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ON APPROXIMATE SYMMETRIES OF A WAVE EQUATION WITH QUADRATIC NON-LINEARITY

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Abstract - Two different approximate symmetry methods and a proposed new one are contrasted using a wave equation with quadratic non-linearity. For each method, the approximate symmetries are calculated first. Then approximate solutions corresponding to some of the symmetries are calculated. It is found that a given specific approximate solution is attainable only by using the new proposed method.

1. INTRODUCTION

Lie Group Theory is a systematic way of finding exact solutions to differential equations. Exact solutions are obtained from the so-called symmetries of the differential equations. For some nonlinear problems, however the symmetries are not rich to obtain useful solutions. If the problem involves a small parameter, then an approximate solution instead of an exact solution can be sought. To combine the power of Lie Group Theory and Perturbation methods to attack such problems, two different approximate symmetry theories have been developed recently.

In Method I, an approximate generator is calculated to find approximate solutions [1,2]. In this method, the dependent variable is not expanded in a perturbation series. In Method II, the dependant variable is expanded in a perturbation series, the equations at each order of approximation are constructed first. The approximate symmetry of the original equation is defined to be the exact symmetry of the outcoming equations at different levels of approximation [3-5]. The equations are assumed to be coupled.

In most of the cases, after expanding the dependent variable, the outcoming equations are linear in unknown variables with some known functions at the right hand sides of the equations. The linear parts of all the equations are the same with the non-homogenous terms being different. Expressing those non-homogenities by arbitrary functions of the independent variables, in Method III, we define the approximate symmetry of the original equation as the exact symmetry of the non-homogenous linear equation. We therefore remove the "coupled-equations" assumption in Method II. This new method requires less algebra and is able to find some approximate solutions that are unattainable by the previous methods. Each method will be applied in the next sections to a simple wave equation with quadratic non-linearity.

2. EQUATION OF MOTION AND AN APPROXIMATE SOLUTION

Consider the wave equation with a quadratic non-linearity

$$\frac{d^2w}{dt^2}+w+\in w^2=0 \tag{1}$$

where $\in$ is a small parameter. If one expands the dependent variable in a perturbation series

$$w=u+\in v \tag{2}$$

and substitutes into the original equation of motion, one obtains

$$\begin{aligned}&\frac{d^2u}{dt^2}+u=0\\&\frac{d^2v}{dt^2}+v=-u^2\end{aligned} \tag{3}$$

The solution of the first equation is

$$u = a \sin t + b \cos t \tag{4}$$

Substituting this solution to the right hand side of second equation, one has

$$v=c\sin t+d\cos t+\frac{ab}{3}\sin 2t-\frac{a^2-b^2}{6}\cos 2t-\frac{a^2+b^2}{2} \tag{5}$$

Therefore the approximate solution is

$$w=a\sin t+b\cos t+\in\left(c\sin t+d\cos t+\frac{ab}{3}\sin 2t-\frac{a^2-b^2}{6}\cos 2t-\frac{a^2+b^2}{2}\right) \tag{6}$$

If boundary conditions permit, the first two terms after the small parameter can be included into the first order solution.

Our aim would be to obtain the approximate solution given in Eq. (6) by using three different approximate symmetry theories. Note that the exact symmetries of Eq. (1) are

$$X=\xi\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial w}\qquad \xi=a,\quad \eta=0 \tag{7}$$

and the above symmetries do not yield useful solutions.

3. METHOD I

In this method [1,2], the dependent variable is not expanded in a perturbation series. Therefore the original equation rather than Eq. (3) is used in finding approximate solutions. The generator is expanded in a series instead

$$X=X_0+\in X_1=\xi_0\frac{\partial}{\partial t}+\eta_0\frac{\partial}{\partial w}+\in\left(\xi_1\frac{\partial}{\partial t}+\eta_1\frac{\partial}{\partial w}\right) \tag{8}$$

where X_0 is the generator corresponding to the unperturbed equation. Details of finding the approximate generator are given in refs [1,2]. The final result is

$$\xi=\xi_0+\in\xi_1=(c_1\sin t+c_2\cos t)w+c_3\sin 2t+c_4\cos 2t+c_5 \\ +\in\left[(C_1\sin t+C_2\cos t)w+C_3\sin 2t+C_4\cos 2t+C_5\right] \tag{9}$$

$$\eta=\eta_0+\in\eta_1=(c_1\cos t-c_2\sin t)w^2+(c_3\cos 2t-c_4\sin 2t+c_6)w+c_7\sin t+c_8\cos t \\ +\in\left[(C_1\cos t-C_2\sin t)w^2+(C_3\cos 2t-C_4\sin 2t+C_6)w+C_7\sin t+C_8\cos t\right] \tag{10}$$

First of all, it should be noted that the symmetries at order $\in$ are repetitions of the unperturbed symmetries. By defining new constants ($\hat{c}_i=c_i+\in C_i$) these symmetries can always be included into the first order symmetries. These symmetries are called as trivial symmetries.

By using any combinations of symmetries given in Eqs. (9) and (10) the approximate solution in equation (6) can not be obtained. A reason might be as follows: That solution is found from Eqs. (3) solved in order. At each order, a linear equation is solved. However, in Method I one uses the original nonlinear equation and this results in a loss of some solutions. Note that in Eq. (6), cos t and sin t are independent solutions added due to linearity. If instead, a nonlinear equation is employed, the addition of those solutions would not be a solution.

4. METHOD II

In this method, the approximate symmetry of Eq.(1) is defined as the exact symmetry of Eqs.(3). The equations are assumed to be coupled. The generator is defined to be

$$X=\xi\frac{\partial}{\partial t}+\eta^1\frac{\partial}{\partial u}+\eta^2\frac{\partial}{\partial v} \tag{11}$$

Performing the standard calculations, the infinitesimals are found to be .

$$\begin{aligned}\xi&=a\\ \eta^1&=bu\\ \eta^2&=2bv+cu+d\cos t+e\sin t\end{aligned} \tag{12}$$

Using parameters a and b, one may write

$$\frac{dt}{a}=\frac{du}{bu}=\frac{dv}{2bv} \tag{13}$$

from which the functions are defined to be

$$u=c_1e^{mt},\quad v=c_3e^{2mt} \tag{14}$$

where m= b/a. Substituting the functions into Eqs.(3), one has

$$m = \pm i, \quad c_3 = \frac{c_1^2}{3} \tag{15}$$

Hence, two different approximate solutions are found as

$$w = c_1 e^{it} + \in \frac{c_1^2}{3} e^{2it} \tag{16}$$

or

$$w = c_2 e^{-it} + \in \frac{c_2^2}{3} e^{-2it} \tag{17}$$

Since Eqs.(3) are nonlinear, the addition of above solutions to obtain solution (4) at the first order would not be possible. Hence, one realizes that approximate solution (6) is unattainable by this method.

5. METHOD III

In this method, similar to Method II, the dependent variable is expanded in a perturbation series and Eqs. (3) are obtained. Contrary to the assumption in Method II, Eqs. (3) are not coupled. When the first equation is solved, the second equation has a known function at the right hand side which can be solved. In this proposed method, therefore, we define the approximate symmetry of Eq. (1) as the exact symmetry of the following linear non-homogenous equation

$$\frac{d^2u}{dt^2} + u = h(t) \tag{18}$$

h(t) of course varies at each order with h=0 at the first order of approximation. The exact symmetries of Eq. (18) are

$$\xi = (c_1 \sin t + c_2 \cos t)w + \alpha_1(t) \tag{19}$$

$$\eta = (c_1 \cos t - c_2 \sin t)w^2 + \alpha_2(t)w + \alpha_3(t) \tag{20}$$

$$2\alpha_2' - \alpha_1'' = 3h(c_1 \sin t + c_2 \cos t) \tag{21}$$

$$\alpha_2'' + 2\alpha_1' = h'(c_1 \sin t + c_2 \cos t) \tag{22}$$

$$\alpha_3'' + \alpha_3 = h(2\alpha_1' - \alpha_2) + h'\alpha_1 \tag{23}$$

Alternatively, equivalence transformations [6,7] can also be employed to find the symmetries of Eq. (18). When h=0 is taken, one obtains correctly the symmetries of unperturbed equation. Using those symmetries, one can easily obtain

$$u = a \sin t + b \cos t \tag{24}$$

as the first order solution. To find the next order solution, choose

$$c_1 = c_2 = \alpha_2(t) = 0 \qquad \alpha_1(t) = \alpha \text{ a const.} \tag{25}$$

so that Eqs. (21) and (22) are satisfied. From Eq. (23)

$$\alpha_3'' + \alpha_3 = h'\alpha \tag{26}$$

where

$$h(t) = -u^2 = -\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2t - ab\sin 2t \tag{27}$$

Substituting for h(t) into Eq. (26), one finally obtains

$$\alpha_3 = \alpha\left(\frac{2ab}{3}\cos 2t + \frac{a^2-b^2}{3}\sin 2t\right) \tag{28}$$

One now has to solve

$$\frac{dt}{\alpha} = \frac{dv}{\alpha\left(\frac{2ab}{3}\cos 2t + \frac{a^2-b^2}{3}\sin 2t\right)} \tag{29}$$

from which

$$v = \frac{ab}{3}\sin 2t - \frac{a^2-b^2}{6}\cos 2t + k \tag{30}$$

Substituting (30) into $\ddot{v} + v = h(t)$, one finally obtains

$$v = \frac{ab}{3}\sin 2t - \frac{a^2-b^2}{6}\cos 2t - \frac{a^2+b^2}{2} \tag{31}$$

The approximate solution would then be

$$w = a\sin t + b\cos t + \in\left(\frac{ab}{3}\sin 2t - \frac{a^2-b^2}{6}\cos 2t - \frac{a^2+b^2}{2}\right) \tag{32}$$

Using this method, therefore, one can obtain the approximate solution given in (6). This approximate solution can not be obtained using the previous methods.

6. CONCLUDING REMARKS

The following conclusions may be drawn from this study:
1) Since Method I and II uses nonlinear equations in finding approximate symmetries, some approximate solutions can not be attainable. This is not the case for Method III.
2) Method III requires less algebra than Method II in calculating symmetries.

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