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TRIGONOMETRY ON ISO-TAXICAB GEOMETRY

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ABSTRACT : In this paper, we will define the iso-taxicab trigonometric functions $\cos_I \theta$, $\sin_I \theta$, $\tan_I$ and $\cot_I \theta$. Then, we will give the subtraction formulas for iso-taxicab trigonometric functions $\cos_I \theta$ and $\sin_I \theta$.

1. INTRODUCTION

The most useful, the most understandable, and the most applicable geometry is, of course, the Euclidean geometry. Why is this true? To answer this question, we definitely has to look at the non-Euclidean geometry. The first non-Euclidean geometry that we can think of is that the hyperbolic geometry. It is known that the hyperbolic geometry is not easy to understand. Both the theory and the applications of hyperbolic geometry are quite sophisticated.

In 1975, by using a different metric in R^2

$$d_T(A,B) = |x_1 - x_2| + |y_1 - y_2|$$

for $A=(x_1,y_1), B=(x_2,y_2)$, E. F. Krause has defined a new geometry, named by taxicab geometry. The taxicab geometry is a non-Euclidean geometry. It satisfies the Euclid's 13 axioms, except one, side-angle-side axiom. It has a wide range of applications in the real world, and it is easy to understand [1].

Clearly, the taxicab geometry arises when travelling by a taxi in a town which has vertical streets and horizontal avenues. And the taxicab plane has squares as a block. The points and the lines of taxicab geometry are the same as in Euclidean geometry.

In [2], K. O. Sowell has introduced the iso-taxicab geometry. As it is mentioned by Sowell, at the origin there are three axes: the x-axis, y-axis and y'-axis, having 60^0 angle with each other. These three axes separate the plane into six regions, called hextants. These hextants will be numbered $I-VI$ in a counterclockwise direction beginning with the hextant where the coordinates of the points are both positive (Figure 1.1).

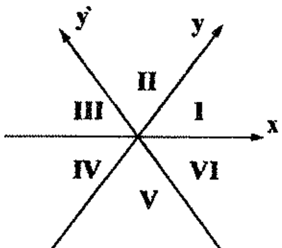

Figure 1.1

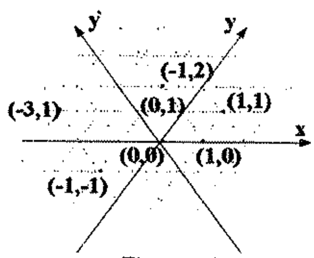

Figure 1.2

Points will still be named by ordered pairs $A=(a_1,a_2)$ of real numbers with respect to the x-axis and y-axis (Figure 1.2). At any point in the plane, three lines may be drawn parallel to the axes which separate the plane into six regions. Two points, then, have a $I-IV$ orientation, a $II-V$ orientation, or a $III-VI$ orientation to one another (Figure 1.3 - 1.4 - 1.5).

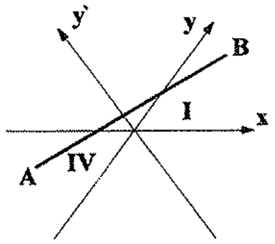

Figure 1.3

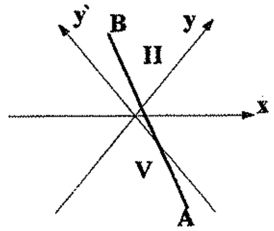

Figure 1.4

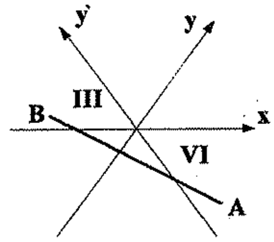

Figure 1.5

The similarities and differences of Euclidean, taxicab and iso-taxicab geometries are also discussed by Sowell [2]. For instance, because angle measure is not depending upon the distance function, angles may be measured as in Euclidean geometry. Only Euclidean plane geometry has a rotation invariant metric; therefore one should expect properties involving angle measurement to be altered in taxicab geometries. Specifically, the side-angle-side axiom may be assumed in neither iso-taxicab geometry nor in taxicab geometry. Of course, any theorem which relies on the side-angle-side axiom is also invalid in each of these taxicab geometries. Among these are the angle-side-angle theorem and the side-side-side theorem. The taxicab circle is a square and $\pi_T = 4$. The iso-taxicab circle is hexagon. Because the circumference of iso-taxicab circle is six times the radius, $\pi_I = 3$.

It is also given in [2] that in iso-taxicab geometry, three distance functions arise depending upon the orientations of the points : For $A=(x_1,y_1)$, $B=(x_2,y_2)$

$$d_I(A,B)=\begin{cases} (i) & |x_1-x_2|+|y_1-y_2| \quad , \ I-IV \text{ orientation} \\ (ii) & |y_1-y_2| \quad , \ II-V \text{ orientation} \\ (iii) & |x_1-x_2| \quad , \ III-VI \text{ orientation} \end{cases}$$

If the points lie on a line parallel to x-axis, the formula (iii) holds; parallel to y or y'-axes, the formula (ii) holds.

2. TRIGONOMETRY ON ISO-TAXICAB GEOMETRY

We first start with defining the iso-taxicab unit circle and then trigonometric function. Then, we will determine $\cos_I(\theta)$, $\sin_I(\theta)$, $\tan_I(\theta)$, and $\cot_I(\theta)$.

Definition 1. *The set of points such that the iso-taxicab distance from the origin is* 1 *defines the iso-taxicab unit circle which we can write as*

$$C_I=\begin{cases} (x,y) \mid & |x|+|y|=1 \quad , \ I-IV \text{ orientation} \\ (x,y) \mid & |y|=1 \quad , \ II-V \text{ orientation} \\ (x,y) \mid & |x|=1 \quad , \ III-VI \text{ orientation} \end{cases}$$

Definition 2. *Let $f:[0,2\pi_I)\to C_I$ be a function such that*

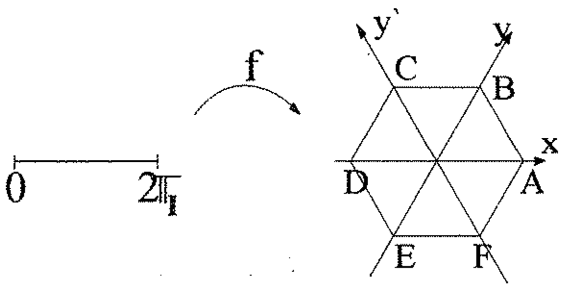

Figure 2.1

$$f(\theta) = P \Leftrightarrow \begin{cases} d_I(P,A) & = \theta \ , \ \theta \in I & = [0, \frac{\pi_T}{3}) \\ 1 + d_I(P,B) & = \theta \ , \ \theta \in II & = [\frac{\pi_T}{3}, \frac{2\pi_T}{3}) \\ 2 + d_I(P,C) & = \theta \ , \ \theta \in III & = [\frac{2\pi_T}{3}, \pi_T) \\ 3 + d_I(P,D) & = \theta \ , \ \theta \in IV & = [\pi_T, \frac{4\pi_T}{3}) \\ 4 + d_I(P,E) & = \theta \ , \ \theta \in V & = [\frac{4\pi_T}{3}, \frac{5\pi_T}{3}) \\ 5 + d_I(P,F) & = \theta \ , \ \theta \in VI & = [\frac{5\pi_T}{3}, 2\pi_T) \end{cases}$$

f is called the iso-taxicab trigonometric function.

It is easy to show that f is one-to-one and surjective function. Using the definition of C_I, we can write

$$f(\theta) = (x,y) \Leftrightarrow \begin{cases} y & = \theta \ , \ \theta \in I \\ 1 - x & = \theta \ , \ \theta \in II \\ 3 - y & = \theta \ , \ \theta \in III \\ 3 - y & = \theta \ , \ \theta \in IV \\ 4 + x & = \theta \ , \ \theta \in V \\ 6 + y & = \theta \ , \ \theta \in VI \end{cases}$$

Thus, using the iso-taxicab trigonometric function, each angle θ can be represented by a real number in the interval $[0, 6)$.

2.1 COSINE AND SINE FUNCTIONS

Definition 3. *Let θ be an angle and Q be a point as in Figure 2.2. Let $P = (x, y)$ be the intersection point of the line $\overline{OQ}$ and the unit circle C_I. Then, the x coordinate of P defines the Cosine function and the y coordinate of P defines the Sine function. Namely,*

$$\cos_I(\theta) = x, \ \sin_I(\theta) = y.$$

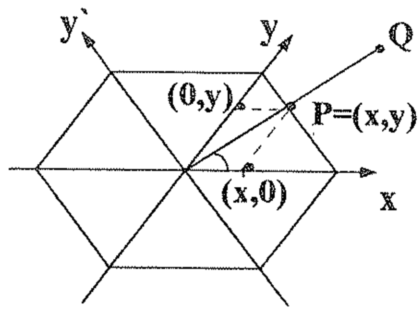

Figure 2.2

So, by using Definition 1 and Definition 2, we have the following table:

$$
\begin{array}{llllll}
\theta \in I & \Rightarrow & \cos_I(\theta) = & 1-\theta & , \ \sin_I(\theta) = & \theta \\
\theta \in II & \Rightarrow & \cos_I(\theta) = & 1-\theta & , \ \sin_I(\theta) = & 1 \\
\theta \in III & \Rightarrow & \cos_I(\theta) = & -1 & , \ \sin_I(\theta) = & 3-\theta \\
\theta \in IV & \Rightarrow & \cos_I(\theta) = & \theta-4 & , \ \sin_I(\theta) = & 3-\theta \\
\theta \in V & \Rightarrow & \cos_I(\theta) = & \theta-4 & , \ \sin_I(\theta) = & -1 \\
\theta \in VI & \Rightarrow & \cos_I(\theta) = & 1 & , \ \sin_I(\theta) = & \theta-6
\end{array}
$$

Therefore, we can draw $\cos_I(\theta)$, and $\sin_I(\theta)$ as follows:

θ	0	$\frac{\pi_I}{6}$	$\frac{\pi_I}{4}$	$\frac{\pi_I}{3}$	$\frac{\pi_I}{2}$	$\frac{2\pi_I}{3}$	π_I	$\frac{4\pi_I}{3}$	$\frac{3\pi_I}{2}$	$\frac{5\pi_I}{3}$	$2\pi_I$
$\cos_I(\theta)$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	-1	-1	0	$\frac{1}{2}$	1	1
$\sin_I(\theta)$	0	$\frac{1}{2}$	$\frac{3}{4}$	1	1	1	0	-1	-1	-1	0

With these values, the graphs are

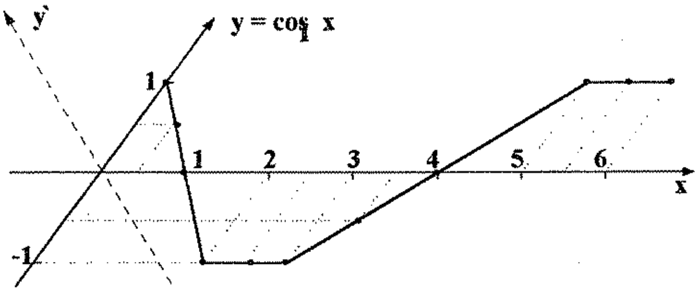

Figure 2.3 (The graph of Cosine)

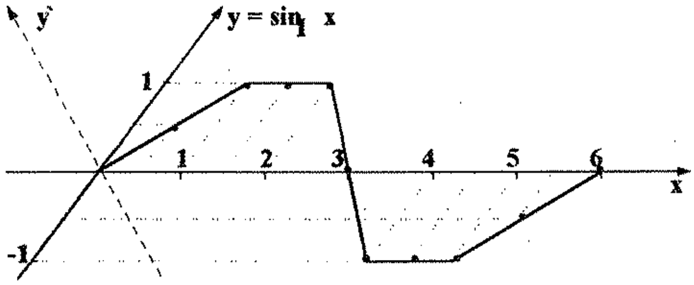

Figure 2.4 (The graph of Sine)

2.2 TANGENT AND COTANGENT FUNCTIONS

Definition 4.

Let P be a point on the iso-taxicab unit circle, but not on the y − axis. Let $T = (1, t)$ be the intersection point of the line $x = 1$ and the line $\overline{OP}$. The ordinate of the point T is called the iso − taxicab tangent of the angle θ and denoted by $\tan_I(\theta)$.

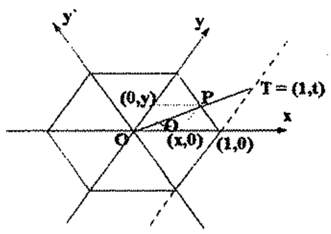

Figure 2.5

It is clear from Figure 2.5 that $\tan_I(\theta) = \dfrac{y}{x} = \dfrac{\sin_I(\theta)}{\cos_I(\theta)}$

Definition 5.

Let P be a point on the iso-taxicab unit circle, but not on the $x-$ axis. Let $C = (c, 1)$ be the intersection point of the line $y = 1$ and the line $\overline{OP}$. The apsisse of the point C is called the $iso - taxicab$ cot angent of the angle θ and denoted by $\cot_I(\theta)$.

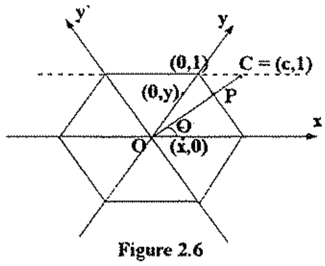

Figure 2.6

It is clear from Figure 2.6 that $\cot_I(\theta) = \dfrac{x}{y} = \dfrac{\cos_I(\theta)}{\sin_I(\theta)}$

From Definition 4 and Definition 5, we have the following table:

$$\tan_I(\theta) = \begin{cases} \frac{\theta}{1-\theta} & , \quad \theta \in I \\ \frac{1}{1-\theta} & , \quad \theta \in II \\ \theta - 3 & , \quad \theta \in III \\ \frac{3-\theta}{\theta-4} & , \quad \theta \in IV \\ \frac{1}{4-\theta} & , \quad \theta \in V \\ \theta - 6 & , \quad \theta \in VI \end{cases}$$

and $\cot_I(\theta) = 1/\tan_I(\theta)$. Thus,

θ	0	$\frac{\pi_I}{6}$	$\frac{\pi_I}{4}$	$\frac{\pi_I}{3}$	$\frac{\pi_I}{2}$	$\frac{2\pi_I}{3}$	π_I
$\tan_I(\theta)$	0	1	3	$+\infty \quad -\infty$	-2	-1	0
$\cot_I(\theta)$	$-\infty \quad +\infty$	1	$\frac{1}{3}$	0	$-\frac{1}{2}$	-1	$-\infty \quad +\infty$

θ	$\frac{4\pi_I}{3}$	$\frac{3\pi_I}{2}$	$\frac{5\pi_I}{3}$	$2\pi_I$
$\tan_I(\theta)$	$+\infty \quad -\infty$	-2	-1	0
$\cot_I(\theta)$	0	$-\frac{1}{2}$	-1	$-\infty \quad +\infty$

With these values, the graphs are

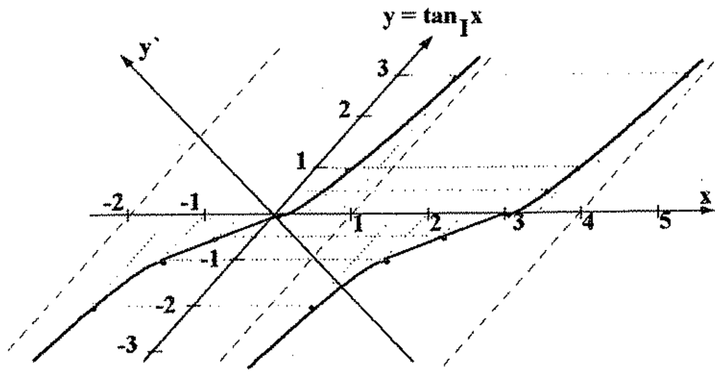

Figure 2.7 (The graph of Tangent)

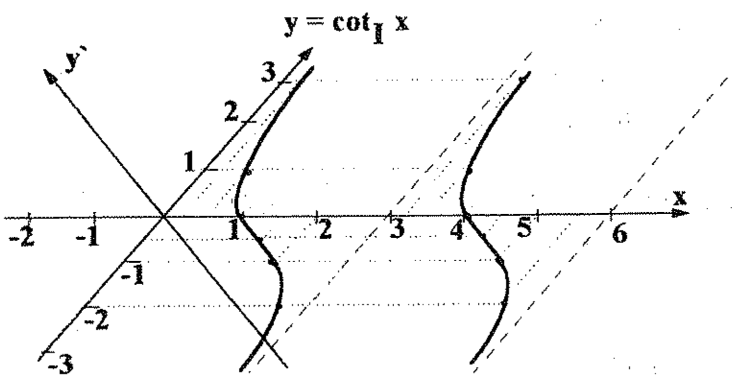

Figure 2.8 (The graph of Cotangent)

2.3 SUBSTRACTION FORMULAS

We first want to recall that we have six quadrants, $I, II, III, IV, V,$ and VI; and two angles, θ_1, θ_2. Thus, we have $6^2 = 36$ possibilities. Then, for each case, we have to apply to the Definition 1 and Definition 2 for the vectors θ_1, θ_2, and the difference vector $\theta_3 = \theta_1 - \theta_2$.

Example 6. *Suppose $\theta_1 \in I$, $\theta_2 \in I$. Then $\theta_3 = \theta_1 - \theta_2 \in I$ (Figure 2.9). Let*

$$\begin{aligned} P &= (x,y) &&= (\cos_I(\theta_1), \sin_I(\theta_1)) \\ P' &= (x',y') &&= (\cos_I(\theta_2), \sin_I(\theta_2)) \\ P'' &= (x'',y'') &&= (\cos_I(\theta_3), \sin_I(\theta_3)) \end{aligned}$$

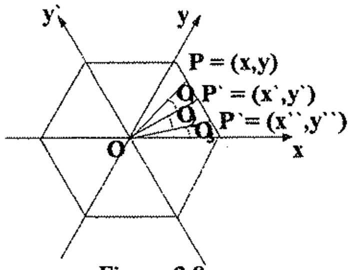

Figure 2.9

Thus, we have

$$\begin{aligned} \text{For } \theta_1 &: y = \theta_1 \ , \ x + y = 1 \\ \text{For } \theta_2 &: y' = \theta_2 \ , \ x' + y' = 1 \\ \text{For } \theta_3 &: y'' = \theta_3 \ , \ x'' + y'' = 1 \end{aligned}$$

Solving these equations for x'' and y'', we get

$$x'' = 1 + x - x' \ , \ y'' = y - y'$$

which implies that

$$\begin{aligned} \cos_I(\theta_1 - \theta_2) &= 1 + \cos_I(\theta_1) - \cos_I(\theta_2) \\ \sin_I(\theta_1 - \theta_2) &= \sin_I(\theta_1) - \sin_I(\theta_2) \ . \end{aligned}$$

We, now, give the table of formulas for 36 cases:

	θ_1	θ_2	θ_3	$\cos_I(\theta_1-\theta_2)$	$\sin_I(\theta_1-\theta_2)$
1	I	I	I	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$	$\sin_I(\theta_1)-\sin_I(\theta_2)$
2	II	II	I	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$	$-\cos_I(\theta_1)+\cos_I(\theta_2)$
3	III	III	I	$1+\sin_I(\theta_1)-\sin_I(\theta_2)$	$-\sin_I(\theta_1)+\sin_I(\theta_2)$
4	IV	IV	I	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$	$-\sin_I(\theta_1)+\sin_I(\theta_2)$
5	V	V	I	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$	$\cos_I(\theta_1)-\cos_I(\theta_2)$
6	VI	VI	I	$1-\sin_I(\theta_1)+\sin_I(\theta_2)$	$\sin_I(\theta_1)-\sin_I(\theta_2)$
7	II	I	I	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$	$-\cos_I(\theta_1)+\cos_I(\theta_2)$
8	II	I	II	$1+\cos_I(\theta_1)-\cos_I(\theta_2)$	1
9	III	I	II	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$	1
10	III	I	III	-1	$\sin_I(\theta_1)+\sin_I(\theta_2)$
11	III	II	I	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$	$2-\sin_I(\theta_1)+\cos_I(\theta_2)$
12	III	II	II	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$	1
13	IV	I	III	-1	$\sin_I(\theta_1)+\sin_I(\theta_2)$
14	IV	I	IV	$-1+\cos_I(\theta_1)+\cos_I(\theta_2)$	$\sin_I(\theta_1)+\sin_I(\theta_2)$
15	IV	II	II	$-1+\sin_I(\theta_1)-\cos_I(\theta_2)$	1
16	IV	II	III	-1	$-\cos_I(\theta_1)-\cos_I(\theta_2)$
17	IV	III	I	$1+\sin_I(\theta_1)-\sin_I(\theta_2)$	$-\sin_I(\theta_1)+\sin_I(\theta_2)$
18	IV	III	II	$1+\sin_I(\theta_1)-\sin_I(\theta_2)$	1
19	V	I	IV	$-1+\cos_I(\theta_1)+\cos_I(\theta_2)$	$-\cos_I(\theta_1)-\cos_I(\theta_2)$
20	V	I	V	$-1+\cos_I(\theta_1)+\cos_I(\theta_2)$	-1
21	V	II	III	-1	$-\cos_I(\theta_1)-\cos_I(\theta_2)$
22	V	II	IV	$-1-\cos_I(\theta_1)+\cos_I(\theta_2)$	$-\cos_I(\theta_1)-\cos_I(\theta_2)$
23	V	III	II	$-\cos_I(\theta_1)-\sin_I(\theta_2)$	1

24	V	III	III	-1	$2-\cos_I(\theta_1)-\sin_I(\theta_2)$
25	V	IV	I	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$	$\cos_I(\theta_1)-\cos_I(\theta_2)$
26	V	IV	II	$1-\cos_I(\theta_1)+\cos_I(\theta_2)$	1
27	VI	I	V	$2+\sin_I(\theta_1)-\sin_I(\theta_2)$	-1
28	VI	I	VI	1	$\sin_I(\theta_1)-\sin_I(\theta_2)$
29	VI	II	IV	$1+\sin_I(\theta_1)+\cos_I(\theta_2)$	$-2-\sin_I(\theta_1)-\cos_I(\theta_2)$
30	VI	II	V	$1+\sin_I(\theta_1)+\cos_I(\theta_2)$	-1
31	VI	III	III	-1	$-\sin_I(\theta_1)-\sin_I(\theta_2)$
32	VI	III	IV	$-1+\sin_I(\theta_1)+\sin_I(\theta_2)$	$-\sin_I(\theta_1)-\sin_I(\theta_2)$
33	VI	IV	II	$-2-\sin_I(\theta_1)-\sin_I(\theta_2)$	1
34	VI	IV	III	-1	$-\sin_I(\theta_1)-\sin_I(\theta_2)$
35	VI	V	I	$-1-\sin_I(\theta_1)+\cos_I(\theta_2)$	$2+\sin_I(\theta_1)-\cos_I(\theta_2)$
36	VI	V	II	$-1-\sin_I(\theta_1)+\cos_I(\theta_2)$	1

A similar computation will give us the formulas of addition which we will leave as a "nice" exercise to reader. We note that there are 42 cases.

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2. K. O.SOWELL, *Taxicab Geometry-A New Slant*, Mathematics Magazine, **62**, *238-248*, 1989.