gökpınar, S., Frictionless Contact between an Elastic Layer and A Rigid Rounded Support, *The Arabian Journal for Science and Engineering*, **10**(3), 243-251, 1985
- [14] Geçit, M.R. and Yapıcı H., Contact Problem for an Elastic Layer Resting on Rigid Flat Supports, *The Arabian Journal for Science and Engineering*, **11**(3), 235-242, 1986
- [15] Dempsey, J.P., Zhao, Z.G. and Li, H., Plane Contact of an Elastic Layer Supported by Winkler Foundation, *J. of Applied Mechanics*, Trans. of ASME, **57**, 974-980, 1990
- [16] Little, R.W., *Elasticity*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973
- [17] Kahya, V., *Frictionless Contact Problem Between an Elastic Layer Bonded to a Rigid Support and a Rigid Stamp* (in Turkish), MSc. Thesis, KTU Natural and Applied Sciences Institute, Trabzon, 1997
- [18] Sneddon, I.N., *Fourier Transforms*, Mc Graw-Hill, New York, 1951
- [19] Erdoğan, F. and Gupta, G.D., On The Numerical Solution of Singular Integral Equations, *Quarterly Journal of Applied Mathematics*, **29**, 525-534, 1972