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ON THE RICCI CURVATURE TENSOR OF (k+1)-DIMENSIONAL SEMI-RULED SURFACES IN E_ν^{n+1}

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Abstract - If we choose a natural companion basis for $(k+1)$-dimensional semi-ruled surfaces in semi-Euclidean space E_ν^{n+1}, then the metric coefficients are $g_{ij} = \varepsilon_i \delta_{ij}$, $1 \le i, j \le k$. In this paper we show that the Ricci curvature tensor of a $(k+1)$-dimensional semi-ruled surface in the semi-Euclidean space E_ν^{n+1} is

$$S = \sum_{j,h=0}^{k} \varepsilon_{jh}(\varepsilon_0 R_{h0j}^0 g_{00} + \sum_{i=0}^{k} R_{hij}^i + \sum_{i=0}^{k} g_{i0}(\varepsilon_i R_{hij}^0 + \varepsilon_0 R_{h0j}^i))\theta_j \otimes \theta_h .$$

Here, $\{\theta_0, \theta_1, \ldots, \theta_k\}$ is the dual basis of the local coordinate basis $\{e_0, e_1, \ldots, e_k\}$.

1. INTRODUCTION

$(k+1)$-dimensional ruled surfaces in E^n are studied by H.Frank and O.Giering, [1], [2]. Several properties of two-dimensional ruled surfaces are also given by C. Thas, [3]. The Ricci curvature tensor of a $(k+1)$-dimensional ruled surface in E^n is calculated by A. Sabuncuoğlu, [5]. In this paper, we will calculate the Ricci curvature tensor of the $(k+1)$-dimensional ruled surface in terms of metric coefficients g_{ij}'s and 1-forms $\theta_0, \theta_1, .., \theta_k$, where $\{\theta_i\}$ is the dual of the coordinate frame field $\{e_0, e_1, .., e_k\}$.

2. FUNDAMENTAL CONCEPTS

Let α be the smooth curve

$$\alpha : I \to E_\nu^{n+1}$$
$$t \to \alpha(t) = (\alpha_1(t), \ldots, \alpha_{n+1}(t))$$

in the $(n+1)$-dimensional semi-Euclidean space E_{ν}^{n+1} where $\{0\} \subset I \subset \Re$. Let $\{e_1(t), e_2(t), .., e_k(t)\}$ be an orthonormal vector system defined at each point $\alpha(t)$ of the curve α. This system spans a subspace of the tangent space $T_{E_{\nu}^{n+1}}(\alpha(t))$ at $\alpha(t) \in E_{\nu}^{n+1}$. If this space is denoted by $E_{k,\mu}(t)$, then it is a k-dimensional subspace of the form

$$E_{k,\mu}(t) = Sp\ \{e_1(t), e_2(t), .., e_k(t)\} \subset E_{\nu}^{n+1}, \qquad 0 \le \mu \le \nu.$$

$E_{k,\mu}(t), \quad \mu \ge 1$, will be called a semi-subspace and

$$\langle e_i(t), e_j(t)\rangle = \varepsilon_i \delta_{ij}, \qquad \varepsilon_i = \begin{cases} +1 \ , & 1 \le i \le k-\mu \\ -1 \ , & k-\mu+1 \le i \le k. \end{cases}$$

is satisfied. For $\mu \ge 1$, there are μ time-like vectors in the semi-subspace $E_{k,\mu}(t)$. Since there is no time-like vector field on $E_{k,0}(t)$ for $\mu = 0$, then $E_{k,0}(t) = E_k(t)$ and it is a Euclidean subspace. If $\mu = 1$, then there is one time-like vector, so $E_{k,1}(t)$ is a time-like subspace.

Throughout this paper, we assume that $E_{k,\mu}(t), \quad \mu \ge 1$, is a semi-subspace.

Definition 1 *While the semi subspace $E_{k,\mu}(t)$ moves along a curve α in E_{ν}^{n+1}, it forms a $(k+1)$-dimensional surface. This surface is called the $(k+1)$-dimensional generalized semi-ruled surface in the $(n+1)$- dimensional semi-Euclidean space E_{ν}^{n+1} and is denoted by M^*.*

On the other hand, let

$$M^* = \bigcup_{t \in I} E_{k,\mu}(t)$$

such that $E_{k,\mu}(t) \subset E_{\nu}^{n+1}$ where $\{e_1(t), e_2(t), ..., e_k(t)\}$ is an orthonormal vector field system defined at every point on the curve α. It is knows that M^* is $(k+1)$-dimensional submanifold of E_{ν}^{n+1}.

Definition 2 *$E_{k,\mu}(t)$ is called the generating space at $\alpha(t)$ of the semi-ruled surface M^* and the curve α is called the base curve of M^*.*

Let M^* be a $(k+1)$-dimensional generalized semi-ruled surface in E_ν^{n+1}. $E_{k,\mu}(t) = Sp\{e_1(t), e_2(t), .., e_k(t)\} \subset E_\nu^{n+1}$, $0 \le \mu \le \nu$, be the genera-ting space and α be the base curve parametrized by the arc-length. Then, M^* can be expressed by the parametric equation

$$\varphi(t, u_1, u_2, .., u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t), \qquad (t, u_1, u_2, .., u_k) \in I \times \Re^k \qquad (1)$$

Let M^* be a $(k+1)$-dimensional semi-ruled surface in the semi-Euclidean space E_ν^{n+1}, let $\{e_1, e_2, .., e_k\}$ be the natural companion bases of the generating space $E_{k,\mu}(s)$ and $e_0 = \varphi_*(\frac{\partial}{\partial s})$ be a vector field such that $\{e_0, e_1, .., e_k\}$ is an orthonormal basis of $\chi(M^*)$. We get

$$\varphi_t = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t)$$

and

$$\varphi_{u_i} = e_i(t), \quad 1 \le i \le k$$

if we take the partial derivation of φ in equality (1). Throughout our this paper we assume that the system

$$\{\dot{\alpha}(t) + \sum_{i=1}^{k} u_i \dot{e}_i(t), e_1, e_2, .., e_k\} \qquad (2)$$

is linearly independent.

Definition 3 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1}, $E_{k,\mu}(t) = Sp\ \{e_1(t), e_2(t), .., e_k(t)\} \subset E_\nu^{n+1}$, $0 \le \mu \le \nu$, be the generating space in M^* and $\dot{e}_i$, $0 \le i \le k$ be the derivative of e_i along α, then the vector subspace*

$$A(t) = Sp\ \{e_1, e_2, .., e_k, \dot{e}_1, .., \dot{e}_k\}$$

is called the asymptotic bundle of M^* *in* $E_{k,\mu}(t)$.

If

$$\dim A(t) = k + m,\ 0 \le m \le k$$

then there exist an orthonormal basis of $A(t)$ containing $E_{k,\mu}(t)$, which is denoted by

$$\{e_1, e_2, .., e_k, e_{k+1}, .., e_{k+m}\}.$$

Since $E_{k,\mu}(t)$, $\mu \ge 1$ is a semi-subspace we have

$$\langle e_i(t), e_i(t)\rangle = \varepsilon_i, \qquad \varepsilon_i = \begin{cases} +1 , & 1 \le i \le k-\mu \\ -1 , & \mathrm{k}-\mu+1 \le i \le k. \end{cases}$$

For $\mu \ge 1$, there are μ time-like vectors in the semi-subspace $E_{k,\mu}(t)$. Since E_ν^{n+1} is a semi-Euclidean space, there are ν time-like vectors in an orthonormal basis of E_ν^{n+1}. Therefore there are μ time-like vectors in basis of the asymptotic bundle $A(t)$ containing $E_{k,\mu}(t)$. Hence,

$$A(t) = Sp\ \{e_1, e_2, .., e_k, a_{k+1}, .., a_{k+m}\}$$

is a semi-subspace.

Theorem 1 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1}, $E_{k,\mu}(t)$ be the generating space and $A(t)$ be the asymptotic bundle of M^*. If* $\dim A(t) = k + m,\ 0 \le m \le k$, *then we choose an orthonormal basis $\{e_1, e_2, .., e_k\}$ of $E_{k,\mu}(t)$ such that*

$$\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + \varepsilon_{k+i} \kappa_i a_{k+i} \quad , \qquad 1 \le i \le m,$$

and

$$\dot{e}_h = \sum_{j=1}^{k} \alpha_{hj} e_j \qquad , \qquad \mathrm{m}+1 \le h \le k,$$

where,

$$\varepsilon_{ij}\alpha_{ij} = -\alpha_{ji} \ , \quad \varepsilon_j = \langle e_j, e_j\rangle, \qquad \varepsilon_{ij} = \varepsilon_i \varepsilon_j$$

and for $r \le \mu$

$$\kappa_1 > \kappa_2 > \ldots > \kappa_{m-r} > 0$$

$$\kappa_{m-r+1} < \kappa_{m-r+2} < \ldots < \kappa_m < 0,$$

see [6].

The basis $\{e_1, e_2, .., e_k\}$ we considered in theorem 2.1 is called the natural companion basis of $E_{k,\mu}(t)$.

Let M^* be an semi-Riemann manifold and R be the Riemann curvature tensor. The Ricci tensor field S is defined in the form

$$S(X,Y) = \sum_{i=0}^{k} \varepsilon_i \langle R(e_i, X)Y, e_i \rangle \tag{3}$$

where $X, Y \in \chi(M^*)$. We have

$$R(e_h, e_l)e_i = \sum_{j=0}^{k} R_{ihl}^{j} e_j, \; 1 \le i, h, l \le k \tag{4}$$

where $\{e_0, e_1, .., e_k\}$ is basis of $\chi(M^*)$ [4].

3. THE RICCI CURVATURE TENSOR OF A (k+1)-DIMENSIONAL SEMI-RULED SURFACES IN E_ν^{n+1}

In this section, we calculate the Ricci curvature tensor of a $(k+1)$-dimensional semi-ruled surface in terms of metric coefficients g_{ij} and 1-forms $\theta_0, \theta_1, .., \theta_k$, where $\{\theta_0, \theta_1, .., \theta_k\}$ is the dual of the coordinate frame field $\{e_0, e_1, .., e_k\}$.

Theorem 2 Let M^* be $(k+1)$*-dimensional semi-ruled surface in* E_ν^{n+1}, $\{e_1, .., e_k\}$ *is the natural companion basis* of $E_{k,\mu}(t)$, *and the metric coefficient of* M^* *be* g_{ij}, $1 \le i, j \le k$. *Then*

$$g_{00} = \sum_{i=1}^{k} \varepsilon_i (\zeta_i + \sum_{j=1}^{k} \alpha_{ij} u_j)^2 + \sum_{h=1}^{m} \varepsilon_{k+h} (\eta_h + \varepsilon_{k+h} \kappa_h u_h)^2 + \varepsilon_{k+m+1} (\eta_{m+1})^2$$

$$\mathsf{g}_{i0} = \varepsilon_i(\zeta_i + \sum_{j=1}^{k}\alpha_{ij}u_j)$$

$$\mathsf{g}_{ij} = \varepsilon_i\delta_{ij}, \qquad 1 \le i, j \le k \tag{5}$$

see [6].

Theorem 3 *The Ricci curvature tensor of a* $(k+1)$*-dimensional semi-ruled surface* M^* *is*

$$S = \sum_{j,h=1}^{k}\varepsilon_{jh}(\varepsilon_0 R_{h0j}^{0}\mathsf{g}_{00} + \sum_{i=0}^{k}R_{hij}^{i} + \sum_{i=0}^{k}\mathsf{g}_{i0}(\varepsilon_i R_{hij}^{0} + \varepsilon_0 R_{h0j}^{i}))\theta_j \otimes \theta_h .$$

Proof Let $X = \sum_{j=0}^{k} x_j e_j$ and $Y = \sum_{i=0}^{k} y_i e_i$. Since R is a tensor field, we have

$$\begin{aligned} R(e_i, X)Y &= R(e_i, \sum_{j=0}^{k} x_j e_j)(\sum_{h=0}^{k} y_h e_h) \\ &= \sum_{j,h=0}^{k} x_j y_h R(e_i, e_j)e_h \end{aligned}$$

By equation (4), we find

$$\begin{aligned} R(e_i, X)Y &= \sum_{j,h=0}^{k} x_j y_h (\sum_{l=0}^{k} R_{hij}^{l} e_l) \\ &= \sum_{j,h,l=0}^{k} x_j y_h R_{hij}^{l} e_l . \end{aligned}$$

From (3), we obtain

$$\begin{aligned} S(X,Y) &= \sum_{i=0}^{k}\varepsilon_i\langle \sum_{j,h,l=0}^{k} x_j y_h R_{hij}^{l} e_l, e_i\rangle \\ &= \sum_{i=0}^{k}\sum_{j,h,l=0}^{k}\varepsilon_i x_j y_h R_{hij}^{l}\langle e_l, e_i\rangle \\ &= \sum_{j,h=0}^{k} x_j y_h (\sum_{i,l=0}^{k}\varepsilon_i R_{hij}^{l}\langle e_l, e_i\rangle \end{aligned}$$

Since $\langle e_l, e_i\rangle = \mathsf{g}_{li}$, $1 \le i, l \le k$, using (5), we find

$$S(X,Y) = \sum_{j,h=0}^{k} x_j y_h \left(\sum_{i,l=0}^{k} \varepsilon_i R^l_{hij} g_{li} \right)$$

$$= \sum_{j,h=0}^{k} x_j y_h (\varepsilon_0 R^0_{h0j} g_{00} + \sum_{i=0}^{k} \varepsilon_i R^0_{hij} g_{i0} + \sum_{l=0}^{k} \varepsilon_0 R^l_{h0j} g_{0l} + \sum_{i=0}^{k} \varepsilon_i R^i_{hij} g_{ii}$$

$$= \sum_{j,h=0}^{k} x_j y_h (\varepsilon_0 R^0_{h0j} g_{00} + \sum_{i=0}^{k} R^i_{hij} + \sum_{i=0}^{k} g_{i0} (\varepsilon_i R^0_{hij} + \varepsilon_0 R^i_{h0j})).$$

Since $\{\theta_0, \theta_1, .., \theta_k\}$ is the dual of the basis $\{e_0, e_1, .., e_k\}$,

$$(\theta_j \otimes \theta_h)(X,Y) = \theta_j(X).\theta_h(Y) = \varepsilon_{jh} x_j y_h$$

with the appropriate substitutions in S(X,Y) we get

$$S(X,Y) = \sum_{j,h=0}^{k} \varepsilon_{jh} \theta_j(X).\theta_h(Y)(\varepsilon_0 R^0_{h0j} g_{00} + \sum_{i=0}^{k} R^i_{hij} + \sum_{i=0}^{k} g_{i0} (\varepsilon_i R^0_{hij} + \varepsilon_0 R^i_{h0j}))$$

$$= \sum_{j,h=0}^{k} \varepsilon_{jh} (\varepsilon_0 R^0_{h0j} g_{00} + \sum_{i=0}^{k} R^i_{hij} + \sum_{i=0}^{k} g_{i0} (\varepsilon_i R^0_{hij} + \varepsilon_0 R^i_{h0j}))(\theta_j \otimes \theta_h)(X,Y)$$

Since this is true for each of $X, Y \in \chi(M^*)$ then

$$S = \sum_{j,h=0}^{k} \varepsilon_{jh} (\varepsilon_0 R^0_{h0j} g_{00} + \sum_{i=0}^{k} R^i_{hij} + \sum_{i=0}^{k} g_{i0} (\varepsilon_i R^0_{hij} + \varepsilon_0 R^i_{h0j}))\theta_j \otimes \theta_h.$$

REFERENCES

1. H. Frank, O. Giering, Verallgemeinerte Regelflachen, Math. Zeit. **150**, 261-271, 1976.

2. H. Frank, O. Giering, Zur Schnittkrümmung Verallgemeinerte Regelflachen, Arch. Math. **32** , 86-91, 1978.

3. C. Thas, Properties of Ruled Surfaces in The Euclidean Space E^n , Academia Sinica, **6**, 133-142, 1978.

4. B. O'Neill, *Semi-Riemannian Geometry*, Academic Press. New York, London, 1983.

5. A. Sabuncuoğlu, Ricci Curvature Tensor of (k+1)-Ruled Surface in E^n, Commun. Fac. Sci. Ank. Series A_1 **37**, 1-4, 1988.

6. C. Ekici, Yarı-Öklidiyen Uzaylarda Genelleştirilmiş yarı-regle Yüzeyler, Osmangazi Üniversitesi Fen Bilimleri Enstitüsünde Hazırlanmış Doktora Tezi, Eskişehir, 1998.