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ON THE CURVATURES OF (k+1)-DIMENSIONAL SEMI-RULED SURFACES IN E_v^{n+1}

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Abstract - In this paper, first we will define the generalized $(k+1)$-dimensional semi-ruled surface M^*, such that the generator space of M^* is the semi-subspace of E_v^{n+1} where E_v^{n+1} is the semi-Euclidean space. Then, we will compute the mean curvature, Riemann curvature, Ricci curvature and scalar curvature of M^*.

1. INTRODUCTION

In this section, we will give some preliminaries. We assume that all manifolds, maps, vector fields, etc. ... are differentiable of class C^∞.

Let M be a semi-Euclidean submonifold of E_v^{n+1}, $\overline{D}$ be a Levi-Civita connection of E_v^{n+1} and D be a Levi-Civita connection of M. If $X,Y \in \chi(M)$ and Π is the second fundamental form of M, then we have the Gauss equation

$$\overline{D}_X Y = D_X Y + \Pi(X,Y) \qquad (1)$$

[4].

Let ξ be a unit normal vector of M. Then, the Weingarten equation is

$$\overline{D}_X \xi = -A_\xi(X) + D_X^\perp \xi \qquad (2)$$

where A_ξ determines at each point a self-adjoint linear map on $T_M(p)$ and $D^\perp$ is a metric connection [3]. We note that, in this paper, A_ξ will be used for the linear map and the corresponding matrix of the linear map.

From equation (1) and (2), it follows

$$\langle \Pi(X,Y),\xi\rangle = \langle A_\xi(X),Y\rangle \tag{3}$$

and

$$\Pi(X,Y) = \sum_{j=1}^{n-m}\langle A_{\xi_j}(X),Y\rangle\xi_j \tag{4}$$

[4].

Let M be an m-dimensional semi-Riemannian submanifold in E_v^{n+1} and A_ξ be a linear map. If $\xi \in \chi^\perp(M)$ is a normal unit vector at the point $p \in M$, then

$$G(p,\xi) = \det A_\xi \tag{5}$$

is called the Lipschitz-Killing vector of M at p in the direction ξ [3].

If $\xi_1,\xi_2,\ldots,\xi_{n-m}$ constitute an orthonormal basis field of the normal bundle $\chi^\perp(M)$, then, the mean curvature H is defined by

$$H = \sum_{j=1}^{n-m}\frac{İzA_{\xi_j}}{boyM}\xi_j \tag{6}$$

[1].

For every $X_i \in \chi(M)$, $i = 1,2,3,4$ the 4th order covariant tensor field defined by R as

$$R(X_1,X_2,X_3,X_4) = \langle X_1, R(X_3,X_4)X_2\rangle \tag{7}$$

is called the Riemann curvature tensor field and its value at a point $p \in M$, is called Riemann curvature of M at p, where M is an m-dimensional semi-Riemannian submanifold in E_v^{n+1} [1].

The Riemann curvature is denoted by

$$K(X,Y)|_p = \langle X, R(X,Y)Y\rangle|_p \tag{8}$$

[1]. If Π is the second fundamental form, then we have

$$\langle X, R(X,Y)Y\rangle = \langle \Pi(X,Y),\Pi(Y,Y)\rangle - \langle \Pi(X,Y),\Pi(X,Y)\rangle. \tag{9}$$

Let M be an m-dimensional semi-Riemann manifold. A 2-dimensional subspace of M the tangent plane $T_M(p)$ of M at p is called tha tangent plane of M at p and is denoted by $\Im$. For all $X_p, Y_p \in \Im$, the real valued function K defined by

$$K(X_p, Y_p) = \frac{\langle R(X_p, Y_p)Y_p, X_p\rangle}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2} \tag{10}$$

is called the sectional curvature function of M at the point p. $K(X_p, Y_p)$ is called the sectional curvature of M at p [4].

Let M be an m-dimensional semi-Riemann manifold and R be the Riemann curvature tensor. The tensor field S defined in the form

$$S(X,Y) = \sum_{i=1}^{m} \varepsilon_i \langle R(e_i, X)Y, e_i\rangle \tag{11}$$

is called the Ricci curvature tensor field, where $\{e_1, e_2, .., e_m\}$ is a system of orthonormal basis of $T_M(p)$ and the value of $S(X,Y)$ at $p \in M$ is called the Ricci curvature [4]. Here

$$\varepsilon_i = \langle e_i, e_i\rangle, \quad \varepsilon_i = \begin{cases} -1 & , if\ e_i\ time-like \\ +1 & , if\ e_i\ space-like. \end{cases}$$

Let M be an m-dimensional semi-Riemann manifold and $\{e_1, e_2, .., e_m\}$ is an orthonormal basis for $T_M(p)$ for $p \in M$. The real number r_{sk} defined in the form

$$r_{sk} = \sum_{i=1}^{m} S(e_i, e_i)$$

or

$$r_{sk} = \sum_{i \neq j} K(e_i, e_j) = 2\sum_{i<j}^{m} K(e_i, e_j) \tag{12}$$

is called the scaler curvature of M where S is the Ricci curvature tensor field of M [4].

2.CURVATURES OF SEMI-RULED SURFACES IN E_v^{n+1}

In this section, we will calculate the Lipschitz-Killing curvature, section curvature, Ricci curvature, mean curvature and the scalar normal curvature of a semi-ruled surface M^* in the semi-Euclidean space E_v^{n+1}.

Let α be the smooth curve

$$\begin{aligned} \alpha : I &\to E_\nu^{n+1} \\ t &\to \alpha(t) = (\alpha_1(t),..,\alpha_{n+1}(t)) \end{aligned}$$

in the $(n+1)$-dimensional semi-Euclidean space E_ν^{n+1} where $\{0\} \subset I \subset \Re$. Let $\{e_1(t), e_2(t),.., e_k(t)\}$ be an orthonormal vector system defined at each $\alpha(t)$ curve α. This system spans a subspace of the tangent space $T_{E_\nu^{n+1}}(\alpha(t))$ at $\alpha(t) \in E_\nu^{n+1}$. If this space is shown by $E_{k,\mu}(t)$, then, it is a k-dimensional subspace of the form

$$E_{k,\mu}(t) = Sp\{e_1(t), e_2(t),.., e_k(t)\} \subset E_\nu^{n+1}, \quad 0 \le \mu \le \nu .$$

$E_{k,\mu}(t)$, $\mu \ge 1$, will be called a semi-subspace and satisfied

$$\langle e_i(t), e_j(t) \rangle = \varepsilon_i \delta_{ij}, \quad \varepsilon_i = \begin{cases} +1 & , 1 \le i \le k - \mu \\ -1 & , k - \mu + 1 \le i \le k. \end{cases}$$

For $\mu \ge 1$, there are μ time-like vectors in the semi-subspace $E_{k,\mu}(t)$. Since there is no time-like vector field on $E_{k,0}(t)$ for $\mu = 0$, then, $E_{k,0}(t) = E_k(t)$ and it is a Euclidean subspace. If $\mu = 1$, then, there is one time-like vector, so $E_{k,1}(t)$ is a time-like subspace.

Throughout this paper, we assume that $E_{k,\mu}(t)$, $\mu \ge 1$, is a semi-subspace.

Definition 1 *While the semi subspace $E_{k,\mu}(t)$ moves along a curve α in E_ν^{n+1}, it forms a $(k+1)$-dimensional surface. This surface is called the $(k+1)$-dimensional generalized semi-ruled surface in the $(n+1)$- dimensional semi-Euclidean space E_ν^{n+1} and is denoted by M^*.*

Definition 2 *$E_{k,\mu}(t)$ is called the genarating space at $\alpha(t)$ of the semi-ruled surface M^* and the curve α is called the base curve of M^*.*

Let M^* be a $(k+1)$-dimensional generalized semi-ruled surface in E_ν^{n+1} $E_{k,\mu}(t) = Sp\{e_1(t),.., e_k(t)\} \subset E_\nu^{n+1}$, $0 \le \mu \le \nu$, be the generating space and α be the base curve parametrized by the arc-length. Then, M^* can be expressed by the parametric equation

$$\varphi(t,u_1,u_2,...,u_k)=\alpha(t)+\sum_{i=1}^{k}u_ie_i(t), \qquad (t,u_1,u_2,...,u_k)\in I\times\Re^k$$

Let M^* be a $(k+1)$-dimensional semi-ruled surface in the semi-Euclidean space E_ν^{n+1}, let $\{e_1,e_2,..,e_k\}$ be the natural companion basis of the generating space $E_{k,\mu}(s)$ and $e_0=\varphi_*(\frac{\partial}{\partial s})$ be a vector field such that $\{e_0,e_1,..,e_k\}$ is an orthonormal basis of $\chi(M^*)$. In addition, suppose that the vector field system $\{\xi_{k+1},\xi_{k+2},..,\xi_n\}$ is an orthonormal basis of $T_{M^*}^{\perp}(p)$ at $p\in M^*$.
Then,

$$\{e_0,e_1,..,e_k,\xi_{k+1},\xi_{k+2},..,\xi_n\}$$

is an orthonormal basis of $T_{E_\nu^{n+1}}(p)$ at $p\in M^*$. Thus, the equations of the derivative can be written in the form

$$\overline{D}_{e_r}\xi_j=\sum_{i=0}^{k}a_{ri}^je_i+\sum_{i=k+1}^{n}b_{ri}^j\xi_i, \quad 0\le r\le k \quad ve \quad k+1\le j\le n \tag{13}$$

Hence, the Weingarten equation can be written as

$$\overline{D}_{e_r}\xi_j=-A_{\xi_j}(e_r)+D_{e_r}^{\perp}\xi_j, \qquad k+1\le j\le n. \tag{14}$$

Then, from equation (13) and (14) the matrix $A_{\xi_j}(e_r)\in\chi(M^*)$ that coresponds to the linear mapping A_{ξ_j} is

$$A_{\xi_j}=\begin{bmatrix} a_{00}^j & a_{01}^j & a_{02}^j & \cdots & a_{0k}^j \\ a_{10}^j & a_{11}^j & a_{12}^j & \cdots & a_{1k}^j \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k0}^j & a_{k1}^j & a_{k2}^j & \cdots & a_{kk}^j \end{bmatrix}_{(k+1)\times(k+1)}, \quad k+1\le j\le n \tag{15}$$

Also $\langle e_i,e_h\rangle=\varepsilon_i\delta_{ih}$, $1\le i,h\le k$, for the orthonormal basis of the generating space $E_{k,\mu}(s)$ where

$$\varepsilon_i = \begin{cases} +1 & ,1 \le i \le k-\mu \\ -1 & ,k-\mu+1 \le i \le k. \end{cases} \quad \text{and} \quad \mu \le \nu$$

If we take the inner product of the both sides of the derivative equations (13) with e_h, we get

$$\langle \overline{D}_{e_r}\xi_j, e_h \rangle = -\langle \overline{D}_{e_r}e_h, \xi_j \rangle = \varepsilon_h a^j_{rh} \tag{16}$$

Since the vectors e_h are parallel in E^{n+1}_ν, $\overline{D}_{e_r}e_h = 0$. Thus, we find

$$a^j_{rh} = 0 \ , \quad 1 \le r, h \le k$$

From (2) and (3), we get

$$a^j_{h0} = \varepsilon_0 \varepsilon_h a^j_{0h}$$

for the components of A_{ξ_j}. Substituting these in (15), we get $\varepsilon_{0i} = \varepsilon_0\varepsilon_i$ and

$$A_{\xi_j} = \begin{bmatrix} a^j_{00} & a^j_{01} & a^j_{02} & \cdots & a^j_{0k} \\ \varepsilon_{01}a^j_{10} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon_{0k}a^j_{k0} & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times(k+1)} \tag{17}$$

where $\langle e_0, e_0 \rangle = \varepsilon_0$. So, we have proved the following three theorems.

Theorem 1 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E^{n+1}_ν. The matrix in* (17) *corresponding to the shape operator A_{ξ_j} of M^* is either symmetric or anti-symmetric depending on the index.*

Theorem 2 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E^{n+1}_ν* and *A_{ξ_j} be the shape operator defined for the unit normal direction ξ_j, $k+1 \le j \le n$. In this case, for* $k \ge 2$

$$\det A_{\xi_j} = 0.$$

Theorem 3 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1}. For every point in M^* and for every normal direction, the Lipschitz-Killing curvature is zero for $k \geq 2$.*

Now, let us observe some theorems and result on the Riemann curvature of a 2-dimensional sector of a semi-ruled surface.

Theorem 4 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1} and $\{e_0, e_1, .., e_k\}$ be an orthonormal vector system at a neigbourhood of a point $p \in M^*$. The 2-dimensional section of M^*, spanned by the vectors $(e_i)|_p$, $1 \leq i \leq k$ and $(e_0)|_p$ has Riemann curvature*

$$K(e_i, e_0) = -\varepsilon_{0i}\langle \overline{D}_{e_i} e_0, \overline{D}_{e_i} e_0 \rangle$$

where $\varepsilon_{0i} = \varepsilon_0 \varepsilon_i$.

Proof. If we consider equations (8) and (9) where R is the Riemann curvature tensor field of the semi-ruled surface M^*, we find the curvature of the section spanned by $\{e_i, e_0\}$ as defined.

We obtain the following.

Corollary 5 *The Riemann curvature of the 2-dimensional section spanned by $(e_i)|_p$, $1 \leq i \leq k$, and vectors $(e_0)|_p$ of a semi-ruled surface M^* can be expressed in terms of the components of the matrix A_{ξ_j} as*

$$K(e_i, e_0) = \sum_{j=k+1}^{n} \varepsilon_{0i} \varepsilon_j (a_{0i}^j)^2$$

where

$$\langle \xi_j, \xi_j \rangle = \varepsilon_j = \begin{cases} +1 & , k+1 \leq j \leq n-y \\ -1 & , n-y+1 \leq j \leq n \end{cases}, \quad \mu + y = \nu.$$

Corollary 6 *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1} and $\{e_1, e_2, .., e_k\}$ be an orthonormal basis for the generating space $E_{k,\mu}(t)$. The Riemann curvature of the 2-dimensional section spanned by $\{e_i, e_j\}$ is*

$$K(e_i, e_j) = 0$$

for $1 \le i, j \le k$. *Here we will state and prove some theorems and results on the Ricci curvature, scalar curvature and mean curvature of the* $(k+1)$*-dimensional semi-ruled surface in* E_ν^{n+1}.

Theorem 7 *Let* M^* *be a* $(k+1)$*-dimensional semi-ruled surface in* E_ν^{n+1} *and* $\{e_1, e_2, .., e_k\}$ *be the orthonormal basis of the generating space* $E_{k,\mu}(t)$. The *Ricci curvature of* M^* *in the direction of the vector fields* e_r, $1 \le r \le k$, *satisfies*

$$S(e_r, e_r) = -\varepsilon_0 \sum_{j=k+1}^{n} \varepsilon_j (a_{0r}^j)^2, \qquad \langle e_0, e_0 \rangle = \varepsilon_0, \qquad \langle \xi_j, \xi_j \rangle = \varepsilon_j.$$

Here, a_{0r}^j *are the components of the matrix* A_{ξ_j}, $k+1 \le j \le n$.

Proof. Substituting $X = Y = e_r$, $1 \le r \le k$, in equation (11) and using the equation (9), we get

$$S(e_r, e_r) = -\varepsilon_0 \langle \Pi(e_0, e_r), \Pi(e_0, e_r) \rangle.$$

Since, $\Pi(e_0, e_r) = \Pi(e_r, e_0)$, $\overline{D}_{e_r} e_0 = \Pi(e_r, e_0)$ and

$$\overline{D}_{e_r} e_0 = -\sum_{j=k+1}^{n} \varepsilon_r a_{0r}^j \xi_j, \qquad 1 \le i \le k,$$

we obtain the desired equality.

Corollary 8 *The Ricci curvature of a* $(k+1)$*-dimensional semi-ruled surface in* M^* *in* E_ν^{n+1} *in the direction of the vector field* e_0 *is given by*

$$S(e_0, e_0) = \varepsilon_0 \sum_{i=1}^{k} \varepsilon_i S(e_i, e_i).$$

Theorem 9 *Let* M^* *be a* $(k+1)$*-dimensional semi-ruled surface in* E_ν^{n+1} *and* $\{e_0, e_1, .., e_k\}$ *be an orthonormal basis for* $\chi(M^*)$. *Then, the scalar curvature of* M^* *is*

$$r_{sk} = 2\sum_{i=1}^{k} \varepsilon_i S(e_i, e_i).$$

Proof. Using the equation (12) and Corollary 2.2., we get

$$r_{sk} = 2\sum_{i=1}^{k} K(e_0, e_i).$$

Here, considering the property $K(e_0, e_i) = K(e_i, e_0)$ of a sectional curvature and Corollary 2.1. and Theorem 2.6., we obtain the desired equality.

Corollary 10. *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1} and e_0 the unit tangent vector of the base curve of M^*. Then, the scalar curvature of M^* is equal to $2\varepsilon_0$ times the Ricci curvature in the direction of the tangent vector field e_0.*

Theorem 11. *Let M^* be a $(k+1)$-dimensional semi-ruled surface in E_ν^{n+1} and e_0 the unit tangent vector of the base curve of M^*. Then, the mean curvature of M^* is*

$$H = \frac{\varepsilon_0}{k+1}\Pi(e_0, e_0).$$

Proof. We substitute $r = h = 0$ *in* (16) and using the derivative equations (1), we get either

$$\langle \overline{D}_{e_0}\xi_j, e_0\rangle = \varepsilon_0 a_{00}^j$$

or

$$\langle \overline{D}_{e_0}e_0, \xi_j\rangle = -\varepsilon_0 a_{00}^j.$$

Considering the equations (14) and (3), we have

$$\langle \Pi(e_0, e_0), \xi_j\rangle = -\varepsilon_0 a_{00}^j.$$

For the matrix A_{ξ_j}, given by the equality (17),

$$trA_{\xi_j} = -a_{00}^j, \qquad k+1 \le j \le n.$$

Substituting this in the definition of H in (6), we get

$$H = \frac{\varepsilon_0}{k+1}\Pi(e_0, e_0).$$

REFERENCES

1. S. Kobayashi-K. Nomizu, Foundations of Differential Geometry, Interacience Publishers a Division of John Wiley Sons, New York, Vol. I-II 1963.

2. C. Thas, Properties of Ruled Surfaces in The Euclidean Space E^n, Academia Sinica, Vol. 6, No: 1, 133-142, 1978.

3. B. O'Neill, Semi-Riemannian Geometry, Academic Press. New York, London, 1983.

4. M. Ergüt, The Drall and the Scalar Normal Curvature of (k+1)-Dimensional Generalised Ruled Surfaces, Commun. Fac. Sci. Ank. Series A_1 38, 115-125, 1989.